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AN EXTRAPOLATION THEOREM FOR PAIRS OF WEIGHTS

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ABSTRACT. In [4] C. J. Neugebauer showed that pairs of weights belonging to A_p classes satisfy an extrapolation property, namely, any sublinear operator which is of weak type (p_0, p_0) for every pair of weights in A_{p_0} is also of weak type (p,p) for any pair in $A_p, 1 < p \leq p_0$. We investigate the corresponding extrapolation property for pairs of weights in $A_{p,q}$ classes starting from appropriate (p_0, ∞) inequalities. As a consequence we are able to derive some double weighted weak type inequalities from weighted results of the type L^{p_0}, BMO.

Let $w(x) \geq 0$ be a locally integrable function defined on $\mathbf{R}^n$. We denote by $w(E)$ the measurewith density $w(x)$ with respect to the Lebesgue measure, i.e. $w(E) = \int_E w(x)\, dx$. The density $w(x)$ is called a weight with respect to dx. The space L^p with respect to the measure $w(x)\ dx$ will be denoted either by L^p_w or $L^p(wdx)$ and its norm by $||f||_{p,w}$. If $w(x) \equiv 1$, we drop w in the notation. We shall say that a pair (u, v) of non-negative functions belongs to the class $A(p,q), 1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, if

$$\sup_Q(|Q|^{-1} \int_Q u(x)^q\, dx)^{1/q}(|Q|^{-1} \int_Q v(x)^{-p'}\, dx)^{1/p'} = C(p,q,u,v) < \infty,$$

where Q stands for any cube in $\mathbf{R}^n$. In particular, when $q = \infty$ the condition $(u,v) \in A(p,\infty)$ becomes

$$\sup_Q(\operatorname{ess}\sup_{x\in Q} u(x))(|Q|^{-1} \int_Q v(x)^{-p'}\, dx)^{1/p'} = C(p,\infty,u,v) < \infty.$$

Given a locally integrable function $f(x)$, the Hardy-Littlewood maximal function $Mf(x)$ is defined as usual as

$$Mf(x) = \sup_{x\in Q}(|Q|^{-1} \int_Q |f(y)|\, dy).$$

For a pair of weights (u,v) belonging to $A(p,p)$, the maximal function $Mf(x)$ satisfies the weak type inequality

$$\int_{\{x:Mf(x)>\lambda\}} u^p(x)\, dx \leq C(\lambda^{-1}||f||_{p,v^p})^p \tag{1}$$

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for any $\lambda > 0$ with C depending on $C(p,p,u,v)$ only. For a proof see [2]. Following B. Muckenhoupt and R. L. Wheeden in [3], we shall say that f belongs to BMO_w if

$$\sup_Q (ess \sup_{x\in Q} w(x))(|Q|^{-2}\int_Q\int_Q |f(y)-f(z)|\,dy\,dz) = ||f||_{BMO_w} < \infty,$$

where Q stands for any cube in $\mathbf{R}^n$. It is often convenient to express this condition on f by means of the sharp function $f^\#$ defined as

$$f^\#(x) = \sup_{x\in Q} |Q|^{-2}\int_Q\int_Q |f(y)-f(z)|\,dy\,dz;$$

in fact, it is of easy verification (see [1]) that

$$||f||_{BMO_w} \sim ||wf^\#||_\infty. \tag{2}$$

Finally, we recall some definitions concerning the Lorentz $L(p,q,\mu)$ spaces. Let f be a measurable function on a measure space $(M,\mathcal{M},\mu)$. The non-increasing rearrangement $f^*(t)$ of f is defined as

$$f^*(t) = inf\{s : \mu(\{x : |f(x)| > s\}) \le t\},$$

for $t > 0$. The function f is said to belong to the Lorentz space $L(p,q,\mu)$ if the quantities

$$||f||_{p,q,\mu} = (q/p\int_0^\infty [t^{1/p}f^*(t)]^q\,\frac{dt}{t})^{1/q},$$

whenever $0 < p < \infty$ and $0 < q < \infty$, and

$$||f||_{p,\infty,\mu} = \sup_{t>0} t^{1/p}f^*(t),$$

when $0 < p \le \infty$ and $q = \infty$, are finite. For more details see [5].

A Theorem of Extrapolation.

B. Muckenhoupt and R. L. Wheeden proved in [3] that the fractional integral of order α, $0 < \alpha < n$,

$$I_\alpha f(x) = \int_{\mathbf{R}^n} f(y)|x-y|^{\alpha-n}\,dy \tag{3}$$

satisfies the inequality

$$||I_\alpha f||_{BMO_w} \le C_w ||f||_{n/\alpha,w^{n/\alpha}} \tag{4}$$

if and only if $(w, w) \in A(n/\alpha, \infty)$. Taking this limit inequality as a starting point it can be obtained by extrapolation (see[1]) that

$$||I_\alpha f||_{q,w^q} \leq C_{w,p,q}||f||_{p,w^p}$$

provided that $1/q = 1/p - \alpha/n$, with $1 < p < n/\alpha$, and $(w, w) \in A(p, q)$. The constant $C_{w,p,q}$ depends on $C(p, q, w, w)$ only. The arguments given in [3] to prove (4) can be used to show that the inequality

$$||I_\alpha f||_{BMO_u} \leq C||f||_{n/\alpha,\ v^{n/\alpha}} \tag{5}$$

holds if and only if $(u, v) \in A(n/\alpha, \infty)$. By (2), the inequality (5) can be rewritten as

$$||u(I_\alpha f)^\#||_\infty \leq C||f||_{n/\alpha,\ v^{n/\alpha}}. \tag{6}$$

By analogy with the one weigh case inequality (5) suggests the search of an extrapolation theorem giving new inequalities taking (5) or (6) as a starting point. It is easy to get convinced that the inequalities involving $A(p, q)$ weights that can be obtained shall be of weak type. It is pertinent to mention at this point a result due to C.J.Neugebauer [4], who proved that if a sublinear operator T maps $L^{p_0}(u^{p_0}dx)$ weakly into $L^{p_0}(v^{p_0}dx)$ for every pair $(u, v) \in A(p_0, p_0)$, then T maps $L^p(u^p dx)$ weakly into $L^p(v^p dx)$ for every pair $(u, v) \in A(p, p)$, with the restriction $1 < p < p_0$. Taking into account the behaviour of $(I_\alpha f)^\#$ stated in (6) and other instances that shall be presented as illustrations, the extrapolation theorem that seems appropriate for our purpose is the following one

Theorem (of weak extrapolation). *Let T be an operator defined on C_0^∞ with values in the space of measurable functions. Let us assume that it verifies*

1. $|T(\lambda f)| = |\lambda||T(f)|$, $\quad |T(f + g)| \leq |T(f)| + |T(g)|$, *and*

2. *for given r and $\beta, 1 \leq r < \beta \leq \infty$ and for every pair of weights (a, b) such that $(a^r, b^r) \in A(\beta/r, \infty)$*

$$||aT(f)||_\infty \leq C||f||_{\beta,b^\beta}$$

with a constant C depending on $C(\beta/r, \infty, a^r, b^r)$ only.

Then, if $r < p < \beta$, $1/q = 1/p - 1/\beta$ and $(u^r, v^r) \in A(p/r, q/r)$, there exists a constant C which depends on $C(p/r, q/r, u^r, v^r)$ such that

$$u^q(\{x : |Tf(x)| > \lambda\}) \leq C(\lambda^{-p} \int |f|^p v^p dx)^{q/p}$$

holds for every $\lambda > 0$.

Proof. Let $f \in C_0^\infty$, $0 < m = \int |f|^p v^p\, dx < \infty$ and $(u^r, v^r) \in A(p/r, q/r)$. We set

$$b(x) = \begin{cases} |f(x)|^{p/\beta - 1}\, v(x)^{p/\beta} m^{1/q} & \text{if } |f(x)| > 0 \\ e^{\pi |x|^2/q} v(x) & \text{if } |f(x)| = 0. \end{cases}$$

Since $v(x) > 0$ a.e., it turns out that $b(x) > 0$ a.e. on $\mathbf{R}^n$. Moreover, $b(x)$ satisfies

$$(i) \quad \|fv\|_p = \|fb\|_\beta \qquad \text{and}$$

$$(ii) \quad \int b^{-q}\, v^q\, dx \leq 2$$

In fact, they are inmediate if $\beta = \infty$ and for $\beta < \infty$, we have

$$\begin{aligned} \int |f|^\beta\, b^\beta\, dx &= \int_{|f|>0} |f|^\beta\, |f|^{p-\beta}\, v^p m^{\beta/q}\, dx \\ &= m^{\beta/q} \int |f|^p\, v^p\, dx \\ &= m^{1+\beta/q} = m^{\beta/p} \end{aligned}$$

and,

$$\begin{aligned} \int b^{-q}\, v^q\, dx &\leq \int_{|f|>0} |f|^{(1-p/\beta)q}\, v^{(1-p/\beta)q} m^{-1}\, dx + \int e^{-\pi|x|^2}\, dx \\ &= \left(\int |f|^p\, v^p\, dx\right) m^{-1} + 1 = 2 \end{aligned}$$

Now let us define

$$a(x) = [M(b^{-r(\beta/r)'})(x)]^{-1/r(\beta/r)'}.$$

Then, it follows inmediately that $(a^r, b^r) \in A(\beta/r, \infty)$ and $C(\beta/r, \infty, a^r, b^r) \leq 1$. Let us consider the set $E_\lambda = \{x : |Tf(x)| > \lambda\}$. Then, we have

$$\begin{aligned} u^q(E_\lambda) &= \int_{E_\lambda} u^q(x)\, dx \\ &= \int \chi_{E_\lambda}(x)\, a(x)^{-1}\, [a(x)\, u(x)^q]\, dx \qquad (7) \\ &\leq \|\chi_{E_\lambda}\|_{1+1/q, 1, au^q}\, \|a^{-1}\|_{q+1, \infty, au^q} \end{aligned}$$

In order to estimate the second factor above, we observe that

$$\lambda^{q+1} \int_{\{x:a^{-1}(x)>\lambda\}} au^q\,dx \leq \lambda^q \int_{\{x:M(b^{-r(\beta/r)'})>\lambda^{r(\beta/r)'}\}} u^q\,dx. \tag{8}$$

Since $(u^r, v^r) \in A(p/r, q/r)$ implies $(u^{q/s}, v^{q/s}) \in A(s,s)$ for $s = 1+(q/r)/(p/r)'$, from (8), (1) and property (ii) of $b(x)$ we get

$$\lambda^{q+1} \int\limits_{\{x:a^{-1}(x)>\lambda\}} a\,u^q\,dx \;\leq\; C\lambda^q\lambda^{-r(\beta/r)'s}\int b^{-r(\beta/r)'s}\,v^q\,dx$$

$$= \; C\int b^{-q}vq\,dx \leq 2C$$

This implies that $||a^{-1}||_{q+1,\infty,au^q} \leq (2C)^{1/(q+1)}$. Let us now estimate $||\chi_{E_\lambda}||_{1+1/q,q,au^q}$. The non-increasing rearrangement of $\chi_{E_\lambda}(x)$ with respect to the measure $au^q\,dx$ is equal to the characteristic function of the interval $(0,R)$, $R = \int_{E_\lambda} au^q\,dx$. Then

$$||\chi_{E_\lambda}||_{1+1/q,1,au^q} = (q/(q+1))\int_0^R t^{q/(q+1)}\,dt/t = R^{q/(q+1)}.$$

On the other hand,

$$R = \int\limits_{E_\lambda} au^q\,dx \leq \lambda^{-1}\int\limits_{E_\lambda}|T(f)|au^q \leq \lambda^{-1}\,\|T(f)a\|_\infty\int\limits_{E_\lambda}u^q\,dx \leq \lambda^{-1}\,\|fb\|_\beta\int\limits_{E_\lambda}u^q\,dx.$$

Then,

$$\|\chi_{E_\lambda}\|_{1+1/q,1,au^q} \leq [\lambda\;\|fb\|_\beta\;u(E_\lambda)]^{q/(q+1)}.$$

Collecting our estimates on the factors of the last term of (7) and property (i) of $b(x)$, we obtain

$$u^q(E_\lambda) \leq \left[2C\;\lambda^{-q}\;\|fv\|_p^q\;u^q(E_\lambda)^q\right]^{1/(q+1)}$$

Since $u^q(E_\lambda) < \infty$,

$$u^q(E_\lambda) \leq (2C)^{q+1}(\lambda^{-p}\int|f|^pv^p\,dx)^{q/p},$$

as we wanted to show. □

Illustrations.

Now, we shall illustrate our theorem considering some particular cases for the operator T.

1. Let $0 \leq \alpha < n$, $1 \leq r < \infty$ and let us define

$$M_\alpha^r f(x) = \sup_{x\in Q}(|Q|^{\alpha/n-1}\int_Q |f(y)|^r\, dy)^{1/r}.$$

When $r = 1$, this operator is the fractional maximal function of order α and we denote $M_\alpha = M_\alpha^1$; moreover if in addition $\alpha = 0$ it reduces to the Hardy-Littlewood maximal function, i.e. $M = M_0^1$. It is simple to verify that $M_\alpha^r = T$ satisfies the hypotheses of the theorem for $\beta = nr/\alpha$. In fact, if $(a^r, b^r) \in A(\beta/r, \infty)$ and Q is a cube and $x \in Q$, by Hölder's inequality, we have

$$|Q|^{\alpha/n-1}\int_Q |f|^r\ dx \leq |Q|^{\alpha/n-1}\left(\int |f|^{rn/\alpha}\, b^{rn/\alpha}\ dx\right)^{\alpha/n}\left(\int_Q b^{-r(n/\alpha)'}\ dx\right)^{1/(n/\alpha)'}$$

Then, if t is a Lebesgue point of a and $t \in Q$, we have

$$\begin{aligned} a(t)\quad &\left(|Q|^{\alpha/n-1}\int_Q |f|^r\ dx\right)^{1/r} \\ &\leq \left(\int |f|^{rn/\alpha}\, b^{rn/\alpha}\ dx\right)^{\alpha/nr}\ \operatorname*{ess\ sup}_{y\in Q}\ a(y)\left(|Q|^{-1}\int_Q b^{-r(n/\alpha)'}\ dx\right)^{1/r(n/\alpha)'} \\ &\leq [C(n/\alpha, \infty, a^r, b^r)]^{1/r}\left(\int |fb|^{rn/\alpha}\ dx\right)^{\alpha/nr}, \end{aligned}$$

if $(a^r, b^r) \in A(\beta/r, \infty)$. Taking the supremum for all cubes Q such that $x \in Q$ it follows that

$$||aM_\alpha^r f||_\infty \leq [C(n/\alpha, \infty, a^r, b^r)]^{1/r}||fb||_\beta.$$

Applying the theorem, we obtain that if a pair of non-negative functions (u, v) satisfies that $(u^r, v^r) \in A(p/r, q/r)$ with $1/q = 1/p - \alpha/rn$, $\quad r < p < nr/\alpha$ then

$$u^q(\{x : M_\alpha^r f(x) > \lambda\}) \leq C(\lambda^{-p}\int |f|^p v^p\, dx)^{q/p},$$

for every $\lambda > 0$. This result is well known. Moreover, it is known that the condition on the weights is also necessary. The only purpose for including it here is to show that the results on M_α^r can be reduced to very basic properties of the Hardy-Littlewood maximal function. Observe that in the proof of the theorem only weak type properties of the Hardy- Littlewood maximal function have been assumed to be known.

2. Another interesting illustration arises from inequality (5). As we mention before, it was precisely this inequality (5) which induced us to ask the question of what inequalities could be derived by extrapolation from it. The verification of (5) can be done as in the case of one weight, i.e. when $u = v$. Therefore, the operator $Tf = (I_\alpha f)^{\#}$, where I_α is defined in (3), satisfies the hypotheses of the theorem with $r = 1$ and $\beta = n/\alpha$ and it follows that if $(u, v) \in A(p, q)$, $1/q = 1/p - \alpha/n$, $1 < p < n/\alpha$ then

$$u^q(\{x : (I_\alpha f)^{\#}(x) > \lambda\}) \leq C(\lambda^{-p} \int |f|^p v^p \, dx)^{q/p}, \tag{9}$$

holds for $\lambda > 0$ and a constant C depending on $C(p, q, u, v)$ only. A careful analysis of the proof of (5) given in [3] shows the pointwise estimate

$$(I_\alpha f)^{\#}(x) \leq C_{\alpha,n} M_\alpha f(x),$$

wich is underlying in their argument. Clearly (9) could also be derived from this inequality. The same considerations are valid for an operator slightly more general. Let $s \geq 1$ and let g be a locally integrable function. We define

$$g^{\#,s}(x) = \sup_{x \in Q} \left(|Q|^{-2} \int_Q \int_Q |g(y) - g(z)|^s \, dy \, dz \right)^{1/s}.$$

If we set $Tf = (I_\alpha f)^{\#,s}$, the same argument indicated for $s = 1$, shows that this operator satisfies the hypotheses of the extrapolation theorem for $r = 1$, as long as $1 \leq s < \frac{n}{n-\alpha}$. The exponent $n/(n - \alpha)$ comes in because we are using that I_α satisfies the weak type inequality $(1, n/(n - \alpha))$, which in turn implies that I_α is of strong type $(1, s)$ over bounded sets if $s < n/(n - \alpha)$. Thus, if $(u, v) \in A(p, q)$, $1/q = 1/p - \alpha/n$, $1 < p < n/\alpha$, then

$$u^q(\{x : (I_\alpha f)^{\#,s}(x) > \lambda\}) \leq C(\lambda^{-p} \int |f|^p v^p \, dx)^{q/p}, \tag{10}$$

holds for every $\lambda > 0$, with C depending on $C(p, q, u, v)$ only.

Next, we shall prove that the conditions on the pair (u, v) are also necessary for (10) to hold. In fact, let Q be a cube with diameter d and let $\tilde{Q}$ be the cube with the same center and sides of lenght $12d$. We denote by Q_1 and Q_2 the translates of Q defined by $Q + e_1$, $Q + e_2$ with $|e_1| = 2d$ and $|e_2| = 5d$. It is simple to verify that

(*i*) $|Q_1| = |Q_2| = |Q|$, $Q \cup Q_1 \cup Q_2 \subset \tilde{Q}$ and $|\tilde{Q}| = (12\sqrt{n})^n |Q|$

(*ii*) For every $y \in Q_1$, $z \in Q_2$ and $t \in Q$

$$|y - t| \leq 3d \text{ and } |z - t| \geq 4d.$$

Now, let $f \geq 0$ be a bounded function with support contained in Q. For $x \in Q$, we have

$$(I_\alpha f)^{\#,s}(x) \geq \left(\left|\tilde{Q}\right|^{-2} \int\limits_{\tilde{Q}} \int\limits_{\tilde{Q}} |I_\alpha f(y) - I_\alpha f(z)|^s \, dy\, dz \right)^{1/s}$$

$$\geq \left(\left|\tilde{Q}\right|^{-2} \int\limits_{\tilde{Q}_2} \int\limits_{\tilde{Q}_1} \left| \int\limits_{\tilde{Q}} f(t) \left\{ |y-t|^{\alpha-n} - |z-t|^{\alpha-n} \right\} dt \right|^s dy\, dz \right)^{1/s}$$

From (ii) we obtain

$$\begin{aligned} |y-t|^{\alpha-n} - |z-t|^{\alpha-n} &\geq (3\,d)^{\alpha-n} - (4\,d)^{\alpha-n} \\ &= (3^{\alpha-n} - 4^{\alpha-n})\, n^{(\alpha-n)/2}\, |Q|^{\alpha/n-1} \end{aligned}$$

therefore, for $x \in Q$,

$$(I_\alpha f)^{\#,s}(x) \geq C_{n,\alpha} |Q|^{\frac{\alpha}{n}-1} \int_Q f(t)\, dt = \lambda.$$

Choosing $f = \chi_Q v_k^{-p'}$, where $v_k = v + 1/k$, we get

$$\int |f|^p v^p \, dx \leq \int_Q v_k^{-p'p+p} \, dx = \int_Q v_k^{-p'} \, dx < \infty.$$

Then, taking into account (10), we obtain

$$\begin{aligned} u^q(Q) &\leq u^q\left(\left\{x : (I_\alpha f)^{\#,s}(x) > \lambda\right\}\right) \\ &\leq C \left(\lambda^{-p} \int\limits_Q v_k^{-p'} \, dx \right)^{q/p} \\ &= C C_{n,\alpha}^{-q} |Q|^{(1-\alpha/n)q} \left(\int\limits_Q v_k^{-p'} \, dx \right)^{q/p} \end{aligned}$$

therefore, if $1/q = 1/p - \alpha/n$, it follows

$$\left(\frac{1}{|Q|} \int_Q u^q \, dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q v_k^{-p'} \, dx \right)^{1/p'} \leq C^{1/q} C_{n,\alpha}^{-1}$$

By Fatou's Lemma, this inequality implies

$$\left(|Q|^{-1}\int_Q u^q\,dx\right)^{1/q}\left(|Q|^{-1}\int_Q v^{-p'}\,dx\right)^{1/p'}\leq C,$$

which shows that $(u,v)\in A(p,q)$.

3. We shall give a third illustration of the theorem which consists in taking $Tf = (Hf)^{\#,r}$ for $r>1$, and H standing for the one dimensional Hilbert transform,

$$Hf(x) = p.v.\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt.$$

the argument given in the proof of Feffermann-Stein inequality also show that

$$|Tf(x)|\leq C_r M^r(f)(x) \quad \text{a.e.},$$

and by the first illustration above this implies that the operator $(Hf)^{\#,r}$ is of weak type (p,p) for every pair of weights (u,v) such that $(u^r,v^r)\in A(p/r,p/r)$, $\quad r<p<\infty$.

We shall show that the conditions on the weights are also necesary for the weak type (p,p) of $(Hf)^{\#,r}$, $\quad r>1$. We shall need the following lemma:

Lemma. *Let $f\geq 0$ be a bounded function with support contained in a finite interval I. There exist two constants ρ_r and C_r, depending on r only, and an interval J such that*

$$\begin{cases} 1.\ I\subset J \text{ and } |J| = \rho_r\,|I| \\ 2.\ \int|f(x)|^r\,dx\leq C_r\int_J|Hf(x)|^r\,dx \quad \text{and} \\ 3.\ \int_J Hf(x)\,dx = 0\end{cases} \tag{11}$$

Proof. Let x_0 be the center of I. By Riesz's inequality, we have

$$\begin{aligned}\int|f(x)|^r\,dx &\leq A_r\int|Hf(x)|^r\,dx \\ &= A_r\int_{|x-x_0|<\delta}|Hf(x)|^r\,dx + A_r\int_{|x-x_0|>\delta}|Hf(x)|^r\,dx.\end{aligned} \tag{12}$$

If $\delta > 2|I|$ and $|x - x_0| > \delta$, it follows that

$$\begin{aligned} |Hf(x)| &\le \int \frac{f(y)}{|x-y|}\,dy \\ &\le 2\,|x-x_0|^{-1}\int f(y)\,dy \\ &\le 2\,|I|^{1/r'}\,\|f\|_r\,|x-x_0|^{-1} \end{aligned}$$

then,

$$\begin{aligned} \int_{|x-x_0|>\delta} |Hf(x)|^r\,dx &\le 2^r\,|I|^{r-1}\,\|f\|_r^r \int_{|x-x_0|>\delta} |x-x_0|^{-r}\,dx \\ &= 2^r(|I|/\delta)^{r-1}\,\|f\|_r^r\,(r-1)^{-1}. \end{aligned} \tag{13}$$

We choose $s \ge 2$ such that $A_r 2^r s^{1-r}/(r-1) \le 1/2$. Then if $\delta = s|I|$, it follows by (12) and (13) that

$$A_r \int_{|x-x_0|>\delta} |Hf(x)|^r\,dx \le (1/2)\int |f(x)|^r\,dx,$$

and

$$\int |f(x)|^r\,dx \le 2A_r \int_{|x-x_0|<\delta} |Hf(x)|^r\,dx.$$

Moreover, if t satisfies $|t - x_0| < |I|$, then for $|x - x_0| < \delta$ we get $|x - t| < 2\delta$. Therefore

$$\int |f(x)|^r\,dx \le 2A_r \int_{|x-t|<2\delta} |Hf(x)|^r\,dx.$$

On the other hand, since

$$\int_{|x-t|<2\delta} Hf(x)\,dx = -\int_I f(x)\log\left|\frac{x-t+2\delta}{x-t-2\delta}\right|dx = \phi(t),$$

it turns out that $\phi(t)$ is a continuous function. From

$$\log\left|\frac{x-t+2\delta}{x-t-2\delta}\right| \begin{cases} \ge 0 & \text{if } x \ge t \text{ and} \\ \le 0 & \text{if } x \le t, \end{cases}$$

we see that if $I = (a, b)$, then $\phi(a) < 0$ and $\phi(b) > 0$ implying the existence of $t_0 \in I$ such that $\phi(t_0) = 0$. Summing up, if $J = (t_0 - 2\delta, t_0 + 2\delta)$, $\delta = s|I|$, properties

(1), (2) and (3) of (11) hold for $C_r = 2A_r$ and $\rho = 4s$. This ends the proof of the Lemma. □

Now, we can prove the necessity. Let $v_k = v + 1/k$. Then v_k^{-1} is bounded and

$$f(x) = \chi_I(x) v_k(x)^{-(p/r)'},$$

is a bounded non negative function with support contained in I. By the lemma, we have that for $x \in J$,

$$\begin{aligned}
(Hf)^{\#,r}(x) &\geq \left(|J^{-2}| \int_J |Hf(y) - Hf(z)|^r \; dy\, dz \right)^{1/r} \\
&\geq \left(|J^{-1}| \int_J \left| Hf(y) - |J|^{-1} \int_J Hf(z)^r \; dz \right|^r \; dy \right)^{1/r} \\
&\geq \left(|J^{-1}| \int_J |Hf(y)|^r \; dy \right)^{1/r} \\
&\geq \left(C_r^{-1} \rho_r^{-1} \, |I|^{-1} \int_J |f(y)|^r \; dy \right)^{1/r} \\
&= \lambda
\end{aligned}$$

Since we assume $(Hf)^{\#,r}$ to be of weak type (p,p) with respect to the weights u^p and v^p, we get that for this value of λ

$$u^p(I) \leq u^p(\{x : (Hf)^{\#,r}(x) > \lambda\}) \leq C \, \lambda^{-p} \int f^p v^p \, dx.$$

However,

$$\int f^p v^p \, dx \leq \int_I v_k^{p-(p/r)'p} \, dx = \int f^r \, dx < \infty.$$

Thus, after some computations, we get

$$\left(|I|^{-1} \int_I (u^r)^{p/r} \, dx \right)^{r/p} \left(|I|^{-1} \int_I (v_k^r)^{-(p/r)'} \, dx \right)^{1/(p/r)'} \leq C_0.$$

By Fatou's lemma, this condition implies that $(u^r, v^r) \in A(p/r, p/r)$, as we wanted to show. □

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