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FOURIER VERSUS WAVELETS: A SIMPLE APPROACH TO LIPSCHITZ REGULARITY

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Abstract: We give a very simple proof of the caracterization of Lipschitz regularity of a function by the size of its Haar coefficients.

It is well known that given a real function f periodic with period 2π satisfying a Lipschitz α condition for $0 < \alpha \leq 1$, its k^{th} Fourier coefficient is bounded by $|k|^{-\alpha}$. More precisely, the following result holds (see for example Chapter 12 of [9]).

(A) *Let f be a 2π periodic real function satisfying a Lipschitz α condition for $0 < \alpha \leq 1$, i.e., there exists a positive finite constant M such that, $|f(x+h) - f(x)| \leq M|h|^{\alpha}$, for every pair of real numbers h and x. Then, there exists a constant C such that, for every $k \in \mathbb{Z}$, $|C_k[f]| \leq C|k|^{-\alpha}$, where $C_k[f] = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$.*

The result is an easy consequence of the fact that $\int_0^{2\pi} e^{-ikx}dx = 0$, for $k \neq 0$. Nevertheles, it does not constitute a characterization of Lipschitz α. This fact can easily be observed by taking the Fourier coefficients of the characteristic function of a subinterval of $[0, 2\pi]$. Moreover there is no way to characterize the regularity of a function in terms of the size of its Fourier coefficients, this is a very deep fact implied by the results in the article "Sur les coefficients de Fourier des fonctions continues" by J.P. Kahane, Y. Katznelson and K. de Leeuw, see [4]. On the other hand, we can easily obtain an analogous of **(A)** for the Haar coefficients. We define the Haar coefficients of a locally integrable function f as $C_{a,b} = \int_{\mathbb{R}} f(x)H_{a,b}(x)dx$, where $H_{a,b}(x) = a^{-1/2}H(\frac{x-b}{a})$, $a > 0$, $b \in \mathbb{R}$ and H is the Haar function i.e., H is defined by 1, for $0 \leq x < 1/2$; by -1, for $1/2 \leq x < 1$ and 0 otherwise. More precisely we get the following result

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(B) *Let f be a Lipschitz α function for $0 < \alpha \leq 1$. Then, there exists a constant C such that $|C_{a,b}| \leq Ca^{1/2+\alpha}$, for every $a > 0$ and $b \in I\!R$.*

Proof of (B):

$$\begin{aligned}|C_{a,b}| =& \left|\int_{I\!R} H_{a,b}(x)f(x)dx\right| \\ =& a^{1/2}\left|\int_0^1 H(u)[f(au+b)-f(b)]du\right| \\ \leq& Ca^{1/2+\alpha}. \blacksquare\end{aligned}$$

Clearly we have the following general version of (B):

(B') *Let $\varphi : I\!R^+ \to I\!R^+$ be a non-decreasing function and let f be a function satisfying a Lipschitz (φ) condition, i.e., there exists a constant C such that $|f(x) - f(y)| \leq C\varphi(|x-y|)$, for every x, y in $I\!R$. Then, there exists a constant C such that*

(1) $$|C_{a,b}| \leq Ca^{1/2}\varphi(a); \qquad a > 0, \quad b \in I\!R.$$

The aim of this note is to give a very simple proof of the converse of the preceding result, moreover, we shall prove the Lipschitz (ψ) regularity of a function whose Haar coefficients $C_{a,b}$ satisfy (1), with $\psi(t) = \int_0^t \varphi(s)/s\, ds$. Notice that if $\psi(t) \leq C\varphi(t)$, condition (1) is equivalent to Lipschitz (φ) regularity, which is certainly the case for $\varphi(t) = t^\alpha$, $0 < \alpha \leq 1$. The proof of this converse can be extended to get a characterization of Lipschitz spaces with some non-isotropic metrics in higher dimensions.

By using the inversion formula for the continuous wavelet transform, Holschneider and Tchamitchian prove in [3] that Lipschitz α regularity of a function is completely characterized by the size of the wavelet coefficient, see also [2]. The inversion formula itself relies on the Fourier transform. Nevertheless the notion of Lipschitz regularity can be naturally extended to metric spaces and generally, wavelet coefficients can be computed for functions defined on spaces of homogeneous type where Fourier transform is not available. Since the work by Campanato [1], Meyers [5], Spanne [6] among others it has become classical the integral characterization of pointwise regular functions such us Lipschitz α or more generally Lipschitz (φ). A simple proof of these facts for one dimension as can be found in the book [8], can be adapted to give a direct proof of the desired result. The adventage of this approach is that it can be used to get an analog of this result for some families of non-isotropic dilations in dimension higher than one whithout an explicit inversion formula.

Let us first observe that the inequality $|C_{a,b}| \leq Ca^{1/2}\varphi(a)$ can be rewritten as

(2) $$|m(I^-) - m(I^+)| \leq C\varphi(|I|),$$

where $I = [b, b+a]$, I^- is the left half of I, I^+ is its right half, $m(J) = \frac{1}{|J|}\int_J f(x)\, dx$, and $|J|$ is the measure of the interval J.

(3) Theorem: *Let $\varphi : I\!R^+ \to I\!R^+$ be a non-decreasing function such that*

$$\int_0^1 \varphi(t)/tdt < \infty.$$

Let f be a locally integrable function. If the Haar coefficients of f satisfy (1)*, then f is Lipschitz (ψ), with $\psi(t) = \int_0^t \varphi(s)/s\, ds$.*

Proof: Let x and y be two real numbers with $x < y$. Let us now construct two sequences of subintervals $\{I_k^-\}$ and $\{I_k^+\}$ of $I = [x,y]$ in the following way: I_1^- is the left half of I and I_1^+ its right half, I_2^- is the left half of I_1^-, I_2^+ the right half of I_1^+. And so, I_k^- is the left half of I_{k-1}^- and I_k^+ the right half of I_{k-1}^+. Notice now that

$$\begin{aligned}|f(x) - f(y)| \leq& |f(x) - m(I_k^-)| + \sum_{i=2}^{k} |m(I_i^-) - m(I_{i-1}^-)| + |m(I_1^-) - m(I_1^+)| \\ &+ \sum_{i=1}^{k-1} |m(I_i^+) - m(I_{i+1}^+)| + |m(I_k^+) - f(y)|.\end{aligned}$$

By an application of (2) with $b = x$ and $a = y - x$ we get that the central term $|m(I_1^-) - m(I_1^+)|$ is bounded by $C\varphi(|I|)$. In order to estimate the general term of the first sum $|m(I_i^-) - m(I_{i-1}^-)|$ with $2 \leq i \leq k$, notice that

$$\begin{aligned}|m(I_i^-) - m(I_{i-1}^-)| &= |m(I_i^-) - 1/2m(I_i^-) - 1/2m(I_{i-1}^- \setminus I_i^-)| \\ &= 1/2|m(I_i^-) - m(I_{i-1}^- \setminus I_i^-)|.\end{aligned}$$

Since I_i^- and $I_{i-1}^- \setminus I_i^-$ are contiguous intervals with the same length, we apply (2) to the last term in the above equality to obtain $|m(I_i^-) - m(I_{i-1}^-)| \leq C\varphi(|I_{i-1}^-|)$. In a similar way, we can estimate the general term of the second sum by $\varphi(|I_i^+|)$. Therefore

$$\begin{aligned}|f(x) - f(y)| \leq& |f(x) - m(I_k^-)| + C\sum_{i=2}^{k} \varphi(|I_{i-1}^-|) + C\varphi(|I|) + C\sum_{i=1}^{k-1} \varphi(|I_i^+|) \\ &+ |m(I_k^+) - f(y)| \\ \leq& |f(x) - m(I_k^-)| + 2C\sum_{i=0}^{\infty} \varphi(\frac{|I|}{2^i}) + |m(I_k^+) - f(y)|.\end{aligned}$$

Now by Lebesgue Differentiation Theorem, when k tends to infinity, and the properties on φ we get

$$\begin{aligned}|f(x) - f(y)| &\leq 2C\sum_{i=0}^{\infty} \varphi(\frac{|I|}{2^i}) \\ &\leq \frac{2C}{log2} \sum_{i=0}^{\infty} \int_{|I|/2^i}^{|I|/2^{i-1}} \varphi(t)/t\, dt \\ &\leq \frac{2C}{log2} \psi(|I|),\end{aligned}$$

for almost every x and y. So that, after redefining f on a null set, we have a Lipschitz (ψ) function. ■

To ilustrate the applicability of this method to the regularity problem in the parabolic setting, even when our method applies to more general situation, we shall restrict ourselves to the case of dilations $T_\lambda x = e^{A\log\lambda}x$ with $\lambda > 0$ and A the diagonal matrix with eigenvalues 1 and 2 in two dimensions. Actually $T_\lambda x = (\lambda x_1, \lambda^2 x_2)$, for $x = (x_1, x_2)$. The associated translation invariant metric $\rho(x)$ on $\mathbb{R}^2$ is the only solution of $|T_{1/\rho(x)}x| = 1$ (see for example [7]). Let us introduce the following two wavelets in $\mathbb{R}^2$

$$\eta_1(x,y) = \chi(x)H(y)$$

$$\eta_2(x,y) = H(x)\chi(y),$$

where χ is the characteristic function of the one dimensional interval [0,1). Performing the usual translations in $\mathbb{R}^2$ and the parabolic dilations induced by A we get an L^2-normalized family of functions $\eta_i^{a,b}(x) = a^{-3/2}\eta_i(T_{1/a}(x-b)) = a^{-3/2}\eta_i(\frac{x_1-b_1}{a}, \frac{x_2-b_2}{a^2})$, for $a > 0$ and $b \in \mathbb{R}^2$.

(4) Theorem: *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function such that*

$$\int_0^1 \varphi(t)/t\,dt < \infty.$$

Let f be a locally integrable function on $\mathbb{R}^2$. Assume that there is a constant C such that

$$(5) \qquad |<f, \eta_i^{a,b}>| \leq Ca^{3/2}\varphi(a); \qquad a > 0, \quad b \in \mathbb{R}^2, \quad i = 1,2$$

then f satisfies the Lipschitz (ψ) condition with respect to ρ, i.e., $|f(x) - f(y)| \leq C\psi(\rho(x-y))$.

Proof: Let us first notice that the inequalities in (5) can be written as follows

$$(5.a) \qquad |m(I \times J^-) - m(I \times J^+)| \leq C\varphi(a),$$

$$(5.b) \qquad |m(I^- \times J) - m(I^+ \times J)| \leq C\varphi(a),$$

where I and J are two real intervals conforming a parabolic rectangle, i.e. $|I|^2 = |J|$, I^- is the left half of I, while I^+ is its right half. Similar notation applies to J. Given $x = (x_1, x_2)$ and $y = (y_1, y_2)$ two points in the plane, in order to estimate $|f(x) - f(y)|$, we introduce the point $z = (y_1, x_2)$ which satisfies both $\rho(x-z) \leq \rho(x-y)$ and $\rho(z-y) \leq \rho(x-y)$, so that we look for the following inequalities

$$(6) \quad |f(x) - f(z)| \leq C\psi(\rho(x-z)) \quad \text{and} \quad |f(z) - f(y)| \leq C\psi(\rho(z-y)).$$

We shall work out with some detail the proof of the first inequality in (6) and we shall only sketch the similar proof of the second. Let us assume that $x_1 < y_1$, call $I = [x_1, y_1]$. As in the proof of Theorem 3 we are lead to two subinterval sequences of I, $\{I_i^-\}$ and $\{I_i^+\}$ with $k_i = |I_i^-| = |I_i^+|$. Let $J_i = [x_2, x_2 + (2k_i)^2]$, $R_i^- = I_i^- \times J_i$ and $R_i^+ = I_i^+ \times J_i$ for $i \in \mathbb{N}$. For each $k \in \mathbb{N}$ we have

$$\begin{aligned}|f(x) - f(z)| \leq &|f(x) - m(R_k^-)| + \sum_{i=2}^{k} |m(R_i^-) - m(R_{i-1}^-)| + |m(R_1^-) - m(R_1^+)| \\ &+ \sum_{i=1}^{k-1} |m(R_i^+) - m(R_{i+1}^+)| + |m(R_k^+) - f(z)|.\end{aligned}$$

By (5.b) the central term in the right hand side above satisfies the desired bound. For the general term in each of the sums, for example for $|m(R_i^-) - m(R_{i-1}^-)|$, we proceed in the following way: decompose R_{i-1}^- into eight equal parabolic rectangles $R_1, \ldots, R_8$ with $R_1 = R_i^-$, so that

$$8m(R_{i-1}^-) = m(R_1) + m(R_2) + \ldots + m(R_8).$$

Clearly we may assume that the R_j's are indexed in such a way that R_j shares one side with R_{j+1}. Now, since

$$|m(R_i^-) - m(R_{i-1}^-)| \leq \sum_{j=1}^{7} |m(R_j) - m(R_{j+1})|,$$

we only need to show that each of the terms $|m(R_j) - m(R_{j+1})|$ satisfies the desired inequality. Let us first observe that if R_j and R_{j+1} have a common vertical side we can apply again (5.b). On the other hand, when R_j and R_{j+1} share a horizontal side the rectangle defined by the union $R_j \cup R_{j+1}$ of both is not a parabolic rectangle, so that we divide both of them in eight equal parabolic rectangle by dividing only the vertical sides of R_j and R_{j+1} in eight equal intervals. Let us write $R_1^*, R_2^*, \ldots, R_{16}^*$ to denote these new rectangles and assume that they are indexed from top to bottom. Hence

$$\begin{aligned}|m(R_j) - m(R_{j+1})| &= 1/8|\sum_{k=1}^{8} m(R_k^*) - \sum_{k=9}^{16} m(R_k^*)| \\ &\leq 1/8 \sum_{i=0}^{7} |m(R_{8-i}^*) - m(R_{9+i}^*)| \\ &\leq 1/8 \sum_{i=0}^{7} \sum_{k=0}^{2i} |m(R_{8-i+k}^*) - m(R_{9-i+k}^*)|.\end{aligned}$$

Now, since each R_j^* is parabolic, the general term in the last sum can be bounded by (5.a). Finally, by the Differentiation Theorem, which is still valid in the parabolic setting, we may obtain the first inequality in (6). The proof of the second follows the same lines provided we change the iteration of the diadic decomposition on the x-axis by the iteration of the procedure of dividing in four equal parts the vertical intervals containing x_2 and y_2. ∎

REFERENCES

- [1] Campanato, S. *"Propietà di hölderianità di alcune classi di funzioni"*. Ann. Scuola Norm. Sup. Pisa 17 (1963), 175-188.
- [2] Daubechies, I. *"Ten lectures on wavelets"*. SIAM CBMS - NSF Regional Conf. Series in Applied Math. (1992).
- [3] Holschneider, R. and Tchamitchian, F.: *"Pointwise analysis of Riemann's nondifferentiable function"*. Centre de Physique Théorique. Marseille. (1989).
- [4] Kahane, J.P.; Katznelson, Y. and De Leeuw, K.: *"Sur les coefficients de Fourier des fonctions continues"*. CRAS Paris, t.285.
- [5] Meyers, G. *"Mean oscillation over cubes and Hölder continuity"*. Proc. Amer. Math. Soc. 15 (1964), 717-724.
- [6] Spanne, S. *"Some function spaces defined using the mean oscillation over cubes"*. Ann. Scuola Norm. Sup. Pisa 19 (1965), 593-608.
- [7] Stein, E. and Wainger, S. *"Problems in harmonic analysis related to curvature"*. Bull. Am. Math. Soc. 84 (1978), 1239-1295.
- [8] Torchinsky, A. *"Real-Variable Methods in Harmonic Analysis"*. Academic Press, Inc. (1986).
- [9] Wheeden, R. and Zygmung, A.: *"Measure and Integral"*. Marcel Dekker INC. (1977).

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