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THE MINIMUM VALUE OF THE α-CONCENTRATION OF PROCACCIA OF ELEMENTS IN FUCHSIAN GROUPS WITH ZERO-TRACE GENERATORS

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ABSTRACT: The α-concentrations of Procaccia of different elements of a fractal set Ω have been interpreted in terms of entropic/energetic relationships between different states in a quantum mechanical system. But in order to study problems in quantum mechanics in terms of fractal geometry, we need the set Ω to be endowed with an infinite-word code. The limit sets of Fuchsian groups studied in [1] have an $(\alpha, f(\alpha))$ decomposition of Procaccia that model each $(\alpha, f(\alpha))$ curve in the Tél classification. In this work we establish a connection between a value of α and the spelling of infinite words in the fractal sets depicted in [1].

SECTION 1. INTRODUCTION.

SECTION 1.1. THE PHYSICAL MOTIVATION.

In order to construct a Cantor set K, we depart from a unit segment, take off the central third of $[0,1]$, then take away the central thirds of the 2 smaller segments remaining, ... and iterate this procedure *ad infinitum*. This is an example of what the physicists call a "mathematical" fractal set. We have $\dim_H(K) = \log(2)/\log(3)$, a number strictly between 0 and 1, where $\dim_H(K)$ is the Hausdorff dimension of the set K. On the other hand, there are other types of fractal sets Ω arising from the study of physical phenomena. Let us consider the forced pendulum, with internal frequency ω. When plotting the winding number W as a function g of ω, we have that, for a certain critical value of the parameters involved, $W = g(\omega)$ is a Cantor-like staircase. It means that $g(\omega)$ is constant in the so-called intervals of resonance I_k, $k \in I\!N$, of the variable ω, each I_k producing a step of the staircase. The complement of the union of the interior of the intervals I_k, $k \in I\!N$, is a fractal set Ω, and $\dim_H(\Omega)$ is again a number strictly between 0 and 1 [2]. This Ω, given by a natural process or a physical phenomenon, is very different from K. It does not have the regularity of self similarity shown in the process of formation of K.

Procaccia, Jensen and others [3] devised a way of decomposing a "natural" —as opposed to "mathematical"— non-regular Ω into self-similar and regular subsets $\Omega_\alpha \subset \Omega$, $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, $\alpha \in I\!R$. If we denote by $f(\alpha)$ the $\dim_H(\Omega_\alpha)$, then the curve $(\alpha, f(\alpha))$ —heuristically and empirically found to be smooth— is considered an important characteristic of the physical phenomenon which yielded the fractal Ω. In recent years (see, e.g. [4]), it has been observed that a plurality of (apparently unconnected) physical phenomena yield fractals Ω which share the same curve $(\alpha, f(\alpha))$. When this happens the curve is termed "universal".

Duong-van [5] gave a tentative answer to the question posed by the phenomenon of universality. First he observed that both α and $f(\alpha)$ are expressed in terms of $f'(\alpha)$ —via a set of formulae [3]. Then he observed that, if we replace α, $f(\alpha)$, and $f'(\alpha)$ by E, S and $\beta = \frac{1}{T}$ (i.e. entropy, internal energy, and inverse temperature —normalized so that the Boltzmann constant is unity— in quantum mechanics), then what we obtain is precisely the set of connections relating these magnitudes in a quantum mechanical system.

This identification suggests that, what is shared by a variety of physical systems with the same $(\alpha, f(\alpha))$, is a set of subtle entropic/energetic relationships, ...which are not apparent off hand... But in order to apply these results, i.e. in order to interpret quantum mechanics results in terms of fractal geometry and vice versa, the fractal Ω under study must be endowed with the so called infinite-word code for each of its elements [6].

Tél [7] has made a classification of all universal $(\alpha, f(\alpha))$ curves known so far.

SECTION 1.2. THE MATHEMATICAL MODEL.

In 1993 [1], a mathematical model was proposed in order to generate different fractals Ω, such that their corresponding $(\alpha, f(\alpha))$ would model each $(\alpha, f(\alpha))$ curve in the Tél classification. Also, each of these Ω has a natural infinite-word code. These fractals are the limit sets $L(G)$ of minimally generated groups G of movements in the hyperbolic half plane $I\!H$, the generators having zero trace. Let us recall that a rigid movement in $I\!H = \{z = x + iy / y > 0\}$ is a transformation $z \to A(z) = \frac{az+b}{cz+d}$, where a, b, c, and d are real numbers and $ad - bc = 1$. If the trace $a + d$ of all generators of a group of movements G is zero, then we need a minimum of three generators —A, B, and C— for the fractal limit set $\Omega = L(G)$ to be non trivial. A careful look at Fig.1 will remind the reader how the limit set $L(G)$ is formed (with three generators A, B, and C in this example) as a fractal set in the real line $I\!R = \partial I\!H$. The infinite-word code for all points in $\Omega = L(G) \subset I\!R = \partial I\!H$ in terms of the generators —each generator a letter— is described in [1] and [8], and will be described briefly below. A natural question arises: given a fixed value α, what can we say about the spelling of the infinite words which are points in $\Omega = L(G)$ sharing this value of α? The object of this work is to spell words with a minimum value of α, for all $L(G)$, where G is as in [1], i.e. minimally generated by movements in $I\!H$ with zero trace. Let $W(n)$ be the set of finite words of length n (n-letters), i.e. the set of intervals in the n^{th} approximation of the fractal (see below). From Cesaratto [8] we can readily infer that in order to search for points with a minimal value of α, we have to search for the segments of smallest size $W_{\min}(n)$ in each $W(n)$, $n \geq n_0$, $n_0 \in I\!N$, n_0 sufficiently large; and we want this smallest n-letter word (or words) $W_{\min}(n)$ to converge, as $n \to \infty$, to an infinite word $W = W(\alpha_{\min})$.

SECTION 2.

The object of this section is to prove

Therorem 2. *Let G be a Fuchsian group minimally generated by operators of zero trace, and let $L(G)$ be its limit set in $\partial I\!H = I\!R$ (as described in Section 1.2). Let $W(n)$ and $W_{\min}(n)$ be also as above. We will prove that there exists an infinite word W such that its first n letters make up a word $W_{\min}(n)$ of the corresponding covering by intervals $W(n)$ of the fractal set $L(G)$. W has, essentially, the same spelling independent of G. Furthermore, we will construct that essentially unique word.*

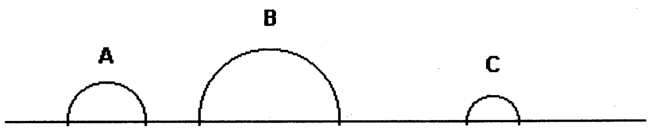
SECTION 2.1. ABOUT NOTATION.

As we said before, G is minimally generated, and $L(G)$ is non trivial; let A, B, and C be the generators. For simplicity let us introduce a pair of abbreviations, while we focus on Fig.1.

1) *Par abus de langage*, and when there is no danger of confusion, A, B, and C will also denote the isometric circles of the corresponding transformations, as shown in Fig.1.a. Let us recall that, if a transformation has zero trace, then it transforms the inside of its isometric circle onto the outside of it, and vice versa —i.e. transformations A, B and C transform the outside of circles A, B and C onto the inside of them, and vice versa.

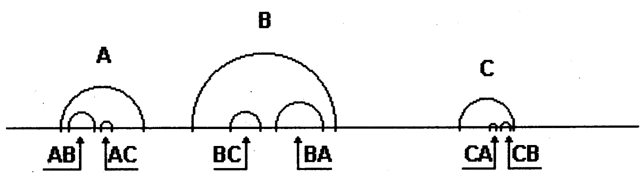
2) Words like, e.g. CA or ABC, which are elements of the group G, will, *par abus de langage* (indeed!), also denote segments CA and ABC indicated in figures 1.c. and 1.d respectively. The longer the word, the shorter the segment, as can be directly observed in Figs.1b...1d: segment ABC is smaller than (and contained in) AB. Thus a correctly spelled infinite word indicates, unequivocally, a point on the real line, in the fractal $L(G)$, and every element in $L(G)$ can be written in this way. The set $W(n)$ of finite words of a certain length n is a covering, by intervals, of $L(G)$. From Cesaratto [8], we can readily infer that a point in $L(G)$ —i.e. an infinite word— has a minimal α ($\alpha_{\min}$) when this point belongs to the smallest segment in $W(n)$, for $n > n_0$, $n_0 \in I\!N$ is sufficiently large. Henceforth, therefore, we will be looking for the smallest segment in $W(n)$, n large, and we will study the spelling of the corresponding word (of n letters)

Fig.1a)



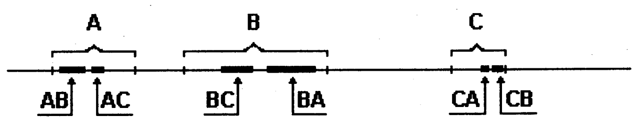
"A, B and C, as semicircles, centred in $\mathbb{R}$, are the isometric circles of the corresponding transformations".

Fig.1b)



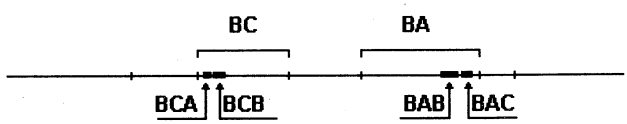
"Circle AB is the image of circle B under transformation A; circle BC is the image of circle C under transformation B... etc".

Fig.1c)



"Segment A is the diameter of circle A. Segment AB, contained in segment A, is the diameter of circle AB, and so on".

Fig.1d)



"This figure is a blow up of Fig.1c). The longer the word, the smaller the corresponding interval. L(G) is precisely the limit set of the smaller and smaller segments, and is made up by all infinite words with a correct spelling".

Figure 1.

which, as n tends to infinity, will give an infinite word —a point in $L(G)$— with a minimal value for α.

SECTION 2.2. AUXILIARY RESULTS.

The object of this section is to prove certain auxiliary results that we will need later.

Lemma 1. *Let X and Y be transformations with zero trace. Let the centre of the isometric circle of Y be at the left of that of X —as seen in Fig.2. Let us consider disjoint segments $\overline{l_0 l_1}$ and $\overline{s_0 s_1}$ in $I\!R$ at the left of the isometric circle of X, $l_0 < l_1 < s_0 < s_1$, and such that $l_1 - l_0 < s_1 - s_0$. Furthermore, let us assume $\mu\left(X\left(\overline{l_0, l_1}\right)\right) < \mu\left(X\left(\overline{s_0, s_1}\right)\right)$ (μ is the usual measure in $I\!R$). Then we have:*

$$\mu\left(Y\left(X\left(\overline{l_0, l_1}\right)\right)\right) < \mu\left(Y\left(X\left(\overline{s_0, s_1}\right)\right)\right). \tag{2.2.1}$$

Proof. Let $Y(z) = \frac{az+b}{cz+d}$, $X(z) = \frac{ez+f}{gz+h}$, and let us write $X\left(\overline{l_0, l_1}\right) = (t_0, t_1)$, $X\left(\overline{s_0, s_1}\right) = (d_0, d_1)$. We want to compare $Y(t_0, t_1)$ with $Y(d_0, d_1)$, its measures can be obtained integrating the derivative of $Y(z)$ between t_0 and t_1, and between d_0 and d_1 respectively. Thus:

$$\mu\left(Y\left(t_0, t_1\right)\right) = \int_{t_0}^{t_1} y'(z)dz = \int_{t_0}^{t_1} \frac{R_Y^2}{\left(z + \frac{d}{c}\right)^2} dz,$$

$$\mu\left(Y\left(d_0, d_1\right)\right) = \int_{d_0}^{d_1} y'(z)dz = \int_{d_0}^{d_1} \frac{R_Y^2}{\left(z + \frac{d}{c}\right)^2} dz,$$

where $R_Y = 1/c$ is the radius of the isometric circle of $y(z)$ (notice that we can take $c > 0$ without changing the transformation $Y(z)$). Making the change of variable $z \to X(z)$ in both integrals, and reordering, we have

$$\mu\left(Y\left(X\left(\overline{l_0, l_1}\right)\right)\right) = \mu\left(Y\left(t_0, t_1\right)\right) = \int_{l_0}^{l_1} \Phi(z)\ dz,$$

$$\mu\left(Y\left(X\left(\overline{s_0, s_1}\right)\right)\right) = \mu\left(Y\left(d_0, d_1\right)\right) = \int_{s_0}^{s_1} \Phi(z)\ dz,$$

where

$$\Phi(z) = \frac{R_X^2 R_Y^2 \tau}{(z - \nu)^2},$$

$$\nu = -\frac{\frac{f}{g} + \frac{dh}{cg}}{\frac{e}{g} + \frac{d}{c}} \qquad \text{and} \qquad \tau = \left(\frac{e}{g} + \frac{d}{c}\right)^{-2}.$$

We now assert that $s_1 < \nu$. To justify our claim it is enough to prove that $\nu > -\frac{h}{g}$ (the centre of the isometric circle of X). We have

$$\nu = -\frac{\frac{f}{g} + \frac{dh}{cg}}{\frac{e}{g} + \frac{d}{c}} = -\frac{\frac{f}{g} + \frac{dh}{cg}}{-\frac{h}{g} + \frac{d}{c}}$$

(recall that Y is a zero-trace operator which means that $e+h=0$). Thus

$$\nu = -\frac{\frac{-1+eh}{gg}+\frac{dh}{cg}}{-\frac{h}{g}+\frac{d}{c}} = \frac{1}{\left(-\frac{h}{g}+\frac{d}{c}\right)g^2} - \frac{h}{g} > -\frac{h}{g},$$

the first equality follows from the fact that $\det(Y)=1$ $(eh-fg=1)$ and the inequality is possible because $(-\frac{h}{g}+\frac{d}{c})>0$ —and it is just the relative location of the centres of X and Y. Therefore, our claim ensures the situation depicted in Fig.3. That is, $l_0<l_1<s_0<s_1<\nu$, hence the area subtended by $\overline{l_0,l_1}$ (denoted with "Al" in Fig.3), is smaller than the one corresponding to $\overline{s_0 s_1}$ ("As" in the same Fig.),which implies (2.2.1). ∎

In the next Lemma with R_X and R_Y we will denote the radii of the isometric circles of the transformations X and Y respectively (letters X and Y will also indicate circles or segments, as before, in a non-ambiguous way).

Lemma 2. *Let X and Y be transformations with zero trace. Let us suppose that $R_X<R_Y$, and that the corresponding isometric circles do not intersect. Then*

$$\mu(\underbrace{XYX\ldots}_{n-\text{letters}}) < \mu(\underbrace{YXY\ldots}_{n-\text{letters}})$$

for any $n\in I\!N$, that is, any n-letter word (letters X and Y only) starting with X is a segment smaller than the corresponding one starting with Y.

Proof. The general proof follows by induction on the length of the word. The case $n=1$ (e.g. $\mu(X)<\mu(Y)$) is just the hypothesis. To show the technique we will sketch the case $n=2$, and what remains is left to the reader. Then our purpose is to prove $\mu(XY)<\mu(YX)$. Let

$$Y(z)=\frac{az+b}{cz+d} \quad \text{and} \quad X(z)=\frac{ez+f}{gz+h} \quad \text{as before.}$$

From the preceding Lemma we can write

$$\mu(XY)=\int_{-d/c-1/c}^{-d/c+1/c}\frac{R_X^2}{\left(z+\frac{h}{g}\right)^2}\,dz \quad \text{and} \quad \mu(YX)=\int_{-h/g-1/g}^{-h/g+1/g}\frac{R_Y^2}{\left(z+\frac{d}{c}\right)^2}\,dz$$

(Recall that we can suppose that c and g are positive, and then $1/c$ and $1/g$ are the radii of the respective isometric circles). Solving the integrals, we have

$$\mu(XY)=\frac{2R_X^2}{(s^2-R_Y^2)g} \quad \text{and} \quad \mu(YX)=\frac{2R_Y^2}{(s^2-R_X^2)c}$$

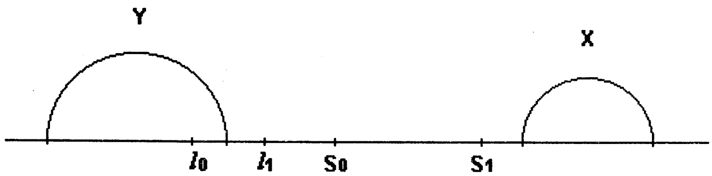


Figure 2.

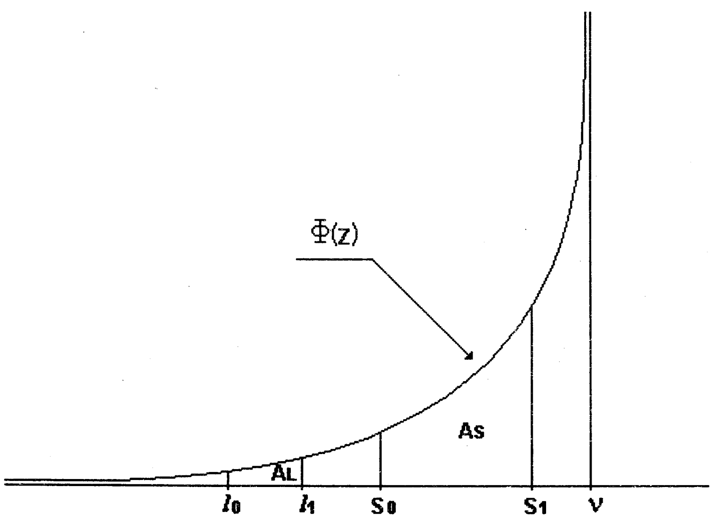


Figure 3.

where (recall that the isometric circles do not intersect) $s = -\frac{h}{g} + \frac{d}{c} > R_X + R_Y$. We only need to prove (recall $R_X = 1/c$, $R_Y = 1/g$)

$$\frac{(s^2 - R_X^2)}{(s^2 - R_Y^2)} < \frac{c}{g} = \frac{R_Y}{R_X}. \tag{2.2.2}$$

Indeed

$$\frac{(s^2 - R_X^2)}{(s^2 - R_Y^2)} = \frac{R_Y^2 - R_X^2}{(s^2 - R_Y^2)} + 1 < \frac{R_Y^2 - R_X^2}{\left((R_X + R_Y)^2 - R_Y^2\right)} + 1 = \frac{R_Y\,(2R_X + R_Y)}{R_X\,(R_X + 2R_Y)} < \frac{R_Y}{R_X},$$

which gives (2.2.2). The Lemma follows. ∎

The following Lemma is elementary. The proof is left to the reader.

Lemma 3. *Let X be a zero-trace operator, let $\overline{t_0 t_1} \subset I\!R$ be a segment outside circle X, and let c be the centre of the isometric circle of X. Then $\mu\left(X\left(\overline{t_0 t_1}\right)\right)$ increases if either R_X or $t_1 - t_0 = \mu\left(\overline{t_0 t_1}\right)$ increases, and decreases if $\left|c - \frac{t_0+t_1}{2}\right|$ increases.*

SECTION 2.3.

The object of this section is to prove a somewhat simpler version of Theorem 2.

In what follows, with R_A, R_B, and R_C we denote the radii of the isometric circles of A, B, and C. Also, finite words made up of letters A, B, and C will consistently denote segments like the ones indicated in Figs.1.c and 1.d.

Theorem 1. *Let A, B, and C be the zero-trace generators of a minimally generated group G (see Fig.1). Let $R_B \geq R_A$, $R_B \geq R_C$. Then for every $n \in I\!N$ we have:*

a) $\mu(\underbrace{ACACA...}_{n-\text{letters}}) \leq \mu$ *(every word of n letters starting with A or B).*

b) $\mu(\underbrace{CACAC...}_{n-\text{letters}}) \leq \mu$ *(every word of n letters starting with C or B).*

Proof. Before proving the theorem, let us notice that taking the smallest of these last two words —e.g. if $\mu(ACACA...) \leq \mu(CACAC...)$— we have the smallest of all possible words of length n. Moreover by Lemma 2 we can assert that if $R_A \leq R_C$ (or if $R_C \leq R_A$) then the smallest word of length n is $\underbrace{ACACA...}_{n-\text{letters}}$ (or $\underbrace{CACAC...}_{n-\text{letters}}$).

Now we can prove Theorem 1: We do it by induction on the length of the word. The case $n = 1$ is merely the hypothesis. The case $n = 2$ follows immediately by Lemma 3. Now we can suppose the theorem valid for words of length n. The location of the words is not arbitrary (see Fig.4) and the proof hinges on this fact. Our first claim is: $\mu(A\underbrace{CAC......}_{n-\text{letters}}) \leq \mu(A\underbrace{............}_{n-\text{letters}})$ (e.g. $A\underbrace{CAC.......}_{n-\text{letters}}$, with $n+1$ letters, is smaller than every other word of $n+1$ letters starting with A). We assume (inductive hypothesis) that

$$\mu(\underbrace{CACAC...}_{n-\text{letters}}) \leq \mu(\underbrace{C.........}_{n-\text{letters}}) \quad \text{and} \quad \mu(\underbrace{CACAC...}_{n-\text{letters}}) \leq \mu(\underbrace{B.........}_{n-\text{letters}}). \tag{2.3.1}$$

Every word of $n+1$ letters starting with A is of the form $A\underbrace{C.........}_{n-\text{letters}}$, or $A\underbrace{B.........}_{n-\text{letters}}$. From this, Eq.(2.3.1), and Lemma 3 we immediately conclude that every segment-word of n letters Z_n located at the left of segment $\underbrace{CACAC...}_{n-\text{letters}}$ fulfills

$$\mu(A\underbrace{CACAC...}_{n-\text{letters}}) \leq \mu(AZ_n). \tag{2.3.2}$$

Eq. (2.3.2) is not obvious for a segment Z located at the right of segment $\underbrace{CACAC...}_{n-\text{letters}}$, for, even if Z_n is larger that $\underbrace{CACAC...}_{n-\text{letters}}$, it is also further away from segment A. Nevertheless $Z_n = C(X_{n-1})$, where X_{n-1} is an $(n-1)$-letter word starting with A or B, and since $\underbrace{CACAC...}_{n-\text{letters}} = C(\underbrace{ACAC.......}_{(n-1)-\text{letters}})$ is a segment left of $C(X_{n-1})$, then, perforce segment $\underbrace{ACAC...}_{(n-1)-\text{letters}}$ is left of segment X_{n-1} (see Fig.5). We are inside all hypotheses of Lemma 1; therefore:

$$\mu(\underbrace{ACACAC...}_{(n+1)-\text{letters}}) = \mu(A(C(\underbrace{ACAC.........}_{(n-1)-\text{letters}}))) \leq \mu(A(C(X_{n-1}))) \leq \mu(A(Z_n)). \tag{2.3.3}$$

Eqs. (2.3.2) and (2.3.3) yield:

$$\mu(\underbrace{ACACAC...}_{(n+1)-\text{letters}}) \leq \mu(\underbrace{A...............}_{(n+1)-\text{letters}}), \tag{2.3.4}$$

where $\underbrace{A....}_{(n+1)-\text{letters}}$ means "any $(n+1)$-letter word starting with A". In the same way, step-by-step, we can see that

$$\mu(\underbrace{CACACA...}_{(n+1)-\text{letters}}) \leq \mu(\underbrace{C...............}_{(n+1)-\text{letters}}), \tag{2.3.4'}$$

where $\underbrace{C...............}_{(n+1)-\text{letters}}$ means "any $(n+1)$-letter word starting with C". It remains to prove that segment $\underbrace{ACACA...}_{(n+1)-\text{letters}}$ is shorter than any segment given by an $(n+1)$-letter word starting with B,... ditto $\underbrace{CACAC...}_{(n+1)-\text{letters}}$ shorter than any $\underbrace{B...............}_{(n+1)-\text{letters}}$.

Now, either $R_A \leq R_C$ or $R_C \leq R_A$. Let us assume the first of the pair. First, we will show that segment $\underbrace{ACACA...}_{(n+1)-\text{letters}}$ is shorter than any $(n+1)$-letter segment starting with BC:

$$\mu(\underbrace{ACACA...}_{(n+1)-\text{letters}}) \leq \mu(BC\underbrace{.................}_{(n-1)-\text{letters}}) \tag{2.3.5}$$

Since $R_B \geq R_A$, then for any n-letter word $\underbrace{C.........}_{n-\text{letter}}$ starting with C we have

$$\mu(A\underbrace{C..........}_{n-\text{letters}}) \leq \mu(B\underbrace{C.........}_{n-\text{letters}}) \qquad (2.3.6)$$

by Lemma 3. Eqs. (2.3.4) and (2.3.6) yield

$$\mu(\underbrace{ACACA...}_{(n+1)-\text{letters}}) \leq \mu(\underbrace{AC...........}_{(n+1)-\text{letters}}) \leq \mu(B\underbrace{C.........}_{n-\text{letters}}), \quad \text{which is (2.3.5).}$$

Let us show that

$$\mu(\underbrace{ACAC.......}_{(n+1)-\text{letters}}) \leq \mu(BA\ X_{n-1})\ , \qquad (2.3.7)$$

where X_{n-1} is an arbitrary word with $n-1$ letters. We have that $R_B \geq R_C$; therefore

$$\mu(CA\ X_{n-1}) \leq \mu(BA\ X_{n-1}). \qquad (2.3.8)$$

From Eqs. (2.3.4') and (2.3.8) we get

$$\mu(\underbrace{CACAC...}_{(n+1)-\text{letters}}) \leq \mu(CA\ X_{n-1}) \leq \mu(BA\ X_{n-1}). \qquad (2.3.9)$$

Let us recall that $R_A \leq R_C$. Then by Lemma 2 and Eq. (2.3.9) we have:

$$\mu(\underbrace{ACAC.......}_{(n+1)-\text{letters}}) \leq \mu(\underbrace{CACAC...}_{(n+1)-\text{letters}}) \leq \mu(BA\ X_{n-1}),$$

which gives (2.3.7).

We have to prove now Eqs. similar to (2.3.5) and (2.3.7) for word $\underbrace{CACAC...}_{(n+1)-\text{letters}}$ instead of $\underbrace{ACACA...}_{(n+1)-\text{letters}}$. That is, we need to show that

$$\mu(\underbrace{CACAC...}_{(n+1)-\text{letters}}) \leq \mu(BA\ X_{n-1}), \qquad (2.3.5')$$

and that

$$\mu(\underbrace{CACAC...}_{(n+1)-\text{letters}}) \leq \mu(BC\ X_{n-1}). \qquad (2.3.7')$$

The proof of Eq. (2.3.5') is analogous, step by step, to the proof of Eq. (2.3.5). It remains to prove Eq. (2.3.7'). Repeating step-by-step the argument that took us to Eq. (2.3.3) we obtain

$$\mu(B(C\ X_{n-1})) > \mu(B(\underbrace{CACAC...}_{n-\text{letters}})) \qquad (2.3.3')$$

which is an equation similar to (2.3.3), with B taking (in (2.3.3')) the place of A in (2.3.3). Since $R_A \leq R_C$, we have (Lemma 3)

$$\mu(\underbrace{CACAC...}_{n-\text{letters}}) \geq \mu(\underbrace{ACACA...}_{n-\text{letters}}). \tag{2.3.10}$$

Now $\underbrace{ACACA...}_{n-\text{letters}}$ is contained in A, which is far away from C; also $\underbrace{CACAC...}_{n-\text{letters}}$ is contained in C, not so far away from B. Besides $R_B \geq R_C$.

These considerations, Eq.(2.3.10), Lemma 3, a glance at Fig.6, and a moment of reflection are sufficient for us to conclude that

$$\mu(B(\underbrace{CACAC...}_{n-\text{letters}})) > \mu(C(\underbrace{ACACA...}_{n-\text{letters}})). \tag{2.3.11}$$

Eqs. (2.3.3') and (2.3.11) yield Eq. (2.3.7'). The proof is complete. ∎

SECTION 2.4 THE GENERAL CASE: SKETCH OF THE PROOF.

Theorem 1 states that, if $R_B \geq R_A$, and $R_B \geq R_C$, then the correct spelling for $W_{\min}(n)$ is $ACACAC...$, or $CACACA...$ But, as can be verified directly, with $A(z) = \frac{0z+1}{-z+0}$, $B(z) = \frac{-3z+5}{-2z+3}$ and $C(z) = \frac{3z-10}{-z+3}$, we have $R_B < R_A = R_C$, and the spelling for $W_{\min}(n)$ is $\underbrace{BACACA...}_{n-\text{letters}}$, or $\underbrace{BCACAC...}_{n-\text{letters}}$. Therefore, except for a possible finite number of letters, the spelling of $W_{\min}(n)$ is still $\underbrace{ACACA...}_{n-\text{letters}}$, or $\underbrace{CACAC...}_{n-\text{letters}}$. This finite number of letters do not change the value of α, that is

$$\alpha(A_1 A_2 A_3 ... A_n ...) = \alpha(B_1 ... B_k A_1 A_2 A_3 ... A_n ...) \quad [9].$$

How much smaller can we make R_B —when compared with R_A and R_C— in order to maintain, except for a finite number of letters, those $\underbrace{ACACA...}_{n-\text{letters}}$ or $\underbrace{CACAC...}_{n-\text{letters}}$ spelling for $W_{\min}(n)$? The equations

$$R_B \geq R_A \qquad \text{and} \qquad R_B \geq R_C \tag{2.4.1}$$

can be generalized to

$$\frac{R_B}{(C_B - C_C)^2} \geq \frac{R_A}{(C_A - C_C)^2} \qquad \text{and} \qquad \frac{R_B}{(C_B - C_A)^2} \geq \frac{R_C}{(C_C - C_A)^2}. \tag{2.4.2}$$

We proceed by induction as before with Eq. (2.4.2) *in lieu* of Eq. (2.4.1). The proof is technically of the same order of difficulty but somewhat long. If Eqs. (2.4.2) are not simultaneously fulfilled, then $\underbrace{ACACA...}_{n-\text{letters}}$ (or $\underbrace{CACAC...}_{n-\text{letters}}$) is no longer the spelling of $W_{\min}(n)$. If Eqs. (2.4.1) are fulfilled, then Eqs. (2.4.2) trivially follow.

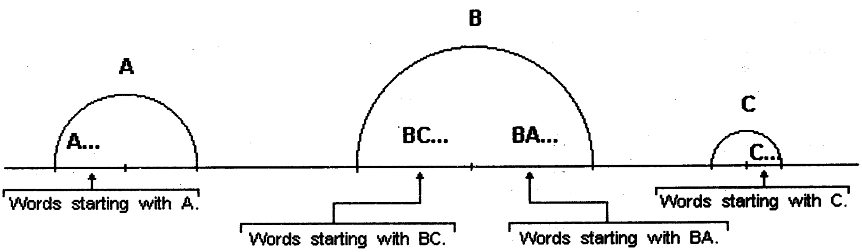


Figure 4.

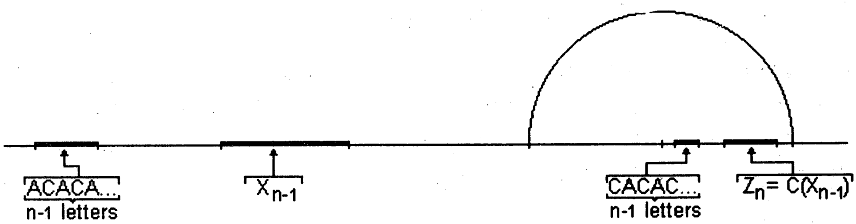


Figure 5.

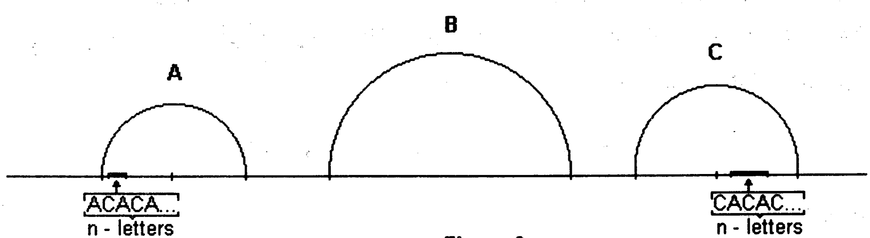


Figure 6.

Equations

$$\frac{R_C}{(C_C - C_A)^2} \geq \frac{R_B}{(C_B - C_A)^2} \quad \text{and} \quad \frac{R_C}{(C_C - C_B)^2} \geq \frac{R_A}{(C_B - C_A)^2} \tag{2.4.3}$$

$$\frac{R_A}{(C_A - C_B)^2} \geq \frac{R_C}{(C_C - C_B)^2} \quad \text{and} \quad \frac{R_A}{(C_A - C_C)^2} \geq \frac{R_B}{(C_B - C_C)^2} \tag{2.4.4}$$

are conditions for $\underbrace{ABAB...}_{n-\text{letters}}$ and $\underbrace{BCBC...}_{n-\text{letters}}$ respectively, to be the spelling (except for a finite number of impurities, Eqs.1 do not allow impurities) of $W_{\min}(n)$.

This completes the sketch of the proof of Theorem 2. ∎

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