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APPROXIMATION AND INTERPOLATION OF FUNCTIONS OF HYPERBOLIC COMPLEX VARIABLE

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ABSTRACT. We develop a theory of approximation and interpolation by polynomials of the functions of hyperbolic complex variable. In the class of the so-called pseudoholomorphic functions, a Jackson-type estimate is obtained and a result on mean convergence of Lagrange interpolation is proved. Then, estimates in approximation of continuous functions by areolar polynomials of Bernstein-type and of DeVore-Szabados-type are given.

1. INTRODUCTION

Let $\alpha, \beta \in \mathbf{R}$ be fixed and q a solution of the equation $q^2 = \alpha q + \beta$. An important result (see e.g.[11]) states that the algebraic ring $C_q = \{z = a + qb; a, b \in \mathbf{R}\}$ is ring-isomorphic with one of the following three:

(i) $C_q, q^2 = -1$, called the field of complex numbers, if $\alpha^2 / 4 + \beta < 0$;

(ii) C_q, $q^2 = 0$, called the ring of dual complex numbers, if $\alpha^2 / 4 + \beta = 0$;

(iii) $C_q, q^2 = +1$, if $\alpha^2 / 4 + \beta > 0$.A number in C_q, $q^2 = +1$ is called binary [11], or double [32], or perplex [10], or anormal-complex [2], or hyperbolic complex [6-8],[24].Throughout in this paper we will use the term of hyperbolic complex number.

Suggested by the classical complex analysis, between 1935-1941 a theory of the functions of hyperbolic complex and dual complex variables was deeply investigated in e.g. [4-8],[24-31]and seems to have some applications in theoretical physics, as was pointed out in the recent papers [10], [13-14].

For all that, the theory of approximation of functions of hyperbolic complex variable by polynomials contains a single result obtained in 1936 in [8]. Because of this reasons, the purpose of the present paper is to give other contributions to this field of investigations. The main idea that can be derived is that, in contrast with what happens in the classical complex analysis, the properties are consequences of known results in the real approximation theory.

Section 2 contains some preliminaries . In Section 3 we firstly give an estimate for the approximation result in [8] and then , by using the Szabados's polynomials in [20] , we obtaind a Jackson-type result .

Also , we consider the problem of approximation by some particular classes of areolar polynomials .

In Section 4 we deal with the interpolation of functions of hyperbolic complex variable.

2. PRELIMINARIES

We will consider some known concepts and results in [6-7] which will be used in the next sections.

Let C_q be with $q^2 = +1$.

DEFINITION 2.1. *The hyperbolic complex numbers* $q_1 = (1+q)/2$, $q_2 = (1-q)/2$ *are called isotropic units. If* $z = x+qy \in C_q$ *then* $z = z'q_1 + z''q_2$ *with* $z' = x+y, z'' = x-y$ *is called the isotropic form of z.*

If $z = x+qy \in C_q$ *then* $\bar{z} = x-qy$ *is the conjugate of* z , $|z| = (x^2+y^2)^{1/2}$ *is the modulus of* z *and* $N_q(z) = z\cdot\bar{z} = x^2-y^2$ *is the hyperbolic norm of* z.

If $z_n, z \in C_q, n \in N$ *we say that* $z_n \to z_0$ if $|z_n - z| \to 0$.

The number $z = x+qy = z'q_1 + z''q_2, \in C_q$ *represents a point* M(x,y) *in the system of the axes* XOY *and on the other hand, a point* $M_{iso}(z', z'')$ *in the so-called isotropic system of coordinates composed by the first and second bisectrix and obtained from* XOY *by a rotation with* $\pi/4$ *in trigonometric sense .*

A rectangle having the sides parallel with the first and second bisectrix will be called isotropic rectangle .

If $a = a_1q_1 + a_2q_2, b = b_1q_1 + b_2q_2 \in C_q$ satisfy $N_q(b-a) \neq 0$, then by R(a,b) will be denoted the isotropic rectangle having $M^{(1)}_{iso}(a_1, a_2), M^{(2)}_{iso}(a_1, a_2)$ (i.e. in the isotropic system of coordonates) as opposite sharp points. Let suppose , for example , that $a_1 < b_1$ and $a_2 < b_2$. In this case , $R(a,b) = \{M_{iso}(z', z''); a_1 \le z' \le b_1, a_2 \le z'' \le b_2\}$.

Let us denote $R^*(a,b) = \{z = z'q_1 + z''q_2 \in C_q; M_{iso}(z', z'') \in R(a,b)\}$.

If $\alpha = A_1q_1 + A_2q_2, \beta = B_1q_1 + B_2q_2 \in C_q$ then

$\alpha + \beta = (A_1 + B_1)q_1 + (A_2 + B_2)q_2$, $\alpha\cdot\beta = (A_1B_1)q_1 + (A_2B_2)q_2$, $\alpha^n = A_1^n q_1 + A_2^n q_2$, $\alpha/\beta = (A_1/B_1)q_1 + (A_2/B_2)q_2$, for $B_1\cdot B_2 \neq 0$.

If $a \in R$ then $a = aq_1 + aq_2$ and if $z = x+qy = Aq_1 + Bq_2$ then $\bar{z} = Bq_1 + Aq_2$.

The number $z = x+qy = Aq_1 + Bq_2 \in C_q$ is divisor of zero iff $N_q(z) = 0$ (or equivalently $A\cdot B = 0$). Also we have $|z| = [(A^2+B^2)/2]^{1/2} \le [|A|+|B|]/\sqrt{2}$.

If $z_n = x_n + qy_n = A_nq_1 + B_nq_2, n \in N, z = x+qy = Aq_1 + Bq_2$, then $z_n \to z$ iff $x_n \to x, y_n \to y$ or iff $A_n \to A, B_n \to B$.

3. UNIFORM APPROXIMATION

Let $\Omega \subset C_q, q^2 = +1, f:\Omega \to C_q$. We can write $f(z) = u(x,y) + q\cdot v(x,y)$, $z = x + qy \in \Omega$ or the isotropic form $f(z) = P_1(z_1,z_2)q_1 + P_2(z_1,z_2)q_2$, $z = z_1q_1 + z_2q_2 \in \Omega$.

The definition of the continuity of f at $z_0 = x_0 + qy_0 = z_0'q_1 + z_0''q_2$ being the same as in the classical complex analysis , can be proved that f is continuous in z_0 iff $u(x,y), v(x,y)$ are continuous in (x_0, y_0) or iff $P_1(z_1,z_2), P_2(z_1,z_2)$ are continuous in (z_0', z_0'') (see [7]).

An important subclass consists in the so-called pseudoholomorphic functions , i.e. of the form $f(z) = f_1(z')q_1 + f_2(z')q_2$, $z = z'q_1 + z''q_2$, with f_1, f_2 continuous functions on their domains of definition .

As was proved in [7] , the natural domains of definition for such functions f , are of the form $R^*(a,b)$. The following result of approximation was proved in [8] .

THEOREM 3.1. *If* $f: R^*(0,1) \to C_q$ *is pseudoholomorphic on* $R^*(0,1)$, $(0 = 0\cdot q_1 + 0\cdot q_2,\ 1 = 1\cdot q_1 + 1\cdot q_2)$ *then the Bernstein polynomials* $B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} f(k/n) z^k (1-z)^{n-k}$ *converge uniformly* (*when* $n \to +\infty$) *to* $f(z)$ *on* $R^*(0,1)$.

REMARK. In the system of the axes of coordinates XOY , $R(0,1)$ represents a quadrangle having the side equal with 1 and $(0,0), (0,\sqrt{2})$ as opposite sharp points , while $R^*(0,1) = \{z = z'q_1 + z''q_2; 0 \le z' \le 1, 0 \le z'' \le 1\} = \{z = x + qy; 0 \le x + y \le 1, 0 \le x - y \le 1\}$ has as geometric image a quadrangle of side equal with $\sqrt{2}/2$ and with (0,0) , (0,1) as opposite sharp points .

The proof of Theorem 3.1. is based on the following relation :

(1) $f(a_k)z^k(1-z)^{n-k} = [f_1(a_k)q_1 + f_2(a_k)q_2]\cdot[z_1^k q_1 + z_2^k q_2]\cdot[(1-z_1)^{n-k}q_1 + (1-z_2)^{n-k}q_2]$
$= f_1(a_k)\cdot z_1^k(1-z_1)^{n-k}q_1 + f_2(a_k)\cdot z_2^k(1-z_2)^{n-k}q_2$, $z = z_1q_1 + z_2q_2, a_k \in R$.

We will give an estimate in the above theorem by using the following.

LEMMA 3.2. *Let* $f(z) = P_1(z',z'')q_1 + P_2(z',z'')q_2$, $z = z'q_1 + z''q_2$, *continuous for* $z \in R^*(0,1)$. *Denoting* $\omega(f,\delta) = \sup\{|f(z_1) - f(z_2)|; z_1, z_2 \in R^*(0,1), |z_1 - z_2| \le \delta\}$, $\omega(P_k;\delta,\delta) = \sup\{|P_k(z_1',z_1'') - P_k(z_2',z_2'')|; |z_1' - z_2'| \le \delta, |z_1'' - z_2''| \le \delta, z_1', z_2', z_1'', z_2'' \in [0,1]\}$, $k = \overline{1,2}$, *the following inequality* :

$$\omega(P_1;\delta,\delta) + \omega(P_2;\delta,\delta) \le 2\sqrt{2}\omega(f;\delta)\ ,\ \delta > 0$$

holds.

PROOF. We have :

$$|f(z_1) - f(z_2)| = |[P_1(z_1',z_1'') - P_1(z_2',z_2'')]q_1 + [P_2(z_1',z_1'') - P_2(z_2',z_2'')]q_2| =$$
$$= \{[(P_1(z_1',z_1'') - P_1(z_2',z_2''))^2 + (P_2(z_1',z_1'') - P_2(z_2',z_2''))^2]/2\}^{1/2}.$$

From here it follows

$$|f(z_1) - f(z_2)| \ge |P_1(z_1',z_1'') - P_1(z_2',z_2'')|/\sqrt{2},\ |f(z_1) - f(z_2)| \ge |P_2(z_1',z_1'') - P_2(z_2',z_2'')|/\sqrt{2}.$$

Passing to supremum with $|z_1 - z_2| \le \delta, z_1, z_2 \in R^*(0,1)$, we get

$$\omega(f;\delta) \ge (1/\sqrt{2})\cdot \sup\{|P_k(z_1',z_1'') - P_k(z_2',z_2'')|; |z_1 - z_2| \le \delta, z_1, z_2 \in R^*(0,1)\} =$$

$$=\left(1/\sqrt{2}\right)\cdot\sup\left\{\left|P_k(z_1',z_1'')-P_k(z_2',z_2'')\right|;\sqrt{\left(z_1'-z_2'\right)^2+\left(z_1''-z_2''\right)^2}\le\delta\sqrt{2},z_1',z_2',z_1'',z_2''\in[0,1]\right\}\ge$$
$$\ge\left(1/\sqrt{2}\right)\cdot\sup\left\{\left|P_k(z_1',z_1'')-P_k(z_2',z_2'')\right|;\left|z_1'-z_2'\right|\le\delta,\left|z_1''-z_2''\right|\le\delta,z_1',z_2',z_1'',z_2''\in[0,1]\right\}=$$
$$=\left(1/\sqrt{2}\right)\omega\left(P_k;\delta,\delta\right),\ k=\overline{1,2},$$

which implies

$$\omega\left(P_1;\delta,\delta\right)+\omega\left(P_2;\delta,\delta\right)\le 2\sqrt{2}\omega\left(f;\delta\right)$$

and the lemma is proved.

REMARK. If $f(z)=f_1(z')q_1+f_2(z'')q_2$ is pseudoholomorphic on $R^*(0,1)$ then entirely analogous we get

$$\omega\left(f_1;\delta\right)+\omega\left(f_2;\delta\right)\le 2\sqrt{2}\omega\left(f;\delta\right),\ \delta>0.$$

THEOREM 3.3. *If* $f(z)=f_1(z')q_1+f_2(z'')q_2$, $z=z'q_1+z''q_2$ *is pseudoholo-morphic on* $R^*(0,1)$ *then*

$\left|B_n(f)(z)-f(z)\right|\le 2k_0\omega\left(f;1/\sqrt{n}\right)$, $n\in\mathbf{N}$, $z\in R^*(0,1)$, *where* k_0 *represent the Sikkema's constant in* [18].

PROOF. By (1) (see [8, p.205, Theorem 1]) we have

$B_n(f)(z)-f(z)=\left[B_n(f_1)(z')-f_1(z')\right]q_1+\left[B_n(f_2)(z'')-f_2(z'')\right]q_2$, which by the Remark of Lemma 3.2. and by [18] implies

$$\left|B_n(f)(z)-f(z)\right|\le\left[\left|B_n(f_1)(z')-f_1(z')\right|+\left|B_n(f_2)(z'')-f_2(z'')\right|\right]/\sqrt{2}\le$$

$\le\left(k_0/\sqrt{2}\right)\left[\omega\left(f_1;1/\sqrt{n}\right)+\omega\left(f_2;1/\sqrt{n}\right)\right]\le 2k_0\omega\left(f;1/\sqrt{n}\right), n\in\mathbf{N}$, $z\in R^*(0,1)$, and the theorem is proved.

Theorem 3.3. can be improved by the following result of Jackson-type.

THEOREM 3.4. *If* $f(z)=f_1(z')q_1+f_2(z'')q_2$, $z=z'q_1+z''q_2\in R^*(0,1)$ *is pseudoholomorphic on* $R^*(0,1)$, *then there exists a sequence of polynomials* $\left(P_n(f)(z)\right)_n$, *degree* $P_n(z)\le n$, *such that*

$$\left|f(z)-P_n(f)(z)\right|\le C\cdot\omega\left(f;1/n\right),\ \forall n\in\mathbf{N}, z\in R^*(0,1),$$

where $C>0$ *is independent of n and f.*

PROOF. By [20], for any $g\in C[0,1]=\{g:[0,1]\to\mathbf{R};\ g\text{ continuous on }[0,1]\}$, there exists a polynomial sequence of the form

$$(2)\quad P_n(g)(x)=\sum_{k=0}^{n}g(k/n)\cdot s_{k,n}(x),\ n\in\mathbf{N}, x\in\mathbf{R}$$

where $s_{k,n}$ are polynomials of degree $\le$n with real coefficients, independent of g, such that

$$\left|g(x)-P_n(g)(x)\right|\le C\omega\left(g;1/n\right),\ x\in[0,1], n\in\mathbf{N}$$

with $C>0$ an absolute constant.

Applying this last estimate, we get

$$\left|f_1(z')-\sum_{k=0}^{n}f_1(k/n)\cdot s_{k,n}(z')\right|\le C\omega\left(f_1;1/n\right),\ z'\in[0,1], n\in\mathbf{N}$$

$$\left|f_2(z'')-\sum_{k=0}^{n}f_2(k/n)\cdot s_{k,n}(z'')\right|\le C\omega\left(f_2;1/n\right),\ z''\in[0,1], n\in\mathbf{N}.$$

Let us denote $P_n(f)(z)=\sum_{k=0}^{n} f(k/n)\cdot s_{k,n}(z)$. Since
$a_p\cdot z^p=\left(a_p q_1+a_p q_2\right)\cdot\left[(z')^p q_1+(z'')^p q_2\right]=a_p(z')^p q_1+a_p(z'')^p q_2,\ \left(a_p\in\mathbf{R}\right)$, and
$f(k/n)=f\left[(k/n)q_1+(k/n)q_2\right]=f_1(k/n)q_1+f_2(k/n)q_2$, we easily get
(3) $s_{k,n}(z)=s_{k,n}(z')q_1+s_{k,n}(z'')q_2$,
and $P_n(f)(z)=P_n(f_1)(z')q_1+P_n(f_2)(z'')q_2$, $z=z'q_1+z''q_2\in R^*(0,1)$.

Therefore, by the Remark of Lemma 3.2 we obtain

$$\left|f(z)-P_n(f)(z)\right|=\left|\left[P_n(f_1)(z')-f_1(z')\right]q_1+\left[P_n(f_2)(z'')-f_1(z'')\right]q_2\right|\leq$$
$$\leq\left[\left|P_n(f_1)(z')-f_1(z')\right|+\left|P_n(f_2)(z'')-f_2(z'')\right|\right]/\sqrt{2}\leq\left(C/\sqrt{2}\right)\left[\omega(f_1;1/n)+\omega(f_2;1/n)\right]\leq$$
$\leq 2C\omega(f;1/n)$, $z\in R^*(0,1)$, which proves the theorem.

REMARK. By [8, Theorem IV], the possibility of uniform approximation by polynomials on izotropic rectangles charactherize the pseudoholomorphic functions. Then a natural question arise: how can be approximated a function which is not pseudoholomorphic, i.e. for example, if f is continuous of general form $f(z)=P_1(z_1,z_2)q_1+P_2(z_1,z_2)q_2$, $z=z_1q_1+z_2q_2\in R^*(0,1)$?

Firstly, we will introduce the following

DEFINITION 3.5. *The functions* $\bar{f}_1(z)=P_1(z_1,z_2)q_1+P_2(z_2,z_1)q_2$, $\bar{f}_2(z)=P_1(z_2,z_1)q_1+P_2(z_1,z_2)q_2$ *will be called pseudoconjugates of the function f.*

The expressions

$$B_n^{(1)}(f)(z,\bar{z})=\sum_{k=0}^{n}\sum_{j=0}^{n}\bar{f}_1\left[(k/n)q_1+(j/n)q_2\right]\cdot p_{n,k}(z)\cdot p_{n,j}(\bar{z})$$

$$B_n^{(2)}(f)(z,\bar{z})=\sum_{k=0}^{n}\sum_{j=0}^{n}\bar{f}_2\left[(k/n)q_1+(j/n)q_2\right]\cdot p_{n,k}(\bar{z})\cdot p_{n,j}(z)$$

where $p_{n,k}(z)=\binom{n}{k}\cdot z^k(1-z)^{n-k}$, $p_{n,j}(z)=\binom{n}{j}\cdot z^j(1-z)^{n-j}$, *are called areolar polynomials of the degree n of Bernstein-type.*

REMARKS. 1). If we define the concept of areolar derivative of a function $f:D\to C_q$, $D\subset C_q, q^2=+1$, by analogy with the classical complex analysis (see e.g.[17, p. 102]), i.e.
$D(f)(z)=(1/2)\{[(\partial u/\partial x)(x,y)-(\partial v/\partial y)(x,y)]+q[(\partial v/\partial x)(x,y)-(\partial u/\partial y)(x,y)]\}=$
$(\partial f/\partial\bar{z})(z)$, $f(z)=u(x,y)+qv(x,y), z=x+qy$, then it is not difficult to see that the succesive areolar derivative of order $n+1$ of $B_n^{(1)}(f)(z,\bar{z})$ and $B_n^{(1)}(f)(z,\bar{z})$ is null, which justifies the name of areolar polynomials of degree n.

2). Let $z=z_1q_1+z_2q_2\in C_q$. By $\bar{z}=z_2q_1+z_1q_2$ and by (1) we obtain

$$B_n^{(1)}(f)(z,\bar{z}) = \sum_{k=0}^{n}\sum_{j=0}^{n}\left[P_1(k/n, j/n)q_1 + P_2(j/n, k/n)q_2\right]\binom{n}{k}\binom{n}{j} z^k (1-z)^{n-k} \bar{z}^j (1-\bar{z})^{n-j} =$$

$$=\left[\sum_{k=0}^{n}\sum_{j=0}^{n} P_1(k/n, j/n) p_{n,k}(z_1) p_{n,j}(z_2)\right] q_1 + \left[\sum_{k=0}^{n}\sum_{j=0}^{n} P_2(j/n, k/n) p_{n,j}(z_1) p_{n,k}(z_2)\right] q_2,$$

$$B_n^{(2)}(f)(z,\bar{z}) = \sum_{k=0}^{n}\sum_{j=0}^{n}\left[P_1(j/n, k/n)q_1 + P_2(k/n, j/n)q_2\right]\binom{n}{k}\binom{n}{j} \bar{z}^k (1-\bar{z})^{n-k} z^j (1-z)^{n-j} =$$

$$=\left[\sum_{k=0}^{n}\sum_{j=0}^{n} P_1(j/n, k/n) p_{n,j}(z_1) p_{n,k}(z_2)\right] q_1 + \left[\sum_{k=0}^{n}\sum_{j=0}^{n} P_2(k/n, j/n) p_{n,k}(z_1) p_{n,j}(z_2)\right] q_2,$$

i.e. $B_n^{(1)}(f)(z,\bar{z}) = B_n^{(2)}(f)(z,\bar{z}) = B_n(P_1)(z_1,z_2)q_1 + B_n(P_2)(z_1,z_2)q_2$, where $B_n(P)(z_1,z_2)$ represents the usual Bernstein polynomial of two real variables z_1, z_2.

Therefore, let us denote $B_n^{(1)}(f)(z,\bar{z}) = B_n^{(2)}(f)(z,\bar{z}) = \overline{B}_n(f)(z,\bar{z})$.

We can prove the

THEOREM 3.6. *If* $f(z) = P_1(z_1,z_2)q_1 + P_2(z_1,z_2)q_2$ *is continuous on* $R^*(0,1)$ *then for all* $n \in \mathbf{N}$ *and* $z = z_1q_1 + z_2q_2 \in R^*(0,1)$ *we have*

$$\left|f(z) - \overline{B}_n(f)(z,\bar{z})\right| \le 4\cdot\omega(f;1/\sqrt{n})$$

PROOF. By a well-known Ipatov's result (see e.g.[19, p.339]) we have

$$\left|B_n(P_k)(z_1,z_2) - P_k(z_1,z_2)\right| \le 2\omega(P_k;1/\sqrt{n},1/\sqrt{n}),\ \forall n \in \mathbf{N}, z_1,z_2 \in [0,1], k \in \overline{1,2}.$$

Then, by Lemma 3.2 we get

$$\left|f(z) - \overline{B}_n(f)(z,\bar{z})\right| = \left|\left[P_1(z_1,z_2) - B_n(P_1)(z_1,z_2)\right]q_1 + \left[P_2(z_1,z_2) - B_n(P_2)(z_1,z_2)\right]q_2\right| \le$$

$$\le\left[\left|P_1(z_1,z_2) - B_n(P_1)(z_1,z_2)\right| + \left|P_2(z_1,z_2) - B_n(P_2)(z_1,z_2)\right|\right]/\sqrt{2} \le 4\cdot\omega(f;1/\sqrt{n}),$$

which proves the theorem.

Using the polynomials given by (2), we can introduce

$$P_n(f)(z,\bar{z}) = \sum_{k=0}^{n}\sum_{j=0}^{n} \bar{f}_1\left[(k/n)q_1 + (j/n)q_2\right] s_{k,n}(z) s_{j,n}(\bar{z}),$$

called areolar polonomial of degree n of DeVore-Szabados-type.

Then by (3) and reasoning as in the proof Theorem 3.6, we immediately obtain the

THEOREM 3.7. *If* $f(z) = P_1(z_1,z_2)q_1 + P_2(z_1,z_2)q_2$, $z = z_1q_1 + z_2q_2$, *is continuous on* $R^*(0,1)$ *then*

$$\left|f(z) - P_n(f)(z,\bar{z})\right| \le C\omega(f;1/n),\ \forall n \in \mathbf{N}, z \in R^*(0,1),$$

where $C>0$ *is an absolute constant.*

REMARKS. 1).Comparing, for example, Theorems 3.3 and 3.4 with the results regarding the approximation by Bernstein-type polynomials in the classical complex analysis (see e.g. [3], [12], [16], [23]) we see that they are essentially different.

2). Similar results with the Theorems 3.3 and 3.6 can be obtained if in the place of the Bernstein polynomials we consider, for example, the Meyer-König and Zeller's operator in [15], or the Baskakov's operator in [1], or the Szasz-Mirakyan's operator in ([21].

3).By simple calculus we obtain

$$\left(\partial\,\overline{B}_n(f)/\partial\,\bar{z}\right)(z,\bar{z}) = \left(\partial\,B_n(P_1)/\partial\,z_2\right)(z_1,z_2)\cdot q_1 + \left(\partial\,B_n(P_2)/\partial\,z_1\right)(z_1,z_2)\cdot q_2,$$

where $f(z)=P_1(z_1,z_2)\cdot q_1+P_2(z_1,z_2)\cdot q_2$, $z=z_1q_1+z_2q_2\in R^*(0,1)$.

Now, if P_1 and P_2 have continuous partial derivative of order one, then we immediately get that $(\partial \overline{B}_n(f)/\partial \bar{z})(z,\bar{z})$ converges (when $n\to+\infty$) uniformly on $R^*(0,1)$ to $(\partial P_1/\partial z_2)(z_1,z_2)\cdot q_1+(\partial P_2/\partial z_1)(z_1,z_2)\cdot q_2=(\partial f/\partial \bar{z})(z)$, taking into account the formulas for $\partial P_1/\partial z_2$, $\partial P_2/\partial z_1$, in [7, chapter I, §6] and the formula for $\partial f/\partial \bar{z}$ in the Remark 1 of Definition 3.5.

Also, since by the standard technique in approximation by real Bernstein polynomials we have

$$\left|(\partial B_n(P_1)/\partial z_2)(z_1,z_2)-(\partial P_1/\partial z_2)(z_1,z_2)\right|\le C\omega(\partial P_1/\partial z_2;1/\sqrt{n},1/\sqrt{n}),$$
$$\left|(\partial B_n(P_2)/\partial z_1)(z_1,z_2)-(\partial P_2/\partial z_1)(z_1,z_2)\right|\le C\omega(\partial P_2/\partial z_1;1/\sqrt{n},1/\sqrt{n}),$$

taking into account the Lemma 3.2 too, we easily get the estimate

$$\left|(\partial \overline{B}_n(f)/\partial \bar{z})(z,\bar{z})-(\partial f/\partial \bar{z})(z)\right|\le C\omega(\partial f/\partial \bar{z};1/\sqrt{n}),\quad z\in R^*(0,1),\ n\in \mathbf{N}.$$

Thus, Remark 3 together with Theorem 3.6 represent a simple constructive solution (containing even quantitative estimates) in the hyperbolic complex analysis, of a similar result in the classical complex analysis [22].

4. INTERPOLATION

Firstly, we deal with the interpolation of pseudoholomorphic functions.

Let $g:[a,b]\to\mathbf{R}$ be and $a\le x_1<\ldots<x_n\le b$. It is known that the Lagrange interpolatory polynomials $L_n(g)(x)$ of degree n-1 which satisfies $L_n(g)(x_k)=g(x_k)$, $k=\overline{1,n}$, is given by

$$L_n(g)(x)=\sum_{k=1}^{n}g(x_k)\cdot l_{k,n}(x)$$

where $l_{k,n}(x)$ are given by

$$l_{k,n}(x)=(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)/(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n).$$

Now, let $f:R^*(a,b)\to C_q, q^2=+1$ be pseudoholomorphic on $R^*(a,b)$, i.e. $f(z)=f_1(z_1)q_1+f_2(z_2)q_2$, $z=z_1q_1+z_2q_2$, with $a,b\in\mathbf{R}$, $a<b$, and $f_1,f_2:[a,b]\to\mathbf{R}$, continuous on $[a,b]$.

Let us consider the complex Lagrange interpolatory polynomial

$$L_n(f)(z)=\sum_{k=1}^{n}f(x_k)\cdot l_{k,n}(z).$$

Obviously $L_n(f)(x_k)=f(x_k)$, $k=\overline{1,n}$, and by simple calculus

$$L_n(f)(z)=\left[\sum_{k=1}^{n}f_1(x_k)l_{k,n}(z_1)\right]q_1+\left[\sum_{k=1}^{n}f_2(x_k)l_{k,n}(z_2)\right]q_2=L_n(f_1)(z_1)q_1+L_n(f_2)(z_2)q_2,$$

$\forall z=z_1q_1+z_2q_2$.

This implies

(4) $f(z)-L_n(f)(z)=[f_1(z_1)-L_n(f_1)(z_1)]q_1+[f_2(z_2)-L_n(f_2)(z_2)]q_2$

and taking into acount that for $z_k - x_i \neq 0, k = \overline{1,2}, i = \overline{1,n}$

$$f_k(z_k) - L_n(f_k)(z_k) = (z_k - x_1)\dots(z_k - x_n)[z_k, x_1, \dots, x_n; f], k = \overline{1,2},$$

by simple calculus we obtain the following

THEOREM 4.1. *If we denote $R_n(f)(z) = f(z) - L_n(f)(z)$, then $R_n(f)(z) = (z - x_1)\dots(z - x_n)[z, x_1, \dots x_n; f]$, $\forall z \in R^*(a,b)$ with $N_q(z - x_i) \neq 0$, $i = \overline{1,n}$, where $[z, x_1, \dots, x_n; f]$ denotes the divided difference of f at the points $z, x_1, \dots, x_n \in R^*(a,b)$, (where if $a_k \in C_q$, $k = \overline{1,m}$ are such that $N_q(a_k - a_i) \neq 0$ for $k \neq i$, then by definition*

$$[a_1, \dots, a_m; f] = \sum_{k=1}^{m} f(a_k) / [(a_k - a_1)\dots(a_k - a_{k-1})(a_k - a_{k+1})\dots(a_k - a_m)]).$$

Now, let $P_n(x)$ be the Legendre's polynomials of degree n and $-1 < x_1^* < \dots < x_n^* < 1$, where x_i^* represent the zeros of $P_n(x)$.

THEOREM 4.2. *Let $f: R^*(-1,1) \to C_q$ be pseudoholomorphic on $R^*(-1,1)$, $f(z) = f_1(z_1)q_1 + f_2(z_2)q_2$, $z = z_1q_1 + z_2q_2 \in R^*(-1,1)$. Let $\gamma: [a,b] \to R^*(-1,1)$, $\gamma(a) = -1$, $\gamma(b) = 1$, be a rectifiable path and let us denote*

$$L_n(f)(z) = \sum_{k=1}^{n} f(x_k^*) \cdot l_{k,n}(z), n \in \mathbf{N}.$$

Then, $\lim_{n\to\infty} \int_\gamma L_n(f)(z)dz = \int_\gamma f(z)dz$.

If moreover, the geometric image of γ is the interval [-1,1], then

$$\lim_{n\to\infty} \int_{[-1,1]} |f(z) - L_b(f)(z)|^2 dz = 0.$$

PROOF. By (4) and by [7, chapter IV] we get

$$\int_\gamma [f(z) - L_n(f)(z)]dz = \left\{ \int_{-1}^{1} [f_1(z_1) - L_n(f_1)(z_1)dz_1] \right\} q_1 + \left\{ \int_{-1}^{1} [f_2(z_2) - L_n(f_2)(z_2)dz_2] \right\} q_2$$

But by the classical theorem of Erdös and Turan [9] we have

$$(5) \qquad \lim_{n\to\infty} \int_{-1}^{1} [f_k(z_k) - L_n(f_k)(z_k)]^2 dz = 0, k = \overline{1,2}.$$

This obviously implies that

$$\lim_{n\to\infty} \int_{-1}^{1} [f_k(z_k) - L_n(f_k)(z_k)]dz_k = 0, k = \overline{1,2}$$

and therefore $\lim_{n\to\infty} \int_\gamma [f(z) - L_n(f)(z)]dz = 0$.

Now, let γ be the path such that the geometric image of γ is the interval [-1,1]. By (4) we obtain

$$|f(z) - L_n(f)(z)|^2 = \left[|f_1(z_1) - L_n(f_1)(z_1)|^2 + |f_2(z_2) - L_n(f_2)(z_2)|^2 \right] / 2 = F_n(z_1, z_2) =$$

$$= F_n(z_1, z_2)q_1 + F_n(z_1, z_2)q_2, \; z = z_1q_1 + z_2q_2 \in R^*(-1,1).$$

Applying the definition of the integral for $F_n(z_1, z_2)$ on

$\gamma(t)=[-1,1]$, $t\in[a,b]$, since $\gamma(t)=\gamma(t)q_1+\gamma(t)q_2$, in [7,chapter IV] we immediately get

$$\int_{[-1,1]} F_n(z_1,z_2)dz=(1/2)\cdot\left[\int_a^b\left|f_1(\gamma(t))-L_n(f_1)(\gamma(t))\right|^2 d\gamma(t)+\int_a^b\left|f_2(\gamma(t))-L_n(f_2)(\gamma(t))\right|^2 d\gamma(t)\right]=$$

$$=(1/2)\cdot\left[\int_{-1}^{1}\left|f_1(u)-L_n(f_1)(u)\right|^2 du+\int_{-1}^{1}\left|f_2(u)-L_n(f_2)(u)\right|^2 du\right]\xrightarrow{n\to\infty}0,\text{ by (5).}$$

The theorem is proved .

REMARK . In the case when $f:R^*(a,b)\to C_q$, $a,b\in R$, $a<b$, is of the form $f(z)=P_1(z_1,z_2)q_1+P_2(z_1,z_2)q_2$, $z\in R^*(a,b)$, with P_1 and P_2 continuous on $R^*(a,b)$, as in Section 3 we can introduce the interpolatory areolar polynomials of Lagrange-type.

$$L_n(f)(z,\bar{z})=\sum_{k=1}^{n}\sum_{j=1}^{n}\bar{f}_1\left(x_k q_1+x_j q_2\right)\cdot l_{k,n}(z)\cdot l_{j,n}(\bar{z})=$$

$$=\sum_{k=1}^{n}\sum_{j=1}^{n}\left[P_1\left(x_k,x_j\right)q_1+P_2\left(x_j,x_k\right)q_2\right]\cdot l_{k,n}(z)\cdot l_{j,n}(\bar{z})=$$

$$=\left[\sum_{k=1}^{n}\sum_{j=1}^{n}P_1\left(x_k,x_j\right)\cdot l_{k,n}(z_1)\cdot l_{j,n}(z_2)\right]q_1+\left[\sum_{k=1}^{n}\sum_{j=1}^{n}P_2\left(x_j,x_k\right)\cdot l_{k,n}(z_2)\cdot l_{j,n}(z_1)\right]\cdot q_2,$$

where $a\le x_1\le\ldots<x_n\le b$ and $\overline{f_1}(z)=P_1(z_1,z_2)q_1+P_2(z_1,z_2)q_2$, $z=z_1q_1+z_2q_2$.

It is easy to check that $L_n(f)\left(x_p,x_p\right)=f\left(x_p\right),\quad \forall p=\overline{1,n}$.

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