**Revista de la**
**Unión Matemática Argentina**
**Volumen 40, Números 3 y 4, 1997.**



# PROPERTIES OF EXTERNAL VISIBILITY.[1]

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**Abstract.** The *external visibility* of a closed set S means the visibility referred to its complementary points. This kind of geometrical study appears naturally in the planning of movements of servomechanisms and robots. The aim of this paper is to connect the external visibility of a certain set S (in particular Stavrakas' half-line property) with new properties which involve points of S instead of points of its complement. We say that a hunk S has the shining boundary property if its complement is free from bounded connected components and for each boundary point of S there exists a ray issuing from it and disjoint with the interior of S. It is proved here the equivalence (for a planar hunk) of this property and Stavrakas' half-line property. Furthermore, in some cases which we specify, Stavrakas' property is equivalent to the fact that each boundary point has nontrivial strong inner stem. These equivalencies yield new versions of some characterizations of starshapedness due to Stavrakas.

## § 1.- BASIC DEFINITIONS AND NOTATIONS

Unless otherwise stated, all the points and sets considered here are included in $\mathbf{R}^n$ the real n-dimensional euclidean space. The interior, closure, boundary, and complement of a set S are denoted by: intS, cl S, bdry S, $\mathcal{C}$S respectively. The open segment joining x and y is denoted (x,y). The substitution of one or both parentheses by square ones indicates the adjunction of the corresponding extremes. We say that *x sees y* via S if $[x,y] \subset S$. The *star of x* in S is the

[1] The preparation of this paper was supported in part by Comisión de Investigaciones Científicas de la Pcia. de Bs. As.

set st(x,S) of all the points of S that see x via S or analogously the *visibility of x in S* is the star of x in S. The *external visibility of S* is the study of the visibility in the complement of S, or in certain cases in the closure of the complement of S. A *star-center of S* is a point $x \in S$ such that st(x,S) = S. *The kernel of S* is the set kerS of all the points star-centers of S, and *S is starshaped* if kerS $\neq \emptyset$. The ray issuing from x and going through y is denoted by $R(x \rightarrow y)$, while $R(xy \rightarrow)$ is the ray issuing from y and going in the same direction to that of $R(x \rightarrow y)$. We say that the ray $R(x \rightarrow y)$ is *inward through y* if there exists $t \in R(xy \rightarrow)$ such that (y,t) is included in intS ($y \in$ bdry S and $x \in$ st(y,S)). Otherwise we say that $R(x \rightarrow y)$ is *outward through y*. (All the rays considered here are closed ones).The *inner stem of y* with respect to S is the set ins(y,S) = $\{y\} \cup \{x \in$ st(y,S) / $R(x \rightarrow y)$ is an outward ray through y$\}$. S is a *regular domain* if int S is connected and S = cl(intS). A bounded regular domain is called a *hunk*. A point $x \in S$ is a *k-extreme point* of S provided for every (k+1)- dimensional simplex $D \subset S$, $x \notin$ relintD, where relintD denotes the interior of D relative to the (k+1)-dimensional space D generates. S is said to have the *half-line property* (hlp) if for each point $x \in CS$ there exists a ray issuing from x and having empty intersection with S. We note $\Omega_n = \{x \in \mathbf{R}^n \, / \, \|x\| = 1\}$. The *algebraic hull of the set A* is defined as follows: ${}^aA = \{y \in \mathbf{R}^n$ / exists $x \in A$ such that $[x,y) \subset A\}$, in other words ${}^aA$ is formed by all the points of $\mathbf{R}^n$ that have linear accessibility through A; and the *convex hull of A* is denoted convA.

## § 2.- STAVRAKAS' HALF-LINE PROPERTY

**Lemma 2.1:** *If $S \subset \mathbf{R}^n$ is a hunk, then S has the half-line property if and only if for each point x of the complement of S holds that the star of x - in CS - is unbounded.*

Proof. $\Rightarrow$) Let $x \in CS$, consider the ray R with vertex in x such that $R \cap S = \emptyset$. $R \subseteq CS$ and immediately $R \subseteq$ st(x,CS), then st(x,CS) is unbounded.

$\Leftarrow$) Suppose that there exists $x \in CS$ such that for every R half-line with x as vertex it is $R \cap S \neq \emptyset$. We pick U neighborhood such that $S \subseteq U$, then K = st(x,CS) is bounded: it verifies $K \subseteq U$ because otherwise we can consider $R = [x,w] \cup R(xw \rightarrow)$ where w is a point $w \neq x$, $w \in K$, $w \notin U$ and R contradicts the initial assumption.❑

**Lemma 2.2:** *$S \subset R^n$ a hunk. If S has the half-line property, then the complement CS is free from bounded connected components.*

Proof. Suppose that there exists $A \subset CS$ a bounded connected component. Let $a \in A$, and consider st(a,CS). This star is connected because it is clearly a path-connected set and it is unbounded by hypothesis. As it intersects A it should verify st(a,CS) $\subset$ A which is absurd. ❑

Notice that the converse is false. Consider for example the planar set $S = S_1 \cap CS_2$ where $S_1$ and $S_2$ are $S_1 = \{(x,y) \in \mathbf{R}^2 / 0 < 1 \leq x^2 + y^2 \leq 4\}$; $S_2 = \{(x,y) \in \mathbf{R}^2 / |x| < 0.1;\ y \geq 0\}$.

We consider $S \subseteq \mathbf{R}^n$ a closed set such that ${}^aS = S$ and ${}^aCS =$ clCS, we say that S has the *shining boundary property* (sbp) if and only if S has its complement free from bounded connected components and if given each boundary point of S there exists a ray issuing from it and disjoint with the interior of S.

**Proposition 2.3:** *Let $S \subseteq R^n$ be a hunk, ${}^aS = S$ and ${}^aCS =$ clCS. If S has the half-line property, then S has the shining boundary property.*

Proof. Suppose S does not have the shining boundary property, then there are two alternatives: (i) CS has a bounded connected component, (ii) there exists p a boundary point of S such that taking any half-line R(p) with vertex in p, it verifies that R(p) intersects intS. For the first case, using lemma 2.2 it results that S does not have the half-line property. For the second case, let us consider K = st(p,clCS). We can check with a standard argument that under this hypothesis K is a bounded set. We will show the existence of a point x in $CS \cap K$ such that st(x,CS) is bounded. Due to lemma 2.1 this will be absurd. We define the following proper subset of K: $A = \{ x \in K /\ \exists\ v \in \Omega_n$ such that $\{ \lambda v + x / \lambda \geq 0 \} \cap \text{int}S = \emptyset \}$. Notice that it is immediate that if $x \in K \cap CA$, then st(x,CS) is bounded because on the contrary we would have st(x, clCS) a closed unbounded starshaped set and, due to theorem 4.1 of [1], x would belong to A. A is properly included in K because $p \in K$ but $p \notin A$ (otherwise $\exists\ v_0 \in \Omega_n$ such that $\{\lambda.v_0 + p / \lambda \geq 0\} \subseteq K$, what means K unbounded). We prove that A is a closed set: let be $\{x_n\}$ a convergent sequence included in A, and let be $\lim x_n = x$. As $x_n \in A$, there exists $v_n \in \Omega_n$ such that $\{\lambda.v_n + x_n / \lambda \geq 0\} \cap \text{int}S = \emptyset$ then we have a sequence $\{v_n\}$ in $\Omega_n$ and standard compactness arguments assert that $\{v_n\}$ (or a subsequence thereof) converge to a certain $v_o$ in $\Omega_n$. It is easy to verify that $\{\lambda.v_o + x / \lambda \geq 0\} \cap \mathrm{int}S = \emptyset$. Then $x \in A$, and A is a closed set.

If $A = \emptyset$ we consider $x \in \mathrm{int}K$, $x \neq p$. Such $x \notin A$. (There exists such x because ${}^aCS = \mathrm{cl}CS$). If $A \neq \emptyset$, A closed set implies that there exists a neighborhood U of p such that $(U \cap K) \cap A = \emptyset$. The fact that $p \in \mathrm{cl}CS$ implies that p belongs to ${}^aCS$ and then there exists $w \in CS$ such that $[w,p) \subset CS$. We pick any $x \in [w,p) \cap U$. Such x does not belong to A. Then in both cases st(x,CS) is bounded. ❑

**Proposition 2.4:** *Let S be a planar hunk, ${}^aS = S$ and ${}^aCS = \mathrm{cl}CS$. If S does not have the half-line property, then S does not have the shining boundary property.*

Proof. Using lemma 2.1, there exists $x \in CS$ such that $K = \mathrm{st}(x,CS)$ is bounded. We will prove that either exists a bounded connected component of CS, or there exists p a boundary point of S such that any ray issuing from it meets intS. We consider the following set:

$A = \{ y \in \mathrm{cl}K / \exists\, v \in \Omega_2$ such that $\{ \lambda.v + y / \lambda \geq 0\} \cap \mathrm{int}S = \emptyset\}$. Naturally there appear four possibilities: (i) $A = \emptyset$ ; (ii) A properly included in clK and $x \in A$; (iii) A properly included in clK ($x \notin A$, $A \neq \emptyset$); and (iv) $A = \mathrm{cl}K$. We consider each of them: (i) if we pick $p \in \mathrm{bdry}K \cap \mathrm{bdry}S$, such a point verifies that any ray issuing from it meets intS. Otherwise p would belong to A which is absurd. (ii) We prove first that if there exists a point $a \in A$, $a \neq x$, then the whole segment [a,x] is included in A: the fact that $a \in A$ implies that $[a,x] \subset \mathrm{cl}CS$. As a and x belong to A, this implies the existence of $v_0$ and $v_1$ in $\Omega_2$ such that $R_0$: $\{\lambda.v_0 + a / \lambda \geq 0\}$ and $R_1$: $\{\lambda.v_1 + x / \lambda \geq 0\}$ are included in clCS. We now consider the polygonal $P = R_0 \cup [a,x] \cup R_1$, P is included in clCS. Suppose that $R_0 \cap R_1 = \emptyset$ and denote $H_1$ and $H_2$ the open regions determined by P. Thus, the plane results a disjoint union of P, $H_1$ and $H_2$. We can suppose -without loss of generality- that intS is included in $H_1$ (as intS is connected, it lies exclusively in one of the $H_i$, $i = 1, 2$). Then if we take $t \in [a,x]$ it is easy to see that there always exists a half-line with origin in t included in $H_2$. If it occurs that $R_0 \cap R_1 = \{w\}$, the plane results a disjoint union of P and the three open regions P determines: $H_2$: the only bounded region, $H_1$: the only unbounded region such that [a,x] is included in its boundary, and $H_3$: the unbounded region that verifies [a,x] is not included in its boundary. Again, intS will be included exclusively in one of the $H_i$ ( $i = 1, 2, 3$). If $\mathrm{int}S \subset H_2$ or $\mathrm{int}S \subset H_3$ it is immediate that for each t in [a,x] we can choose a half-line with origin in t lying in a half-plane; or in the case that $\mathrm{int}S \subset H_1$ we consider $R(t \rightarrow w)$. In each of these cases, the half-line considered does not meet intS, and then t belongs to A. Now, A properly included in clK implies that there exists $c \in clK$ such that $c \notin A$. Let us consider the first point (going from x towards c) $t \in bdryS \cap R(x \to c)$, point that should exist, otherwise the ray $R(x \to c)$ would make K unbounded. Due to the previous considerations $t \notin A$ because $c \in [t, x]$ and $c \notin A$. (iii) A properly included in clK means that we can take $a \in A$, $a \neq x$. With an argument analogous to the previous one we can consider the first point (going from a towards x) $t \in bdryS \cap R(a \to x)$ and such t does not belong to A. (iv) We will show that K is a connected component of CS and as it is bounded the thesis follows. In this case there exists a half-line $R_v(x)$: $\{ \lambda.v + x / \lambda \geq 0 \}$ such that $R_v(x) \subset clCS$ but the fact that K is bounded means that $R_v(x)$ cannot be wholly included in CS then, there exists $u \in R_v(x) \cap bdryS$. Notice that it cannot exist another direction w, $(w \neq v)$ such that $R_w \subset clCS$ because otherwise if we consider the polygonal $P = R_w \cup R_v$ using an argument analogous to the previous ones we would be able to choose a new direction z such that $R_z \subset CS$. Again, it is easy to see that if $a \in A$, $(a \neq x)$ the only half-line with origin in a that does not intersect intS should pass through u. As K is clearly a connected set it should exist $C \subset CS$, a connected component of CS which contains K. As C results an open set and then path-connected, if there exists $c \in C$ such that $c \notin K$ we can consider an arc $\Gamma$ joining x with c, $\Gamma \subset CS$; but the only way to "leave" K is going through u which is absurd. Then K = C. ❑

**Theorem 2.5:** *Let S be a planar hunk, $^aS = S$ and $^aCS = clCS$. S has the half-line property if and only if S has the shining boundary property.*

Proof. Propositions 2.3 and 2.4. ❑

## § 3.- EMISSION OF OUTWARD RAYS.

In this paragraph we intend to connect the half-line property with the emission of outward rays. For a planar set S we will prove that if S has the half-line property then for each boundary point x of S it holds that the inner stem of x is nontrivial. To do this, we define the strong inner stem of a boundary point of S which is a certain subset of the inner stem of the point, and we prove that this new set is nontrivial. The converse is false as we can see if we consider the planar hunk S defined as $S = \{(x,y) \in \mathbf{R}^2 / 0 < \alpha \leq x^2 + y^2 \leq \beta\}$ (notice that

${}^{a}S = S$ and ${}^{a}CS$ = clCS) and any boundary point of the interior circle has non trivial inner stem, but S does not have the half-line property. A counterexample where the complement of S is free from bounded connected components can be easily constructed.

**Previous definitions.**

We note $A+B = \{a + b / a \in A, b \in B\}$, $\lambda.A = \{\lambda.a / a \in A\}$, ($\lambda \in \mathbf{R}$) and $A-B = A + (-1).B$, where A, B are subsets of $\mathbf{R}^n$. We say that $C \subset \mathbf{R}^n$ is a *cone with vertex a* if $\forall\ \lambda \in \mathbf{R}, \lambda > 0$ it holds $\lambda.(C - \{a\}) \subset C - \{a\}$. Given $A \subset \mathbf{R}^n$, $a \in A$ we define I(A,a) the *inscribed cone in A from a* as the cone formed by $\{a\}$ and every half-line included in A having a as origin. In our case, we consider I(clCS,p) and we define the *set of external directions to S from p* as: $\text{exd}(S,p) = [I(\text{clCS},p) - \{p\}] \cap \Omega_n$ where $p \in$ bdryS.

**Proposition 3.1:** *1) $A \subset \mathbf{R}^n$ a closed set, $a \in$ bdryA, then I(A,a) is a closed cone.*

*2) $S \subset \mathbf{R}^2$ hunk, $p \in$ bdryS then:*

*a) exd(p,S) is an arcwise connected set.*

*b) if $p \in$ bdry(convS), then exd(p,S) contains a half circle of $\Omega_2$.*

*c) if $p \in$ int(convS), then I(clCS,p) is a convex cone and exd(p,S) is a closed arc included in a half circle.*

Proof. 1) It results easily since A is a closed set.

2)a) Suppose that there exist $v_1$ and $v_2$ in exd(p,S), $v_1 \neq v_2$ such that both of the arcs determined by them are not completely included in exd(p,S). Then if we denote $L_i$: $\{\lambda.v_i + p / \lambda \geq 0\}$ ($i = 1, 2$) in both regions determined by $L_1 \cup L_2$ there must exist interior points of S which we note $x_1$ and $x_2$. Notice that $L_i \subset$ clCS and $p \in$ bdryS, then there is no way to connect $x_1$ with $x_2$ with an arc wholly included in intS which is absurd.

2)b) If $p \in$ bdry(convS) there exists L a line through p which supports convS then, if $L^+$ and $L^-$ denote the closed half-planes determined by L we have that if $S \subset L^+$ then $L^- \subset$ I(clCS,p). Then the assertion is immediate.

2)c) As $p \in$ int(convS) no line L through p leaves convS in one of the half-planes determined by L, then the cone I(clCS,p) is properly included in one half-plane. Hence, using part a), it results that the cone is convex and exd(p,S) verifies the thesis. ❑

We will show in an example below a set in $\mathbf{R}^3$ where the item 2)a) does not hold. This is one of the reasons why we work in the plane. We consider now the set J(A,a) formed by {a} and every half-line with origin in a and opposite direction to those which compose I(A,a). We define the *strong inner stem of p in S* ($p \in$ bdryS) as the set: sins(p,S) = J(clCS,p) $\cap$ st(p,S). Notice that it is immediate that sins(p,S) $\subset$ ins(p,S).

**Theorem 3.2:** *Let S be a planar hunk, $^aS = S$ and $^aCS$ = clCS. S has the half-line property if and only if for any $p \in$ bdryS it holds that sins(p,S) is nontrivial and CS is free from bounded connected components.*

Proof. $\Leftarrow$) We will prove that S has the shining boundary property then the thesis will follow from theorem 2.5. We know that there exists a point x, x different from p such that $x \in$ J(clCS,p) $\cap$ st(p,S). The fact that $x \in$ J(clCS,p) implies that x belongs to a certain half-line $R_{-v}(p)$ with origin p and a direction -v such that $R_v(p)$ verifies that it does not intersect intS. This result plus the hypothesis that CS does not contain any bounded connected component derives in the thesis.

$\Rightarrow$) If S has the hlp, then it has the sbp as it was shown in theorem 2.5, then in particular CS is free from bounded connected components. Then we have to prove that each boundary point of S verifies that its strong inner stem is nontrivial. We consider two cases: (a) $p \in$ bdry(convS) and (b) $p \in$ int(convS). (a) Given any point x in the star of p in S, different from p (such a point exists because $^aS = S$ ) it verifies that R(xp $\rightarrow$ ) does not intersect intS because there exists L a support line of convS through p. This means that if convS $\subset L^+$, then R(xp $\rightarrow$ ) $\subset L^-$ where $L^+$ and $L^-$ denote the closed half-planes determined by L. (b) I(clCS,p) is a closed convex cone and exd(p,S) results a closed arc in $\Omega_2$ as we have shown in 3.1. Consider $v_1$ and $v_2$ the extremal directions of this arc. (Eventually they may coincide). Let us denote $L_i$': $\{\lambda v_i + p / \lambda \geq 0\}$; - $L_i$': $\{ \lambda(-v_i) + p / \lambda \geq 0 \}$ and $L_i = L_i' \cup (- L_i')$ (for i = 1, 2). Then, the plane appears divided in four regions (or two in the case $v_1 = v_2$) which we denote I = I(clCS,p), J = J(clCS,p), $R_1 = L_1^- \cap L_2^+$, $R_2 = L_1^+ \cap L_2^-$, where $L_i^+$ are the closed half-planes determined by $L_i$ such that $L_i^+ \cap I = L_i$ , and $L_i^-$ are the closed complements of $L_i^+$ (i = 1, 2). We also consider: $S_i = S \cap R_i$ (i = 1, 2) and $S_3 = S \cap J$. (In the case $v_1 = v_2$ the configuration is simplified but the construction and the following argument are analogous). If st(p,S) $\cap S_3$ is nontrivial, every point of this intersection would belong to sins(p,S). If this does not occur, let us suppose -without loss of generality- that p has linear accessibility through S by points of $S_1$. Both facts that p has linear accessibility by points of S and that $v_1$ is an extremal direction of exd(S,p) assure the existence of interior points of S in $S_1$ and $S_2$. Then, due to the connectedness of intS we have interior points of S in $S_3$ and a fortiori in $-L_1$'. Now, as we are under the hypothesis that every point of $S_3$ does not belong to st(p,S), then, in particular, this is verified by every point of $-L_1$'. These two remarks let us take a point $y \in -L_1'$ such that $y \in CS \cap convS$. We can take U a neighborhood of y such that $U \subset CS \cap convS$. If we consider a point $y' \in U \cap R_1$ it verifies st(y',CS) is a bounded set what is absurd because S has the half-line property. ❑

**Corollary 3.3:** *Let S be a planar hunk, ${}^aS = S$ and ${}^aCS = clCS$. If S has the half-line property then for any p boundary point of S it holds that ins(p,S) is nontrivial.*
Proof. Immediate from theorem 3.2 and the fact that sins(p,S) is included in ins(p,S). ❑

These characterizations of the sets that enjoy the half-line property yield planar results equivalent to Stavrakas' ones [3]:
**Theorem 3.4:** *Let S be a planar compact set such that ${}^aS = S$ and ${}^aCS = clCS$. If $\cap \{ st(x,S) / x$ is a 0-extreme point $\} \neq \varnothing$, then the following statements are equivalents:*

*(i) S has the shining boundary property.*

*(ii) $KerS = \cap \{ st(x,S) / x$ is a 0-extreme point$\}$*

Proof. Immediate from theorem 2 of [3] and theorem 2.5. ❑

**Corollary 3.5:** *Let S be a planar compact set, ${}^aS = S$ and ${}^aCS = clCS$. S is starshaped if and only if S has the shining boundary property and the intersection of the stars of the 0-extreme points is nonempty.*
Proof. Immediate from corollary 1 of [3] and theorem 2.5. ❑

Finally we show an example of a hunk $S \subset \mathbf{R}^3$ that enjoys the half-line property, the shining boundary property but that it contains a boundary point such that the strong inner stem of it is trivial. Then, there is no way of improving the planar results. We denote $X = (x,y,z) \in \mathbf{R}^3$

$S_1 = \{X / x^2 + y^2 + (z - 1)^2 \leq 1, z \leq 1\} \cup \{X / -2 \leq x \leq 2, -2 \leq y \leq 2, 1 \leq z \leq 2\} \cup$

$\cup \{X / -2 \leq x \leq 2, 1 \leq y \leq 2, 0 \leq z \leq 2\} \cup \{ X / -2 \leq x \leq 2, -2 \leq y \leq -1, 0 \leq z \leq 2\}$

$S_2$ is the symmetrical to $S_1$ with respect to the plane $z = 0$. Then the set considered is $S = S_1 \cup S_2$. The origin p is a boundary point of S and ins(p,S) = {p}. Notice that exd(p,S) is formed by two arcs lying on the plane $z = 0$ that do not form an arcwise set, so the proposition 3.1 2)c) cannot be generalized.

It remains open the possibility of getting a generalization of the equivalence between the half-line property and the shining boundary property to spaces of dimension higher than two; or on the contrary to show a counterexample.

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*Recibido en Agosto de 1995*