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MOLECULAR CHARACTERIZATION OF HARDY-ORLICZ SPACES

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Abstract: We give a molecular characterization of the Hardy-Orlicz spaces $H_w(\mathbb{R}^n)$ (Theorem 2.18), which generalizes similar results for the Hardy spaces $H^p(\mathbb{R}^n)$ for $p \leq 1$. This result is applied to provide a proof of the boundedness of singular integral operators on $H_w(\mathbb{R}^n)$. (Theorem 3.10).

INTRODUCTION. The purpose of this work is to study the Hardy-Orlicz spaces H_w. The usual Hardy spaces H^p can be obtained as particular cases taking $w(t) = t^p$. In [V] Viviani gives an atomic decomposition of H_w. The molecular theory can be found in [GC-RF]. Several authors have used this technique to deal with operators defined on Hardy spaces, see for instance [C], [C-W], [M], [M-S], [T-W].

In this paper we obtain a molecular characterization for H_w with a general w, see section 2, Theorem (2.18). Then, in section 3, we apply this result to study the boundedness of singular integral operators on $H_w(\mathbb{R}^n)$. One of the main difficulties is to define a suitable gauge, that is a notion of molecular "norm", in the context of Orlicz spaces. The one we introduce in (1.41) it is not the same as that considered in the papers above when $w(t) = t^p$. However, in view of Theorem (2.18), they turn out to be equivalent. In the first section we give the notation, definitions and some properties that we shall use in the sequel. We introduce the maximal spaces H_w, the atomic spaces $H^{\rho,q}$, $1 < q \leq \infty$ and the molecular spaces $\mathcal{M}_{(\rho,q,\varepsilon)}, 1 < q \leq \infty,\ \varepsilon > 0$.

1. NOTATION AND DEFINITIONS

Let w be a positive function defined on $\mathbb{R}^+ = \{x \in \mathbb{R}; x > 0\}$. We shall say that w is of lower type l (respectively, upper type l), if there exists a positive constant C such that

$$w(st) \leq Ct^l w(s)$$

for every $0 < t \leq 1$ (respectively, $t \geq 1$). It is easy to see that if w is of positive lower type l, then $\lim_{t\to 0^+} w(t) = 0$, therefore we define $w(0) = 0$.

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We shall say that a positive function w defined on $I\!R^+$ is quasi-increasing (respectively, quasi-decreasing) if there exists a constant C such that

$$w(s) \leq Cw(t)$$

for every $s \leq t$ (respectively $s \geq t$).

We shall understand that two positive functions are equivalent if their quotient is bounded above and below by two positive constants.

Let w be a function of positive lower type l such that $w(s)/s$ is non-increasing. Then the following functions are well defined

$$w^{-1}(s) = \sup\{t : w(t) \leq s\} \quad , \tag{1.1}$$

$$\rho(t) = \frac{t^{-1}}{w^{-1}(t^{-1})} \quad , \tag{1.2}$$

$$\tilde{w}(t) = \int_0^t \frac{w(s)}{s}\, ds \quad , \tag{1.3}$$

$$\tilde{w}^{-1}(s) = \sup\{t : \tilde{w}(t) \leq s\} \quad \text{and} \tag{1.4}$$

$$\bar{\rho}(t) = \frac{t^{-1}}{\tilde{w}^{-1}(t^{-1})} \quad . \tag{1.5}$$

We state the basic properties of these functions, the proofs can be found in [V].
(1.6) The lower type l is less than or equal to one.
(1.7) w is of upper type 1 with constant $C = 1$.
(1.8) w^{-1} is of lower type 1 and of upper type $1/l$.
(1.9) $\tilde{w}$ is a continuous function equivalent to w.
(1.10) $\tilde{w}$ is strictly increasing.
(1.11) $\tilde{w}$ is subadditive.
(1.12) $\tilde{w}(s)/s$ is non-increasing.
(1.13) $\tilde{w}$ is of lower type l and of upper type 1 with constant $C = 1$.
(1.14) $\tilde{w}^{-1}$ coincides with the ordinary inverse function of $\tilde{w}$ and is equivalent to w^{-1}.

(1.15) ρ is a function of upper type $1/l - 1$ equivalent to the non decreasing function
$\bar{\rho}$.
(1.16) $\rho(t)/t^p$ is quasi-decreasing for $p \geq 1/l - 1$.

In order to introduce the atomic spaces $H^{\rho,q}$ and the molecular spaces $\mathcal{M}_{(\rho,q,\varepsilon)}$, $1 < q \leq \infty$, $\varepsilon > 0$, we need the following definition.

(1.17) DEFINITION. *Let w be a function of positive lower type l. Assume that* $\mathbf{b} = \{b_j\}$ *is a sequence of functions in* $L^q(I\!R^n), 1 \leq q \leq \infty$, *and* $\mathbf{c} = \{c_j\}$ *is a sequence of positive constants such that*

$$(1.18) \qquad \sum_j c_j w(||b_j||_q c_j^{-1/q}) = A < \infty.$$

We define

$$(1.19) \qquad \Lambda_q(\mathbf{b},\mathbf{c}) = \inf\left\{\lambda > 0 : \sum_j c_j w\left(\frac{||b_j||_q c_j^{-1/q}}{\lambda^{1/l}}\right) \leq 1\right\}.$$

We observe that

$$(1.20) \qquad \Lambda_q(\mathbf{b},\mathbf{c}) = 0 \text{ if and only if } b_j \equiv 0 \text{ for every } j.$$

If L is the lower type constant of w, then

$$(1.21) \qquad 0 \leq \Lambda_q(\mathbf{b},\mathbf{c}) \leq max(LA, 1)\ .$$

If we also assume that $w(s)/s$ is non-increasing, we have

$$(1.22) \qquad 0 \leq \Lambda_q(\mathbf{b},\mathbf{c}) \leq max(LA, A^l)$$

and

$$(1.23) \qquad \sum_j c_j w\left(\frac{||b_j||_q c_j^{-1/q}}{\Lambda_q(\mathbf{b},\mathbf{c})^{1/l}}\right) = 1.$$

Moreover, arguing in the same way as in the proof of Lemma (4.7) in [V], we can show that if $\alpha_j = ||b_j||_q c_j^{-1/q}/w^{-1}(c_j^{-1})$, then

$$(1.24) \qquad \sum_j \alpha_j \leq C\ (\Lambda_q(\mathbf{b},\mathbf{c}) + 1)^{1/l^2}\ ,$$

with C independent of $\mathbf{b}$ and $\mathbf{c}$. If $\Lambda_q(\mathbf{b},\mathbf{c}) \geq \beta > 0$, we get

$$(1.25) \qquad \sum_j \alpha_j \leq C_\beta(\Lambda_q(\mathbf{b},\mathbf{c}))^{1/l^2}\ ,$$

where C_β depends on β but not on $\mathbf{b}$ and $\mathbf{c}$.

REMARK. *In the following we shall assume that*

(1.26) *w is a function of positive lower type l such that $w(s)/s$ is non increasing and*
$\rho(t)$ is defined by (1.2).

Given $G \in \mathbb{N}$, we define the G-maximal function of a distribution f on $\mathcal{S}$ by

$$f_G^*(x) = \sup |f(\psi)|,$$

where the supremum is taken over all functions ψ belonging to $C_c^\infty(\mathbb{R}^n)$ satisfying $dist(x, supp(\psi)) < |supp(\psi)|$ and

$$\int |\psi(x)|\, dx + |supp(\psi)|^{G+1} \sum_{|\alpha|=G+1} \int |D^\alpha \psi(x)|\, dx = 1.$$

(1.27) DEFINITION. *Let $G \in \mathbb{N}$ such that $Gl > 1$.*
We define

$$H_w = H_w(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : \int w(f_G^*(x))\, dx = A < \infty \right\}$$

and we denote

$$||f||_{H_w} = \inf \left\{ \lambda > 0 : \int w\left(\frac{f_G^*(x)}{\lambda^{1/l}}\right) dx \leq 1 \right\}.$$

It is easy to verify that if $f \in H_w$, then

(1.28) $\quad 0 \leq ||f||_{H_w} \leq max(LA, A^l)$,

(1.29) $\quad ||f||_{H_w} = 0$ if and only if $f \equiv 0$ and

(1.30) $\quad \int w\left(\frac{f_G^*(x)}{||f||_{H_w}^{1/l}}\right) dx = 1.$

It is easy to see that H_w is a complete topological vector space with respect to the quasi-distance induced by $||\ ||_{H_w}$. Moreover H_w is continuosly included in $\mathcal{S}'$. Clearly, when $w(t) = t^p$, $0 < p \leq 1, w$ satisfies (1.26) with $l = p$ and $H_w(\mathbb{R}^n) = H^p(\mathbb{R}^n)$.

In this work we shall denote $N = [n(1/l - 1)]$, where $[x]$ stands for the biggest integer less than or equal to x.

(1.31) DEFINITION. *A (ρ, q) atom, $1 < q \leq \infty$ is a real valued function a on $\mathbb{R}^n$ satisfying:*

(1.32) $$\int a(x) x^\beta\, dx = 0,$$

for every multi-index $\beta = (\beta_1, \ldots, \beta_n)$ such that $|\beta| = \beta_1 + \ldots + \beta_n \leq N$, where $x^\beta = x_1^{\beta_1} \cdot x_2^{\beta_2} \ldots \cdot x_n^{\beta_n}$, *(1.33) the support of a is contained in a ball B and*

$$(1.34)\qquad \begin{cases} \|a\|_q|B|^{-1/q} \le [|B|\rho(|B|)]^{-1} & \text{if } q < \infty, \text{ or} \\ \|a\|_\infty \le [|B|\rho(|B|)]^{-1} & \text{if } q = \infty. \end{cases}$$

Clearly, when $w(t) = t^p$, $p \in (0,1]$, we have that $\rho(t) = t^{\frac{1}{p}-1}$ and a (ρ, q) atom is a (p, q) atom in the usual sense.

Let us observe that, in view of (1.24), if $\{b_j\}$ is a sequence of multiples of (ρ, q) atoms such that there exists a sequence of balls $\{B_j\}$ satisfying $supp(b_j) \subset B_j$ and (1.18) with $c_j = |B_j|$, then the series $\sum_j b_j$ converges in $\mathcal{S}'$.

(1.35) DEFINITION. *We define $H^{\rho,q} = H^{\rho,q}(\mathbb{R}^n)$, $1 < q \le \infty$, as the linear space of all distributions f on $\mathcal{S}$ which can be represented by*

$$(1.36)\qquad f = \sum_j b_j \qquad \text{in } \mathcal{S}',$$

where $\{b_j\}$ is a sequence of multiples of (ρ, q) atoms such that there exists a sequence of balls $\{B_j\}$ satisfying $supp(b_j) \subset B_j = B(x_j, r_j)$ and (1.18) with $c_j = |B_j|$. We denote $\mathbf{b} = \{b_j\}$, $\mathbf{B} = \{|B_j|\}$ and let

$$\|f\|_{H^{\rho,q}} = \inf \Lambda_q(\mathbf{b}, \mathbf{B}),$$

where $\Lambda_q(\cdot,\cdot)$ is as in (1.19) and the infimum is taken over all possible representations of f of the form (1.36).

(1.37) REMARK. *It can be proved that $H_w(\mathbb{R}^n) \equiv H^{\rho,q}(\mathbb{R}^n)$, $1 < q \le \infty$. Moreover, if we define $H^{\rho,q,k}$, $k \ge N$, as in (1.35) but taking atoms satisfying (1.32) for all $|\beta| \le k$, we also have $H_w \equiv H^{\rho,q,k}$, $1 < q \le \infty$. In particular, this implies that definition (1.27) does not depend on G. The atomic descomposition of H_w and the density of L^2 in H_w will be important tools in this work.*

The Remark can be proved following the lines of [V]. However, in our case, since the space of homogeneous type involved is $\mathbb{R}^n$, it is possible to consider Hardy-Orlicz spaces for a larger range of ρ, q, by using atoms with vanishing moments as in (1.32). The necessary modifications can be carry out.

We are now in conditions to introduce the main object of study of this work, the (ρ, q, ε) molecules and the molecular Hardy-Orlicz spaces.

(1.38) DEFINITION. *Assume that $\varepsilon > 0, x_0 \in \mathbb{R}^n$ and $1 < q \le \infty$. A (ρ, q, ε) molecule centered at x_0 is a real valued function M on $\mathbb{R}^n$ satisfying*

$$(1.39)\qquad \|M\|_q \|M\rho(|\cdot - x_0|^n)|\cdot - x_0|^{n(\varepsilon + \frac{1}{q'})}\|_q \le C,$$

where $q' = q(q-1)^{-1}$, *and*

$$\int M(x)x^{\beta}\,dx = 0 \tag{1.40}$$

for every multi-index β *such that* $|\beta| \le N$.

Given M, a (ρ, q, ε) molecule centered at x_0, and B, a ball with the same center, we denote

$$M^B = M\mathcal{X}_B \text{ and}$$

$$M^{CB} = \frac{M\mathcal{X}_{CB}\rho(|\cdot - x_0|^n)(|\cdot - x_0|^{n(\varepsilon+\frac{1}{q'})}}{\rho(|B|)|B|^{\varepsilon+\frac{1}{q'}}}\ .$$

(1.41) DEFINITION. *Assume* $1 < q \le \infty$ *and* $0 < \varepsilon$. *We define* $\mathcal{M}_{(\rho,q,\varepsilon)} = \mathcal{M}_{(\rho,q,\varepsilon)}(\mathbb{R}^n)$, *as the class of distributions* f *on* $\mathcal{S}$ *which can be represented by*

$$f = \sum_j M_j \qquad \text{in } \mathcal{S}', \tag{1.42}$$

where $\{M_j\}$ *is a sequence of* (ρ, q, ε) *molecules centered in* $\{x_j\}$, *such that there exists a sequence of balls* $\{B_j\} = \{B(x_j, r_j)\}$ *satisfying*

$$\sum_j |B_j| w(||M_j^{B_j}||_q |B_j|^{-1/q}) + \sum_j |B_j| w(||M_j^{CB_j}||_q |B_j|^{-1/q}) < \infty.$$

Let $\mathbf{M^B} = \{M_j^{B_j}\}$, $\mathbf{M^{CB}} = \{M_j^{CB_j}\}$ *and* $\mathbf{B} = \{|B_j|\}$. *We define*

$$||f||_{\mathcal{M}_{(\rho,q,\varepsilon)}} = \inf(\Lambda_q(\mathbf{M^B}, \mathbf{B}) + \Lambda_q(\mathbf{M^{CB}}, \mathbf{B}),)$$

where $\Lambda_q(\cdot,\cdot)$ *is as (1.19) and the infimum is taken over all possible representations of* f *of the form (1.42).*

2. MOLECULAR CHARACTERIZATION OF H_w

In order to prove the molecular characterization of H_w (Theorem 2.18), we need some previous lemmas. Let us observe that, in view of the equivalences stated in (1.9) and (1.14), we can assume, without lost of generality, that w satisfies (1.9) through (1.13).

(2.1) LEMMA. *Assume that μ is a Borel measure on $\mathbb{R}^n$ and E is a bounded set such that $\mu(E) = 1$. Suppose that $\{x^\alpha\}_{|\alpha|\le m}$ is linearly independent on E and V is the linear space generated by $\{x^\alpha \mathcal{X}_E(x)\}_{|\alpha|\le m}$. If $u \in L^q(E), 1 \le q \le \infty$, then there exists a unique $v \in V$ such that*

$$(2.2) \qquad \int (u(x)\mathcal{X}_E(x) - v(x))x^\beta \, d\mu(x) = 0, \qquad \textit{for every } \beta, \ |\beta| \le m.$$

In addition

$$v(x) = \sum_{|\alpha|\le m} \int u(y)\mathcal{X}_E(y) y^\alpha \, d\mu(y) \cdot \ v_\alpha(x),$$

where v_α is the unique element of V which satisfies

$$(2.3) \qquad \int v_\alpha(x) x^\beta \, d\mu(x) = \delta_{\alpha,\beta} \qquad \textit{for every } \beta, \ |\beta| \le m.$$

PROOF. Let $v(x) = \sum_{|\alpha|\le m} c_\alpha x^\alpha \mathcal{X}_E(x), \ c_\alpha \in \mathbb{R}$. Clearly, v satisfies (2.2) if and only if

$$\sum_{|\alpha|\le m} c_\alpha \int_E x^\alpha x^\beta \, d\mu(x) = \int_E u(x) x^\beta \, d\mu(x), \qquad \text{for every } \beta, \ |\beta| \le m.$$

Then, since $\{x_\alpha\}_{|\alpha|\le m}$ is linearly independent on the bounded set E, there exists a unique $v \in V$ which satisfies (2.2). On the other hand, arguing as before, we have that for each α, $|\alpha| \le m$, there exists a unique $v_\alpha \in V$ which satisfies (2.3). Thus, if $\sum_{|\alpha|\le m} d_\alpha v_\alpha = 0, \ d_\alpha \in \mathbb{R}$, we have

$$d_\beta = \sum_{|\alpha|\le m} d_\alpha \int v_\alpha(x) x^\beta \, d\mu(x) = 0, \qquad \text{for every } \beta, \ |\beta| \le m.$$

Therefore, $\{v_\alpha\}_{|\alpha|\le m}$ is a basis of V and we can write $v = \sum_{|\alpha|\le m} a_\alpha v_\alpha, \ a_\alpha \in \mathbb{R}$. Finally, in view of (2.3) and (2.2), it follows that

$$a_\beta = \sum_{|\alpha|\le m} a_\alpha \int v_\alpha(x) x^\beta \, d\mu(x) = \int v(x) x^\beta \, d\mu(x) = \int u(x)\mathcal{X}_E(x) x^\beta \, d\mu(x)$$

for every $\beta, |\beta| \le m$.

(2.4) LEMMA. *Suppose that M is a (ρ, q, ε) molecule centered at x_0, with $1 < q \le \infty$ and $\varepsilon > \frac{1}{l} - 1$. Let σ be a positive constant and $B_k = B(x_0, 2^k\sigma)$, with k a* *non-negative integer. Then there exists a sequence of multiples of (ρ, q) atoms $\{b_k\}, supp(b_k) \subset B_k$, such that*

$$M = \sum_{k\geq 0} b_k \qquad in\ \mathcal{S}', \tag{2.5}$$

$$||b_0||_q \leq C||M^{B_0}||_q \qquad if\ k=0,\ or \tag{2.6}$$

$$||b_k||_q \leq C||M^{CB_0}||_q \quad 2^{-n(\varepsilon+\frac{1}{q'})k} \qquad \text{if } k \geq 1, \tag{2.7}$$

where C is a constant independent of M and σ. When $w(t) = t^p, p \in (0,1]$, we have, without restriction for $\varepsilon > 0$, (2.5), (2.6) and

$$||b_k||_q \leq C||M^{CB_0}||_q 2^{-n(\varepsilon+\frac{1}{p}-\frac{1}{q})k}, \qquad \text{for } k \geq 1. \tag{2.8}$$

PROOF. Clearly, we can suppose that M is a (ρ, q, ε) molecule centered at 0. Let $E_0 = B_0, E_k = B_k - B_{k-1}, k \geq 1$, and $M_k = M\mathcal{X}_{E_k}$. Let V_k be the linear space generated by $\{x^\alpha \mathcal{X}_{E_k}\}_{|\alpha|\leq N}$. From Lemma (2.1), with $E = E_k, d\mu = \frac{1}{|E_k|}\,dx, m = N$ and $u = M_k$, there exists a unique $P_k \in V_k$ which verifies

$$\int (M_k(x) - P_k(x))x^\beta\,dx = 0 \tag{2.9}$$

for every $\beta, |\beta| \leq N$. Moreover,

$$P_k = \sum_{|\alpha|\leq N} \frac{1}{|E_k|} \int M_k(x)\ x^\alpha\,dx.\ Q_{\alpha k}\ , \tag{2.10}$$

where $Q_{\alpha k}$ is the unique element of V_k such that

$$\int Q_{\alpha k}(x)\ x^\beta\,dx = |E_k|\delta_{\alpha\beta} \ \text{ for every } \beta, |\beta| \leq N. \tag{2.11}$$

If we denote $m_{\alpha k} = \frac{1}{|E_k|}\int M_k(x)x^\alpha\,dx$, then we can write

$$M(x) = \sum_{k\geq 0} M_k(x) = \sum_{k\geq 0}(M_k(x) - P_k(x)) + \sum_{k\geq 0}\sum_{|\alpha|\leq N} m_{\alpha k}Q_{\alpha k}(x).$$

Since $\sum_{r\geq 0} |E_r|m_{\alpha r} = \int M(x)x^\alpha\,dx = 0$, applying summation by parts, we obtain

$$\begin{aligned}
\sum_{k\geq 0}\sum_{|\alpha|\leq N} m_{\alpha k}Q_{\alpha k}(x) &= \sum_{|\alpha|\leq N}\sum_{k\geq 0}(m_{\alpha k}|E_k|)(|E_k|^{-1}Q_{\alpha k}(x)) \\
&= \sum_{|\alpha|\leq N}\sum_{k\geq 0}\Big(\sum_{r\geq k} m_{\alpha r}|E_r| - \sum_{r\geq k+1} m_{\alpha r}|E_r|\Big)(|E_k|^{-1}Q_{\alpha k}(x)) \\
&= \sum_{|\alpha|\leq N}\sum_{k\geq 0}(|E_{k+1}|^{-1}Q_{\alpha k+1}(x) - |E_k|^{-1}Q_{\alpha k}(x)) \sum_{r\geq k+1} m_{\alpha r}|E_r| \\
&= \sum_{|\alpha|\leq N}\sum_{k\geq 0} \eta_{\alpha k}R_{\alpha k}(x)\ ,
\end{aligned}$$

where $\eta_{\alpha k} = \sum_{r\geq k+1} m_{\alpha r}|E_r|$ and $R_{\alpha k}(x) = |E_{k+1}|^{-1}Q_{\alpha k+1}(x) - |E_k|^{-1}Q_{\alpha k}(x)$. Then, since $supp(M_k - P_k) \subset E_k$ and $supp(\eta_{\alpha k}R_{\alpha k}) \subset E_k \cup E_{k+1}$, it follows that

$$M = \sum_{k\geq 0}(M_k - P_k) + \sum_{|\alpha|\leq N}\sum_{k\geq 0}\eta_{\alpha k}R_{\alpha k}, \qquad \text{locally in } L^q. \tag{2.12}$$

Clearly, by (2.9) and (2.11), $M_k - P_k$ and $\eta_{\alpha k}R_{\alpha k}$ are multiples of (ρ, q) atoms. Furthermore, by (2.11), we get

$$|Q_{\alpha k}(x)| \leq C(2^k\sigma)^{-|\alpha|}. \tag{2.13}$$

Thus, by using (2.10) and Hölder's inequality, we have

$$|P_k(x)| \leq C\int \frac{|M_k(x)|}{|E_k|}\,dx \leq C\left(\int |M_k(x)|^q \frac{dx}{|E_k|}\right)^{\frac{1}{q}},$$

which inmediately yields

$$||M_k - P_k||_q \leq C||M_k||_q, \qquad \text{for every } k \geq 0. \tag{2.14}$$

Then, for $k \geq 1$, since ρ is increasing and of upper type $\frac{1}{l} - 1$, we obtain

$$\begin{aligned} ||M_k - P_k||_q \leq C||M_k||_q &\leq C\frac{||M\mathcal{X}_{CB_0}\rho(|\cdot|^n)|\cdot|^{n(\varepsilon+\frac{1}{q'})}||_q}{\rho((2^{k-1}\sigma)^n)(2^{k-1}\sigma)^{n(\varepsilon+\frac{1}{q'})}} \\ &\leq C||M^{CB_0}||_q 2^{-n(\varepsilon+\frac{1}{q'})k}\ . \end{aligned} \tag{2.15}$$

On the other hand, applying Hölder's inequality, (2.15) and the restriction on ε, we have

$$\begin{aligned} |\eta_{\alpha k}| &\leq \sum_{r\geq k+1}\int |M_r(x)|x|^{|\alpha|}\,dx \\ &\leq \sum_{r\geq k+1} ||M_r||_q(2^r\sigma)^{|\alpha|}|E_r|^{\frac{1}{q'}} \\ &\leq C\sigma^{|\alpha|}||M^{CB_0}||_q|B_0|^{\frac{1}{q'}}\sum_{r\geq k+1}2^{r(|\alpha|-n\varepsilon)} \\ &\leq C\sigma^{|\alpha|}||M^{CB_0}||_q|B_0|^{\frac{1}{q'}}2^{(|\alpha|-n\varepsilon)k}. \end{aligned} \tag{2.16}$$

From (2.13), we get

$$|R_{\alpha k}(x)| \leq C(2^k\sigma)^{-|\alpha|-n}$$

and, applying (2.16), we obtain

$$||\eta_{\alpha k}\ R_{\alpha k}||_\infty \leq C||M^{CB_0}||_q|B_0|^{\frac{-1}{q}}2^{-n(\varepsilon+1)k}.$$

Hence, since $supp\ (\eta_{\alpha k}\ R_{\alpha k}) \subset B_{k+1}$, it follows that

$$(2.17) \qquad ||\eta_{\alpha k}\ R_{\alpha k}||_q \leq C||M^{CB_0}||_q\ 2^{-n(\varepsilon+\frac{1}{q'})k}.$$

Finally, if we define $b_0 = M_0 - P_0$, $b_1 = \sum_{|\alpha|\leq N} \eta_{\alpha 0} R_{\alpha 0}$, $b_k = M_{k-1} - P_{k-1} + \sum_{|\alpha|\leq N} \eta_{\alpha k-1} R_{\alpha k-1}$, $k \geq 2$, by (2.12), (2.14), (2.15) and (2.17), we get (2.5), (2.6) and (2.7). When $w(t) = t^p, p \in (0,1]$, we can improve (2.15) and get

$$||M_k - P_k||_q \leq C||M_k||_q \leq C||M^{CB_0}||_q\ 2^{-n(\varepsilon+\frac{1}{p}-\frac{1}{q})k}, \quad \text{for } k \geq 1.$$

Thus, arguing as before, but without restriction on ε, we have

$$||\eta_{\alpha k} R_{\alpha k}||_q \leq C||M^{CB_0}||_q\ 2^{-n(\varepsilon+\frac{1}{p}-\frac{1}{q})k}, \quad \text{for } k \geq 0$$

which proves (2.8).

(2.18) THEOREM. *Assume that w is a function of positive lower type l such that $w(s)/s$ is non increasing. Let $\rho(t)$ be the function defined by $\rho(t) = t^{-1}/w^{-1}(t^{-1})$. Then $H_w \equiv \mathcal{M}_{(\rho,q,\varepsilon)}$ with $1 < q \leq \infty$ and $\varepsilon > \frac{1}{l} - 1$. When $w(t) = t^p, p \in (0,1]$, we have $H_w \equiv \mathcal{M}_{(\rho,q,\varepsilon)}$ for $1 < q \leq \infty$ and every $\varepsilon > 0$.*

PROOF. By (1.37) is sufficient to prove that $H^{\rho,q} \equiv \mathcal{M}_{(\rho,q,\varepsilon)}$.
First inclusion: $H^{\rho,q} \subset \mathcal{M}_{(\rho,q,\varepsilon)}$. Let f be a distribution in $H^{\rho,q}$. Assume that $\mathbf{b} = \{b_j\}$ is a sequence of multiples of (ρ,q) atoms such that $f = \sum_j b_j$ is a representation of f as in (1.35). Clearly, b_j is a (ρ,q,ε) molecule centered at x_j. Moreover, if we denote $M_j = b_j$, in view of (1.35), (1.38) and (1.41), we have

$$||f||_{\mathcal{M}_{(\rho,q,\varepsilon)}} \leq \Lambda_q(\mathbf{M^B},\mathbf{B}) + \Lambda_q(\mathbf{M^{CB}},\mathbf{B}) \leq \Lambda_q(\mathbf{b},\mathbf{B}).$$

Thus, we have that $f \in \mathcal{M}_{(\rho,q,\varepsilon)}$ and

$$||f||_{\mathcal{M}_{(\rho,q,\varepsilon)}} \leq ||f||_{H^{\rho,q}}.$$

Second inclusion: $\mathcal{M}_{(\rho,q,\varepsilon)} \subset H^{\rho,q}$. Let f be a distribution in $\mathcal{M}_{(\rho,q,\varepsilon)}$. According to definition (1.41) suppose that $\{M_j\}$ is a sequence of (ρ,q,ε) molecules centered at $\{x_j\}$ and $\{B_j\}$ is a sequence of balls, $B_j = B(x_j,r_j)$, such that

$$(2.19) \qquad f = \sum_j M_j \ \text{ in } \mathcal{S}' \text{ and } 0 < \Lambda_q(\mathbf{M^B},\mathbf{B}) + \Lambda_q(\mathbf{M^{CB}},\mathbf{B}) < \infty.$$

In view of (1.20), we can assume that $\Lambda_q(\mathbf{M^B},\mathbf{B}) > 0$ and $\Lambda_q(\mathbf{M^{CB}},\mathbf{B}) > 0$. Applying lemma (2.4) to each M_j with $\sigma = r_j$, from (2.19), we have

$$(2.20) \qquad f = \sum_j \sum_{k\geq 0} b_k^j \qquad \text{in } \mathcal{S}',$$

where b_k^j is a multiple of a (ρ,q) atom, $supp\ (b_k^j) \subset B_k^j = B(x_j, 2^k r_j)$, and $||b_0^j||_q \leq C||M_j^{B_j}||_q$ if $k = 0$ or $||b_k^j||_q \leq C||M_j^{CB_j}||_q 2^{-n(\varepsilon+\frac{1}{q'})k}$ if $k \geq 1$. Let $\eta \geq 1$ be a constant to be determinated later. Since w is an increasing function of lower type l and of upper type 1 we have

$$\sum_j \sum_{k\geq 0} |B_k^j| w \left(\frac{||b_k^j||_q \, |B_k^j|^{-1/q}}{[\eta(\Lambda_q(\mathbf{M^B},\mathbf{B}) + \Lambda_q(\mathbf{M^{CB}},\mathbf{B}))]^{1/l}} \right)$$
$$\leq \frac{C}{\eta} \left[\sum_j |B_j| w \left(\frac{||M_j^{B_j}||_q \, |B_j|^{-1/q}}{\Lambda_q(\mathbf{M^B},\mathbf{B})^{1/l}} \right) \right.$$
$$\left. + \sum_j \sum_{k\geq 1} 2^{kn(1-(\varepsilon+1)l)} |B_j| w \left(\frac{||M_j^{CB_j}||_q \, |B_j|^{-1/q}}{\Lambda_q(\mathbf{M^{CB}},\mathbf{B})^{1/l}} \right) \right],$$

which, by the restriction on ε, is less than or equal to

$$\frac{C}{\eta} \left(\sum_j |B_j| \, w \left(\frac{||M_j^{B_j}||_q \, |B_j|^{-1/q}}{\Lambda_q(\mathbf{M^B},\mathbf{B})^{1/l}} \right) + \sum_j |B_j| \, w \left(\frac{||M_j^{CB_j}||_q \, |B_j|^{-1/q}}{\Lambda_q(\mathbf{M^{CB}},\mathbf{B})^{1/l}} \right) \right) \leq \frac{2C}{\eta}.$$

Choosing $\eta = 2C$, we get

$$\sum_j \sum_{k\geq 0} |B_k^j| \, w \left(\frac{||b_k^j||_q |B_k^j|^{-1/q}}{[2C(\Lambda_q(\mathbf{M^B},\mathbf{B}) + \Lambda_q(\mathbf{M^{CB}},\mathbf{B}))]^{1/l}} \right) \leq 1. \tag{2.21}$$

From (2.20), (2.21) and the observation above (1.35), we obtain

$$||f||_{H^{\rho,q}} \leq C(\Lambda_q(\mathbf{M^B},\mathbf{B}) + \Lambda_q(\mathbf{M^{CB}},\mathbf{B})). \tag{2.22}$$

Then, since we have (2.22) for every possible representation of f in the form (2.19), we get

$$||f||_{H^{\rho,q}} \leq C||f||_{\mathcal{M}_{(\rho,q,\varepsilon)}}.$$

Note that the restriction $\varepsilon > \frac{1}{l} - 1$ was only used in the proof of the inclusion $\mathcal{M}_{(\rho,q,\varepsilon)} \subset H^{\rho,q}$. When $w(t) = t^p, p \in (0,1]$, we can apply (2.8) and, following the same lines as above, we get $H_w \equiv \mathcal{M}_{(\rho,q,\varepsilon)}$ with $\varepsilon > 0$ and $1 < q \leq \infty$.

3. APPLICATION OF THE MOLECULAR CHARACTERIZATION OF H_w

In this section we shall assume that T is a singular integral operator in $\mathbb{R}^n$ with a kernel K of class C^{k+1} outside the origin with k a non-negative integer, satisfying

$$\left| \int_{r<|x|<R} K(x)\, dx \right| \leq C, \qquad 0 < r < R, \tag{3.1}$$

$$\lim_{r\to 0} \int_{r<|x|<1} K(x)\, dx \qquad \text{exists, and} \tag{3.2}$$

$$(3.3) \qquad |D^\beta K(x)| \leq C|x|^{-n-|\beta|} ,$$

for every multi-index β such that $|\beta| \leq k+1$, and every $x \neq 0$. It is well known that, under these conditions, T is a bounded operator on L^q, $1 < q < \infty$. Moreover, if we define the maximal operator

$$T^* f(x) = \sup_{\delta>0} |T_\delta f(x)| ,$$

where

$$T_\delta f(x) = \int_{\delta<|y|} K(y) f(x-y)\, dy ,$$

we have that T^* is bounded on L^q, $1 < q < \infty$ and

$$(3.4) \qquad Tf(x) = \lim_{\delta\to 0} T_\delta f(x) \quad \text{a.e. } x$$

The purpose of this section is to show the boundedness of T on H_w. The main tool will be the molecular characterization obtained in section 2. In [H-V], Harboure and Viviani, using another technique, proved a similar result in the context of the spaces of homogeneous type. In that work, the cancellation property of the kernel K is stronger than (3.1). Moreover, since in our case the space involved is $\mathbb{R}^n$, we can impose more regularity to the Kernel and by using atoms with vanishing moments as in (1.37), it is possible to consider Hardy Orlicz spaces for a larger range of w.

(3.5) LEMMA. *Let w and ρ be as in theorem (2.18). Let T be a singular integral operator with a kernel K satisfying (3.1), (3.2) and (3.3) with $k+1 > n(\frac{1}{l}-1)$. Assume that b is a function belonging to L^q, $1 < q < \infty$, with vanishing moments up to the order k and $supp(b) \subset B = B(x_0, r)$. Let $0 < \varepsilon < 1 - \frac{1}{l} + \frac{k+1}{n}$, then Tb is a (ρ, q, ε) molecule centered at x_0 and*

$$(3.6) \qquad ||Tb||_q \leq C||b||_q,$$

$$(3.7) \qquad ||Tb\ \rho(|\cdot - x_0|^n)|\cdot - x_0|^{n(\varepsilon+\frac{1}{q'})}||_q \leq C\rho(|B|)\ |B|^{\varepsilon+\frac{1}{q'}}\ ||b||_q ,$$

where C is a constant independent of b.

PROOF. Since T conmutes with translations we may assume that b is supported in a ball $B = B(0,r)$. Clearly Tb satisfies (3.6). Let $\tilde{B} = B(0, 2r)$, then

$$||Tb\ \rho(|\cdot|^n)(|\cdot|^n)^{\varepsilon+\frac{1}{q'}}||_q^q = \left(\int_{\tilde{B}} + \int_{C\tilde{B}}\right) |Tb(x)\ \rho(|x|^n)|x|^{n(\varepsilon+\frac{1}{q'})}|^q\, dx = I_1 + I_2.$$

Since ρ is increasing and of upper type, applying (3.6), we have that I_1 is bounded by

$$C[\rho(|B|)\ |B|^{\varepsilon+\frac{1}{q'}}\ ||b||_q]^q.$$

On the other hand, if $x \in C\tilde{B}$ we have

$$Tb(x) = \int_B (K(x-y) - P(x-y))b(y)\,dy,$$

where P is the Taylor polynomial of K at x of degree k. The typical estimate for the remainder in Taylor's formula for this function, (3.3) and Hölder's inequality yield

$$(3.8) \qquad |Tb(x)| \le C \int_B |b(y)| \frac{|y|^{k+1}}{|x|^{n+k+1}}\,dy \le C \frac{||b||_q |B|^{\frac{k+1}{n}+\frac{1}{q'}}}{|x|^{n+k+1}} \qquad , x \in C\tilde{B}.$$

From this estimate and (1.16) we have that I_2 is bounded by

$$C[\rho(|B|)|B|^{\frac{k+1}{n}+\frac{1}{q'}+1-\frac{1}{l}}||b||_q]^q \int_{C\tilde{B}} |x|^{n(\varepsilon-\frac{1}{q}-\frac{k+1}{n}+\frac{1}{l}-1)q}\,dx.$$

Then, since $\varepsilon < 1 - \frac{1}{l} + \frac{k+1}{n}$, I_2 is less than or equal to

$$C[\rho(|B|)\;|B|^{\varepsilon+\frac{1}{q'}}\;||b||_q]^q \;,$$

which completes the proof of (3.7). In order to prove that Tb has vanishing moments up to the order k we shall use the following partition of unity. Take functions $\phi_j(t)$, $j = 0, 1, 2, \ldots, C^\infty$ in $(0, \infty)$ satisfying $\phi_j \ge 0$, $\sum_{j=0}^{\infty} \phi_j(t) = 1$ for every t in $(0, \infty)$. Moreover, we can assume that $supp(\phi_0) \subset [0, 2r]$, $supp(\phi_j) \subset [2^{j-1}r, 2^{j+1}r]$ for $j \ge 1$ and $|\phi_j^{(k)}(t)| \le C_k t^{-k}$ for every $t > 0$, every $k = 0, 1, 2, \ldots$ and every j, with C_k depending only on k. Now, we define for each j, $K_j(x) = K(x)\phi_j(|x|)$, and observe that all the K_j's satisfy the same estimates as K with a uniform constant. Moreover, we have

$$\sum_{j\ge 0} \mathcal{X}_{supp(K_j * b)}(x) \le 4, \text{ at each } x \in I\!R^n.$$

Then we can write

$$(3.9) \qquad \int Tb(x)x^\beta\,dx = \int \sum_{j\ge 0} K_j * b(x)x^\beta\,dx, \quad \text{for every } \beta, |\beta| \le k.$$

Clearly,

$$|\sum_{j=0}^{n} K_j * b(x)x^\beta| \le \sum_{j=0}^{n} |K_j * b(x)||x|^{|\beta|}\chi_{\tilde{B}}(x) + \sum_{j=0}^{n} |K_j * b(x)||x|^{|\beta|}\chi_{C\tilde{B}}(x) = A_1 + A_2$$

For $j \ge 1$, by Hölder's inequality, there exists a constant C, independent of j, such that

$$||K_j * b||_\infty \le C|B|^{-1/q}||b||_q \;.$$

On the other hand, arguing as in (3.8), for $x \in C\tilde{B}$, we get

$$|K_j * b(x)| \leq C\frac{||b||_q|B|^{\frac{k+1}{n}+1/q'}}{|x|^{n+k+1}} \qquad \text{for } j \geq 0,$$

where C is again independent fo j. Thus, since the overlap of the supports of $K_j * b$ is uniformly bounded we have that

$$A_1 \leq C(|K_0 * b(x)| + ||b||_q|B|^{-1/q})|x|^{|\beta|}\chi_{\tilde{B}}(x)$$

and

$$A_2 \leq C\frac{||b||_q|B|^{\frac{k+1}{n}+1/q'}}{|x|^{n+k+1-|\beta|}}\chi_{C\tilde{B}}(x).$$

Then, by (3.9) and the dominated convergence theorem, we obtain

$$\int Tb(x)x^\beta\, dx = 0, \qquad |\beta| \leq k,$$

since $K_j * b$ has vanishing moments up to order k.

(3.10) THEOREM. *Let w and ρ be as in theorem (2.18). Let T be a singular integral operator with a kernel K satisfying (3.1), (3.2) and (3.3) with $k+1 > 2n(\frac{1}{l}-1)$. Then there exists a constant C such that*

$$||Tf||_{H_w} \leq C||f||_{H_w}.$$

PROOF. By (1.37) and (2.18) it is enough to show that

$$||Tf||_{\mathcal{M}_{(\rho,q,\varepsilon)}} \leq C||f||_{H^{\rho,q,k}}\ , \tag{3.11}$$

for every $f \in L^2 \cap H^{\rho,q,k}$, where $\frac{1}{l}-1 < \varepsilon < 1 - \frac{1}{l} + \frac{k+1}{n}$ and $1 < q < \infty$. Let $f \in L^2 \cap H^{\rho,q,k}$ and $\mathbf{b} = \{b_j\}$ be a sequence of multiples of (ρ, q) atoms with vanishing moments up to the order k, $sup(b_j) \subset B_j = B(x_j, r_j)$, such that

$$f = \sum_j b_j \text{ in } \mathcal{S}'. \tag{3.12}$$

From the previous lemma we have that Tb_j is a (ρ, q, ε) molecule centered at x_j satisfying (3.6) and (3.7). Let $M_j = Tb_j$. Arguing in a similar way as it was done in the proof of Theorem (2.20) in [H-V], it can be shown that

$$Tf = \sum_j M_j \qquad \text{in } S'. \tag{3.13}$$

Let η a positive constant to be determinated. In view of (1.35), (1.38) and (1.41), applying (3.6), we have

$$\sum_j |B_j|w\left(\frac{||M_j^{B_j}||_q|B_j|^{-1/q}}{(\eta\Lambda_q(\mathbf{b},\mathbf{B}))^{1/l}}\right) \leq \sum_j |B_j|w\left(\frac{C||b_j||_q|B_j|^{-1/q}}{(\eta\Lambda_q(\mathbf{b},\mathbf{B}))^{1/l}}\right).$$

Then taking $\eta = C^l$ we obtain

$$\Lambda_q((\mathbf{M})^{\mathbf{B}}, \mathbf{B}) \leq C^l \Lambda_q(\mathbf{b}, \mathbf{B}).$$

In a similar way, from (3.7), we get

$$\Lambda_q((\mathbf{M})^{\mathbf{CB}}, \mathbf{B}) \leq C^l \Lambda_q(\mathbf{b}, \mathbf{B}).$$

Then, by (3.13), we have

$$||Tf||_{(\rho,q,\varepsilon)} \leq C\Lambda_q(\mathbf{b}, \mathbf{B}),$$

which completes the proof of the Theorem.

(3.14) REMARK. *When $w(t) = t^p$, $p \in (0,1]$, since from (2.18), $H_w \equiv \mathcal{M}(\rho, q, \varepsilon)$, with $\varepsilon > 0$, we have that T is a bounded operator in H_w with the only restriction $k+1 > n(\frac{1}{l} - 1)$.*

REFERENCES

[C] R. COIFMAN, *"Characterizations of Fourier Transforms of Hardy Spaces"*, Proc. Mat. Acad. Sci. U.S.A., Vol. 71, 4133–4134 (1974).

[C-W] R. COIFMAN y G. WEISS, *"Analyse Harmonique non Conmutative sur Certain Espaces Homogenes"*, Lect. Notes in Math. 242, Springer Verlag, Berlín (1971).

[GC-RF] J. GARCIA CUERVA and J. L. RUBIO DE FRANCIA, *"Weighted Norm Inequalities and Related Topics"*, North-Holland, Amsterdan, New York, Oxford, (1985).

[H-V] E. HARBOURE and B. VIVIANI, *"Boundedness of Singular Integral Operators on H_w"*, Rev. de la Unión Matemática Argentina, Vol. 38, 219–244, (1993).

[M] R. MACIAS, *"A Singular Integral Result for Banach Spaces Valued Functions"*, Atas do 2do. Seminário Brasileiro de Análise, São Paulo, 181–186, (1975).

[M-S] R. MACIAS and C. SEGOVIA, *"Singular Integrals on Generalized Lipschitz and Hardy Spaces"*, Studia Mathematica, T. LXV. (1979).

[T-W] M. TAIBLESON and G. WEISS, *"The Molecular Characterization of Certain Hardy Spaces"*, Asterique, 77, 67-151 (1980)

[V] B. VIVIANI, *"An Atomic Descomposition of the Predual of $BMO(\rho)$"*, Rev. Mat. Iberoamericana, Vol. 3, 401–425, (1987).

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