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DUAL SPACES FOR ONE-SIDED WEIGHTED HARDY SPACES

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ABSTRACT. Let $H_+^p(w)$ be the Hardy spaces introduced in [3] defined for one-sided weights w, see [4], and a suitable one-sided maximal function for distributions on the real line. The purpose of this paper is to give a characterization of the dual spaces of $H_+^p(w)$ in terms of certain classes of weighted BMO of Lipschistz spaces. The method used here is similar to that of J. García-Cuerva in [1] for $H^p(w)$ spaces, where w belongs to A_q classes of B. Muckenhoupt. For the case of $w(x)>0$ almost everywhere, the characterization obtained generalizes the one given in [1], see Theorem (2.4).

1. NOTATIONS, DEFINITIONS AND PREREQUISITES

Given a Lebesgue measurable set $E \subset \mathbb{R}$, we denote its Lebesgue measure by $|E|$ and the characteristic function of E by χ_E.

Let f be a measurable function defined on $\mathbb{R}$. The one-sided Hardy-Littlewood maximal functions M^-f and M^+f are given by

$$M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)|dt \quad \text{and} \quad M^+f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(t)|dt \, .$$

As usual, a weight w is a measurable and non-negative function. If $E \subset \mathbb{R}$ is a measurable set, we denote its w-measure by $w(E) = \int_E w(t)dt$.

A weight w belongs to the class A_q^+, $1 \leq q < \infty$, if there exists a constant c such that

$$\sup_{h>0} \left(\frac{1}{h} \int_{x-h}^{x} w(t)dt \right) \left(\frac{1}{h} \int_{x}^{x+h} w(t)^{-\frac{1}{q-1}} dt \right)^{q-1} \leq c \, ,$$

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for all real number x. We observe that w belongs to A_1^+ if and only if $M^-w(x) \leq c\, w(x)$ holds for almost every x.

Given w belonging A_q^+, $1 \leq q < \infty$, we can define $x_{-\infty} \geq -\infty$ and $x_\infty \leq +\infty$, such that

$$(1.1) \qquad \begin{aligned} &\text{(i)} \quad w(x) = 0 \text{ a.e. in } (-\infty, x_{-\infty}), \\ &\text{(ii)} \quad w(x) = \infty \text{ a.e. in } (x_\infty, \infty) \text{ and,} \\ &\text{(iii)} \quad 0 < w(x) < \infty \text{ for almost every } x \in (x_{-\infty}, x_\infty). \end{aligned}$$

We always have $x_{-\infty} \leq x_\infty$. In order to avoid the non-interesting case of $x_{-\infty} = x_\infty$, it is necessary and sufficient that there exists a measurable set E satisfying $0 < w(E) < \infty$.

Let f be a measurable function with support contained in an interval I (I not necessarily bounded). We shall say that f belongs to $L^r(I,w)$, $0 < r \leq \infty$, if $\|f\|_{L^r(I,w)} = \left(\int |f(x)|^r w(x) dx\right)^{1/r}$ is finite. If $I = I\!R$ or $w \equiv 1$ we simply write $L^r(w)$ or $L^r(I)$ respectively, and $L^r(I\!R)$ shall be denoted by L^r. Given a positive integer γ, we say that a function f belongs to $L^r_\gamma(I,w)$ if $f \in L^r(I,w)$ and, if $|I| < \operatorname{dist}(x_{-\infty}, I)$, then we require f to have null moments up to the order $\gamma - 1$, i.e., $\int f(x) x^k dx = 0$ holds for every integer k, $0 \leq k \leq \gamma - 1$.

The following lemma contains the basic results for A_q^+ weights and one-sided maximal functions that we shall need in this paper.

Lemma 1.2.

(i) Let $1 \leq q_1 < q_2 < \infty$. If the weight w belongs to the class $A_{q_1}^+$, then it also belongs to $A_{q_2}^+$.

(ii) Let $1 < q < \infty$. The one-sided Hardy-Littlewood maximal M^+ is bounded on $L^q(w)$ if and only if w belongs to A_q^+.

(iii) Given $w \in A_q^+$, $1 \leq q < \infty$ for every $a \in I\!R$, the w-measure of the interval (a, ∞) is equal to infinite.

(iv) Let $w \in A_q^+$, $1 \leq q < \infty$. Let $\alpha < \beta$ be the end points of the bounded interval I. Then, the interval $\widetilde{I}$ with end points $\alpha - |I|$ and α, satisfies

$$w(\widetilde{I}) \leq c_w\, w(I)$$

where the constant c_w does not depend on I.

A proof of *(ii)* may be found in [4]or in [2]. As for parts *(i)* and *(iii)* the proofs are easy. Part *(iv)* is an immediate consequence of *(ii)*.

Let w belong to A_q^+, $1 \le q < \infty$, and let $x_{-\infty}$ be defined as in (1.1) for the weight w. As usual, $C_0^\infty(I\!R)$ denotes the set of all functions with compact support having derivates of all orders. We shall denote by $\mathcal{D}(x_{-\infty}, \infty)$ the space of all functions in $C_0^\infty(I\!R)$ with support contained in $(x_{-\infty}, \infty)$ equipped with the usual topology and by $\mathcal{D}'(x_{-\infty}, \infty)$ the space of distributions on $(x_{-\infty}, \infty)$.

Given a positive integer γ and $x \in I\!R$, we shall say that a function ψ in $C_0^\infty(I\!R)$, belongs to the class $\Phi_\gamma(x)$ if there exists a bounded interval $I_\psi = [x, \beta]$ containing the support of ψ such that $D^\gamma \psi$ satisfies

$$|I_\psi|^{\gamma+1} \|D^\gamma \psi\|_\infty \le 1 .$$

Let F be a distribution in $\mathcal{D}'(x_{-\infty}, \infty)$. We define as in [3]the one-sided maximal function $F_{+,\gamma}^*$, as

(1.3) $$F_{+,\gamma}^*(x) = \sup\{| < f, \psi > | : \psi \in \Phi_\gamma(x)\} ,$$

for every $x > x_{-\infty}$.

Fixed w belonging to A_q^+ $(1 \le q < \infty)$, a positive integer γ and, $0 < p \le 1$ such that $(\gamma+1)p \ge q > 1$ or $(\gamma+1)p > q$ if $q = 1$, the distribution F in $\mathcal{D}'(x_{-\infty}, \infty)$ belongs to $H_{+,\gamma}^p(w)$ if the "p-norm"

$$\|F\|_{H_{+,\gamma}^p(w)} = \left(\int_{x_{-\infty}}^\infty F_{+,\gamma}^*(x)^p w(x) dx \right)^{1/p} ,$$

is finite.

In the sequel we shall suppose that w belongs to A_q^+, γ is a positive integer, $0 < p \le 1$ and, that they satisfy $(\gamma+1)p \ge q$ if $q > 1$ or $(\gamma+1)p > q$ if $q = 1$.

Lemma 1.4. *Let $I \subset (x_{-\infty}, \infty)$ be an interval and let f belong to $L_\gamma^\infty(I)$. Then for any $x > x_{-\infty}$, we have*

$$f_{+,\gamma}^*(x) \le c_\gamma \|f\|_\infty \left[M^+ \chi_I(x)\right]^{\gamma+1} .$$

Moreover,

$$\|f\|_{H_{+,\gamma}^p(w)} \le c_{\gamma,w} \, \|f\|_\infty \, w(I)^{1/p} .$$

The constants c_γ and $c_{\gamma,w}$ do not depend on f.

This lemma can be found in [3] as Lemma (3.2). Thus, as in [3] we have the following definition of p-atom with respect to a weight w.

A function $a(x)$ defined on $\mathbb{R}$ is called a p-atom with respect to w if there exists an interval I containing the support of $a(x)$, such that

(i) I is contained in $(x_{-\infty}, \infty)$ and $w(I) < \infty$,

(ii) $a(x) \in L^{\infty}_{\gamma}(I)$ and,

(iii) $\|a\|_{\infty} \leq w(I)^{-1/p}$.

We shall say that I is the interval associated to the atom $a(x)$.

The following theorems are of fundamental importance in the theory of the $H^p_{+,\gamma}(w)$ spaces. Their proofs can be found in section 5 of [**3**].

Theorem 1.5. (Decomposition into atoms). *If F belongs to $H^p_{+,\gamma}(w)$, then there exists a sequence $\{a_k\}$ of p-atoms with respect to w and a sequence $\{\lambda_k\}$ of real numbers such that*

$$F = \sum \lambda_k \, a_k \qquad \text{in} \qquad \mathcal{D}'(x_{-\infty}, \infty)$$

and,

$$c'_p \|F\|^p_{H^p_{+,\gamma}(w)} \leq \sum |\lambda_k|^p \leq c_p \|F\|^p_{H^p_{+,\gamma}(w)}$$

holds.

Remark 1.6. By Lemma (1.4) and Theorem (1.5) we have that the set D of all functions f such that there exists an interval $I \subset (x_{-\infty}, \infty)$ with $w(I) < \infty$ and $f \in L^{\infty}_{\gamma}(I)$, is dense in $H^p_{+,\gamma}(w)$.

Theorem 1.7. *Under the hypotheses of Theorem (1.5) and if, in addition, we assume that $x_{-\infty} = -\infty$, then the p-atoms $\{a_k\}$ in the decomposition can be taken in such that way that the corresponding associated intervals are bounded and therefore all the p-atoms in the decomposition have null moments up to the order $\gamma - 1$.*

Remark 1.8. If $x_{-\infty} = -\infty$, by Lemma (1.4) and Theorem (1.7) we have that the set D_1 of all functions f such that there exists a bounded interval $I \subset (x_{-\infty}, \infty)$ with $w(I) < \infty$ and $f \in L^{\infty}_{\gamma}(I)$, is dense in $H^p_{+,\gamma}(w)$.

We shall denote $[H^p_{+,\gamma}(w)]^*$ the dual space of $H^p_{+,\gamma}(w)$ formed by all the real valued continuous linear functionals L with the norm

$$\|L\| = \sup\{|L(F)| : \|F\|_{H^p_{+,\gamma}(w)} \leq 1\}.$$

Let γ be a positive integer and let $\mathcal{P}_\gamma$ be the linear space of all real polynomials of degree less than γ. For any bounded interval I, we define the inner product on $\mathcal{P}_\gamma$ by the formula

$$(P,Q)_I = \int_I P(x)\, Q(x)\, dx \ .$$

Let $\{e_k\}_{k=0}^{\gamma-1}$ be an orthonormal basis of $\mathcal{P}_\gamma$ for the case when $I=[0,1]$. It is easy to verify that for any $I=[a,b]$, the polynomials

$$e_{k,I}(x) = |I|^{-1/2}\, e_k((x-a)/|I|) \ , \qquad 0 \le k \le \gamma - 1 \tag{1.9}$$

form an orthonormal basis of $\mathcal{P}_\gamma$ with the inner product $(\,\cdot\,,\,\cdot\,)_I$. Given a function f such that $f\chi_I \in L^1$, we define its orthogonal projection on $\mathcal{P}_\gamma$, as

$$P_I(f)(x) = \sum_{k=0}^{\gamma-1} \left(\int_a^b f(y)\, e_{k,I}(y)\, dy \right) e_{k,I}(x) \ . \tag{1.10}$$

We observe that, by (1.9),

$$\sup_{x\in I} |e_{k,I}(x)| = |I|^{-1/2} \sup_{x\in[0,1]} |e_k(x)| \le c_\gamma |I|^{-1/2} \ , \tag{1.11}$$

holds for every integer k, $0 \le k \le \gamma-1$. Then, if $f\chi_I \in L^\infty$, by (1.10) and (1.11), we have that

$$|P_I(f)(x)| \le c_\gamma \|f\chi_I\|_\infty \ , \tag{1.12}$$

holds for every $x\in I$, with a constant c_γ depending on γ only.

We shall need a result that allows us to compare $P_I(f)$ and $P_J(f)$. To be more precise we state the following lemma.

Lemma 1.13. *Let $I \subset J$ be two bounded intervals such that $|J| \le 5|I|$. Then, if $f\chi_J \in L^1$, we have that*

$$|P_I(f)(x) - P_J(f)(x)| \le c_\gamma \frac{1}{|I|} \int_I |f - P_J(f)| dx \ ,$$

holds for every x belonging to J.

Proof. Let $\{e_k\}_{k=0}^{\gamma-1}$ be the orthonormal basis of the subspace $\mathcal{P}_\gamma$ defined above and let $\{e_{k,I}\}_{k=0}^{\gamma-1}$ be the orthonormal basis given in (1.9). Thus

$$\begin{aligned} P_I(f)(x) - P_J(f)(x) &= P_I[f - P_J(f)](x) \\ &= \sum_{k=0}^{\gamma-1} \left(\int_I [f - P_J(f)](s) e_{k,I}(s) ds \right) e_{k,I}(x) \ . \end{aligned}$$

Consequently, if x belongs to J we get

$$|P_I(f)(x) - P_J(f)(x)| \leq$$
$$\leq \sum_{k=0}^{\gamma-1} \int_I |[f - P_J(f)](s)| \, ds \|e_{k,I}\chi_I\|_\infty \, \|e_{k,I}\chi_J\|_\infty .$$

By (1.11), we have $\|e_{k,I}\chi_I\|_\infty \leq c'_\gamma |I|^{-1/2}$. Moreover, since $I \subset J$ and $|J| \leq 5|I|$, it follows that if $x \in J$ then $|x-a|/|I| \leq 5$, which implies that $\|e_{k,I}\chi_J\|_\infty \leq |I|^{-1/2} \sup_{|y|\leq 5} |e_k(y)| \leq c''_\gamma |I|^{-1/2}$. Therefore, for every $x \in J$ we obtain

$$|P_I(f)(x) - P_J(f)(x)| \leq \gamma \, c'_\gamma c''_\gamma |I|^{-1} \int_I |[f - P_J(f)](s)| ds ,$$

as we wanted to show. ■

We shall say that a function ℓ, defined on $(x_{-\infty}, x_\infty)$, belongs to $BMO_+(p, \gamma, w)$ if for every interval $I \subset (x_{-\infty}, \infty)$ with $w(I) < \infty$, we have

(i) $\ell\chi_I$ belongs to L^1,

(ii) if $|I| \geq \operatorname{dist}(x_{-\infty}, I)$ then $\int_I |\ell(x)| dx \leq c \, w(I)^{1/p}$ and,

(iii) if $|I| < \operatorname{dist}(x_{-\infty}, I)$ then the orthogonal projection $P_I(\ell)$ is well defined and

$$\int_I |\ell(x) - P_I(\ell)(x)| dx \leq c \, w(I)^{1/p} .$$

holds.

The constant c does not depend on the intervals I and the least constant c for which (ii) and (iii) hold, shall be denoted by $\|\ell\|_{BMO_+(p,\gamma,w)}$.

Remark 1.14. Let ℓ belong to $BMO_+(p, \gamma, w)$ and let A belong to $L^\infty_\gamma(I)$, where $I \subset (x_{-\infty}, \infty)$ is an interval with $w(I) < \infty$. If $|I| \geq \operatorname{dist}(x_{-\infty}, I)$, by the definition of $BMO_+(p, \gamma, w)$, we have that

$$\left| \int A(x)\ell(x)dx \right| \leq \|A\|_\infty \int_I |\ell(x)| dx \leq \|A\|_\infty \|\ell\|_{BMO_+(p,\gamma,w)} w(I)^{1/p} .$$

In the case that $|I| < \operatorname{dist}(x_{-\infty}, I)$, since, by definition of $L^\infty_\gamma(I)$, the function A has null moments up to the order $\gamma - 1$, we get

$$\left| \int A(x)\ell(x)dx \right| = \left| \int A(x)[\ell(x) - P_I(\ell)(x)]dx \right|$$
$$\leq \|A\|_\infty \int_I |\ell(x) - P_I(\ell)(x)| dx$$
$$\leq \|A\|_\infty \|\ell\|_{BMO_+(p,\gamma,w)} w(I)^{1/p} .$$

Remarks.

(a) If there exists $\beta > x_{-\infty}$ such that $w((x_{-\infty},\beta)) < \infty$, then $(BMO_+(p,\gamma,w),$ $\|\cdot\|_{BMO_+(p,\gamma,w)})$ is a normed space.

(b) If we have that $w((x_{-\infty},\beta)) = \infty$ holds for every $\beta > x_{-\infty}$ then $\|\cdot\|_{BMO_+(p,\gamma,w)}$ is a seminorm. Indeed, $\|\ell\|_{BMO_+(p,\gamma,w)}$ is equal to zero if and only if ℓ belongs to $\mathcal{P}_\gamma$, the set of all polynomials of degree less than γ. Therefore defining, as usual, for $\widetilde{\ell}$ belonging to $BMO_+(p,\gamma,w)/\mathcal{P}_\gamma$ the application

$$\|\widetilde{\ell}\|_{BMO_+(p,\gamma,w)/\mathcal{P}_\gamma} = \|\ell'\|_{BMO_+(p,\gamma,w)} ,$$

where $\ell - \ell' \in \mathcal{P}_\gamma$, we obtain the normed space $(BMO_+(p,\gamma,w)/\mathcal{P}_\gamma,$ $\|\cdot\|_{BMO_+(p,\gamma,w)/\mathcal{P}_\gamma})$.

We shall say that a function ℓ defined on $(x_{-\infty},x_\infty)$, belongs to $BMOF_+(p,\gamma,w)$ if for every bounded interval $I \subset (x_{-\infty},\infty)$ with $w(I) < \infty$, we have

(i) $\ell\chi_I$ belongs to L^1 and,

(ii) $\int_I |\ell(x) - P_I(\ell)(x)|dx \le c\, w(I)^{1/p}$ holds with a constant c not depending on the intervals I.

The least constant c for which (ii) holds shall be denoted by $\|\ell\|_{BMOF_+(p,\gamma,w)}$.

Remarks.

(a) The application $\|\cdot\|_{BMOF_+(p,\gamma,w)}$ is a seminorm and, as usual, it induces a norm $\|\cdot\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma}$ in the quotient space $BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma$.

(b) If we have that $w((x_{-\infty},\beta)) = \infty$ holds for every $\beta > x_{-\infty}$, then the space $BMOF_+(p,\gamma,w)$ coincides with $BMO_+(p,\gamma,w)$.

2. STATEMENT OF THE RESULTS

In this paragraph we state the results that characterize the dual space of $H^p_{+,\gamma}(w)$, which is the purpose of the paper.

Theorem 2.1. *Let $w \in A_q^+$, $r > q$, γ a positive integer and $0 < p \le 1$ such that $(\gamma+1)p \ge q$ if $q > 1$ or $(\gamma+1)p > 1$ if $q = 1$. If L belongs to $[H^p_{+,\gamma}(w)]^*$ we have that*

(i) if there exists $\beta > x_{-\infty}$ such that $w((x_{-\infty},\beta)) < \infty$, then there exists a unique ℓ belonging to $BMO_+(p,\gamma,w)$ such that

$$L(f) = \int \ell(x)\, f(x)\, dx$$

holds for every $f \in L^r_\gamma(I,w)$ where $I \subset (x_{-\infty},\infty)$ is any interval with $w(I) < \infty$. Moreover,

$$\|\ell\|_{BMO_+(p,\gamma,w)} \leq c_{\gamma,r,p,w}\|L\| \ .$$

(ii) if we have that $w((x_{-\infty},\beta)) = \infty$ holds for every $\beta > x_{-\infty}$, then there exists a unique class $\widetilde{\ell}$ belonging to $BMO_+(p,\gamma,w)/\mathcal{P}_\gamma$ such that for any ℓ' belonging to $\widetilde{\ell}$, we have that

$$L(f) = \int \ell'(x)\ f(x)\ dx$$

holds for every $f \in L^r_\gamma(I,w)$, where $I \subset (x_{-\infty},\infty)$ is any interval with $w(I) < \infty$. Moreover

$$\|\widetilde{\ell}\|_{BMO_+(p,\gamma,w)/\mathcal{P}_\gamma} \leq c_{\gamma,r,p,w}\|L\| \ .$$

Theorem 2.2. *Let $w \in A^+_q$, γ a positive integer and $0 < p \leq 1$ such that $(\gamma+1)p \geq q$ if $q > 1$ or $(\gamma+1)p > 1$ if $q = 1$. Then, we have*

(i) *if there exists $\beta > x_{-\infty}$ such that $w((x_{-\infty},\beta)) < \infty$, given ℓ belonging to $BMO_+(p,\gamma,w)$, the functional*

$$L(f) = \int \ell(x)\ f(x)\ dx$$

is well defined on the dense set D (see Remark (1.6)) and,

$$\|L\| \leq c_{p,\gamma,w}\ \|\ell\|_{BMO_+(p,\gamma,w)} \ .$$

(ii) *if we have that $w((x_{-\infty},\beta)) = \infty$ holds for every $\beta > x_{-\infty}$, given $\widetilde{\ell}$ belonging to $BMO_+(p,\gamma,w)/\mathcal{P}_\gamma$ and ℓ' in the class $\widetilde{\ell}$, the functional*

$$L(f) = \int \ell'(x)\ f(x)\ dx$$

is well defined on the dense set D, L is independent of $\ell' \in \widetilde{\ell}$ and

$$\|L\| \leq c_{p,\gamma,w}\|\widetilde{\ell}\|_{BMO_+(p,\gamma,w)/\mathcal{P}_\gamma} \ .$$

Theorem 2.3. *Let $w \in A^+_q$, γ a positive integer and $0 < p \leq 1$ such that $(\gamma+1)p \geq q$ if $q > 1$ or $(\gamma+1)p > 1$ if $q = 1$. Then, we have* *(i) if there exists* $\beta > x_{-\infty}$ *satisfying* $w((x_{-\infty},\beta)) < \infty$, *then there exists a bijective linear application* i *from* $[H^p_{+,\gamma}(w)]^*$ *into* $BMO_+(p,\gamma,w)$ *such that if* $i(L) = \ell$, *then*

$$L(f) = \int \ell(x)\, f(x)\, dx$$

holds for every $f \in D$. *Moreover,*

$$c_1\|L\| \le \|\ell\|_{BMO_+(p,\gamma,w)} \le c_2\|L\| \, .$$

(ii) if we have $w((x_{-\infty},\beta)) = \infty$ *holds for every* $\beta > x_{-\infty}$, *then there exists a bijective linear application* i *from* $[H^p_{+,\gamma}(w)]^*$ *into* $BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma$ *such that if* $i(L) = \widetilde{\ell}$ *and* ℓ' *belongs to* $\widetilde{\ell}$, *then*

$$L(f) = \int \ell'(x)\, f(x)\, dx$$

holds for every $f \in D$. *Moreover,*

$$c_1\|L\| \le \|\widetilde{\ell}\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma} \le c_2\|L\| \, .$$

Theorem 2.4. *Let* $w \in A^+_q$, γ *a positive integer and* $0 < p \le 1$ *such that* $(\gamma+1)p \ge q$ *if* $q > 1$ *or* $(\gamma+1)p > 1$ *if* $q = 1$. *If* $x_{-\infty} = -\infty$ *then the conclusions of part (ii) of Theorem (2.3) hold for every* f *belonging to the dense set* D_1 *(see Remark (1.8)) even if there exists* β *such that* $w((-\infty,\beta)) < \infty$.

3. PROOFS OF THE RESULTS

Lemma 3.1. *Let* $w \in A^+_q$, $\gamma \ge 1$ *an integer and,* $0 < p \le 1$ *such that* $(\gamma+1)p \ge q > 1$ *or* $(\gamma+1)p > q = 1$ *and* $r \ge q > 1$ *or* $r > q = 1$. *Let* $I \subset (x_{-\infty},\infty)$ *be an interval with* $w(I) < \infty$ *and let* f *belong to* $L^r_\gamma(I,w)$. *Then* $f \in H^p_{+,\gamma}(w)$ *and*

$$\|f\|_{H^p_{+,\gamma}(w)} \le c_{\gamma,r,p,w}\|f\|_{L^r(I,w)}w(I)^{\frac{1}{p}-\frac{1}{r}} \, .$$

Proof. Let $\alpha < \beta$ be the end points of I.

If $\max(x_{-\infty},\alpha-|I|) \le x$, by definition (1.3), we have $f^*_{+,\gamma}(x) \le M^+f(x)$. Then, by Hölder's inequality and applying Lemma (1.2), we obtain

$$\begin{aligned}\int_{\max(x_{-\infty},\alpha-|I|)}^{\infty} f^*_{+,\gamma}(x)^p w(x)dx &\le \left(\int_{-\infty}^{\beta} M^+f(x)^r w(x)dx\right)^{\frac{p}{r}} w(\widetilde{I}\cup I)^{1-\frac{p}{r}} \\ &\le c_{r,p,w}\|f\|^p_{L^r(I,w)}w(I)^{1-\frac{p}{r}} \, . \end{aligned} \tag{3.2}$$

If there exists x such that $x_{-\infty} < x < \alpha - |I|$, then f has null moments up to the order $\gamma - 1$ and the interval I is bounded. Let ψ belong to the class $\Phi_\gamma(x)$ and I_ψ the interval associated with ψ in this class. We have

$$< f, \psi > = \int_I f(t) \left[\psi(t) - \sum_{s=0}^{\gamma-1} \frac{D^s \psi(\alpha)}{s!} (t-\alpha)^s \right] dt .$$

We may assume that $I \cap I_\psi \neq \emptyset$, then $\alpha - x \leq |I_\psi|$ and we get

$$\begin{aligned} | < f, \psi > | &\leq \frac{\|D^\gamma \psi\|_\infty}{\gamma!} |I|^\gamma \int_I |f(t)| dt \\ &\leq c_\gamma \left(\frac{|I|}{\alpha - x} \right)^{\gamma+1} \frac{1}{|I|} \int_I |f(t)| dt . \end{aligned}$$

Since for every x such that $x_{-\infty} < x < \alpha - |I|$, the one-sided maximal function $M^+ \chi_{\widetilde{I}}$ satisfies: $\frac{|I|}{\alpha - x} \leq M^+ \chi_{\widetilde{I}}(x)$, it follows that

$$f^*_{+,\gamma}(x) \leq c_\gamma [M^+ \chi_{\widetilde{I}}(x)]^{\gamma+1} \frac{1}{|I|} \int_I |f(t)| dt .$$

Now, by Hölder's inequality and taking into account that $w \in A_r^+$, we have

$$\begin{aligned} \frac{1}{|I|} \int_I |f(t)| dt &\leq \|f\|_{L^r(I,w)} \frac{1}{|I|} \left(\int_I w(t)^{-r'/r} dt \right)^{1/r'} \\ &\leq c_{r,w} \|f\|_{L^r(I,w)} w(\widetilde{I})^{-1/r} , \end{aligned}$$

which implies that

$$f^*_{+,\gamma}(x) \leq c_{\gamma,r,w} \|f\|_{L^r(I,w)} w(\widetilde{I})^{-1/r} [M^+ \chi_{\widetilde{I}}(x)]^{\gamma+1} .$$

Then, by Lemma (1.2), we get

$$\int_{x_{-\infty}}^{\alpha - |I|} f^*_{+,\gamma}(x)^p w(x) dx \leq c_{\gamma,r,p,w} \|f\|^p_{L^r(I,w)} w(I)^{1-\frac{p}{r}} . \tag{3.3}$$

By (3.2) and (3.3), this lemma is proved. ∎

Remark. The estimation for the p-norm $\|f\|_{H^p_{+,\gamma}(w)}$ in Lemma (1.4) also follows from Lemma (3.1).

Lemma 3.4. *Let $w \geq 0$ and $r > 1$. Let I be an interval with $w(I) < \infty$. Then, if $g\chi_I \in L^{r'}(I,w)$ we have that $g\chi_I \in L^1(I,w)$. In particular, the orthogonal projection $P_I(gw)$ is well defined.*

The proof is an immediate consequence of Hölder's inequality.

Lemma 3.5. *Let $w \in A_q^+$ and $r \geq q > 1$ or $r > q = 1$. We assume that $I \subset (x_{-\infty}, \infty)$ is an interval satisfying the condition $|I| < dist(x_{-\infty}, I)$. Then, if $f \in L^r(I,w)$ we have that $f \in L^1(I)$. In particular, the orthogonal projection $P_I(f)$ is well defined.*

Proof. Let us observe the condition $|I| < \text{dist}(x_{-\infty}, I)$ implies that I is a bounded interval and if we define $\widetilde{I}$ as in Lemma (1.2) it follows that $w(\widetilde{I}) > 0$. By Hölder's inequalitiy and the A_r^+ condition, $r > 1$, we get

$$\int_I |f(x)|dx \leq \left(\int_I |f(x)|^r w(x) dx\right)^{1/r} \left(\int_I w(x)^{-r'/r} dx\right)^{1/r'}$$
$$\leq c_{r,w} \, |I| \, \|f\|_{L^r(I,w)} w(\widetilde{I})^{-1/r} < \infty ,$$

as we wanted to show. ∎

Proof of Theorem (2.1).

Part (i). We consider a sequence $\{\beta_k\}_{k\geq 1} \uparrow x_\infty$, such that for every $k \geq 1$, the interval $I_k = (x_{-\infty}, \beta_k)$ satisfies $w(I_k) < \infty$. In the case of $w((x_{-\infty}, x_\infty)) < \infty$, we take $\beta_k = x_\infty$, $k \geq 1$. Given $f \in L^r(I_k, w)$, by Lemma (3.1), we have

$$|L(f)| \leq \|L\| \, \|f\|_{H^p_{+,\gamma}(w)}$$
$$\leq c_{\gamma,r,p,w} \, \|L\| \, \|f\|_{L^r(I_k,w)} w(I_k)^{\frac{1}{p} - \frac{1}{r}} .$$

Therefore, L induces a continuous linear functional on $L^r(I_k, w)$. Then, by Riesz's Representation Theorem, there exists a unique $g_k \in L^{r'}(I_k, w)$ such that

$$L(f) = \int f(x) \; g_k(x) \; w(x) \; dx$$

holds for every $f \in L^r(I_k, w)$. The uniqueness of g_k, implies that the restriction $g_{k+1}|_{I_k}$ is equal to g_k almost everywhere in I_k; then, there exists a unique function g defined on $(x_{-\infty}, x_\infty)$ such that for every interval $I \subset (x_{-\infty}, \infty)$ with $w(I) < \infty$, we have

(3.6) $\int_I |g(x)|^{r'} w(x) dx < \infty$ and

(3.7) $L(f) = \int f(x) \; g(x) \; w(x) dx$, for every $f \in L^r(I, w)$.

Let us prove that $\ell = gw$ belongs to $BMO_+(p, \gamma, w)$. Let $I \subset (x_{-\infty}, \infty)$ be an interval with $w(I) < \infty$ and $\text{dist}(x_{-\infty}, I) \leq |I|$. The function $f = sg(\ell)\chi_I$ belongs to $L^r(I, w)$. Besides, by (3.7), Lemma (3.1) and taking into account that $\|f\|_{L^r(I,w)} \leq \|f\|_{L^\infty} w(I)^{1/r}$, we have

$$\int_I |\ell| dx = \int \ell \; f \; dx = L(f) \leq c_{\gamma,r,p,w} \|L\| w(I)^{1/p} . \tag{3.8}$$

Now, we assume that $|I| < \text{dist}(x_{-\infty}, I)$. By (3.6) and Lemma (3.4), the orthogonal projection $P_I(\ell)$ is well defined. The function $f = sg[\ell - P_I(\ell)]\chi_I$ belongs to $L^r(I,w)$ and by Lemma (3.5) we get

$$\begin{aligned}\int_I |\ell - P_I(\ell)| dx &= \int_I [\ell - P_I(\ell)] f \, dx \\ &= \int_I [\ell - P_I(\ell)][f - P_I(f)] \, dx \\ &= \int_I \ell [f - P_I(f)] \, dx \ .\end{aligned}$$

Applying (3.7), Lemma (3.1) and (1.12), we obtain

$$\begin{aligned}\int_I |\ell - P_I(\ell)| \, dx &= L[(f - P_I(f))\chi_I] \\ &\le \|L\| \, c_{\gamma,r,p,w} \, \|(f - P_I(f))\chi_I\|_{L^\infty} w(I)^{1/p} \\ &\le c'_{\gamma,r,p,w} \, \|L\| \, w(I)^{1/p} \ .\end{aligned} \tag{3.9}$$

From (3.8) and (3.9) it follows that $\ell \in BMO_+(p;\gamma,w)$.

Part (ii). Now, for every $\beta > x_{-\infty}$, $w((x_{-\infty},\beta))$ is infinite. This condition implies that $x_{-\infty} = -\infty$. Let $\{\alpha_k\}_{k\ge1} \downarrow -\infty$ and $\{\beta_k\}_{k\ge1} \uparrow x_\infty$ be two sequences such that for every $k \ge 1$, the interval $I_k = (\alpha_k, \beta_k)$ satisfies $w(I_k) < \infty$. If there exists α satisfying $w((\alpha, x_\infty)) < \infty$, we take $\beta_k = x_\infty$, $k \ge 1$. By Lemma (3.1), L induces a continuous linear functional on $L^r_\gamma(I_k,w)$, which, by Hahn-Banach, can be extended to $L^r(I_k,w)$. By Riesz Representation Theorem, the extension is represented by a function g_k belonging to $L^{r'}(I_k,w)$. Suppose there exist functions g_k and g'_k in $L^{r'}(I_k,w)$ such that

$$\int f(x)\, g_k(x)\, w(x)\, dx = \int f(x)\, g'_k(x)\, w(x)\, dx \ ,$$

holds for every $f \in L^r_\gamma(I_k,w)$. We want to show that $g = g_k - g'_k$ is equal to Pw^{-1} almost everywhere in I_k, where P is a polynomial of degree less than γ. In fact, given $f \in L^r(I_k,w)$, the function $[f - P_{I_k}(f)]\chi_{I_k}$ belongs to $L^r_\gamma(I_k,w)$; then, using Lemma (3.4), we have

$$\begin{aligned}0 &= \int_{I_k} [f - P_{I_k}(f)] g w \, dx \\ &= \int_{I_k} [f - P_{I_k}(f)] \left[g - \frac{P_{I_k}(gw)}{w} \right] w \, dx \\ &= \int_{I_k} f \left[g - \frac{P_{I_k}(gw)}{w} \right] w \, dx \ .\end{aligned}$$

Thus, since $I_k \subset (x_{-\infty}, x_\infty)$ it follows that $g = \frac{P_{I_k}(gw)}{w}$ a.e. in I_k.

Taking into account that $I_k = (\alpha_k, \beta_k) \uparrow (x_{-\infty}, x_\infty)$, we can define a function g on $(x_{-\infty}, x_\infty)$ such that for every $I \subset (x_{-\infty}, x_\infty)$ with $w(I) < \infty$, the properties (3.6) and

$$L(f) = \int f(x)\, g(x)\, w(x)\, dx, \qquad \text{for every } f \in L^r_\gamma(I, w)$$

also holds.

In this part (ii), if we have an interval I with $w(I) < \infty$, then $|I| < \operatorname{dist}(x_{-\infty}, I) = \infty$ and arguing as in (3.9), it follows that $\ell = gw \in BMO_+(p, \gamma, w)$. ∎

Let f be a locally integrable function on $(x_{-\infty}, \infty)$ belonging to $H^p_{+,\gamma}(w)$. For every integer n, we define the open set

$$\Omega_n = \{x : x > x_{-\infty},\ f^*_+(x) > 2^n\}$$

and we denote its component intervals by $I_{n,i}$, $i \geq 1$, where $I_{n,1}$ is, if there exists, the connected component that starts at $x_{-\infty}$, and $I_{n,1} = \emptyset$ otherwise. In addition, for every $i > 1$ and $j \geq 1$, we define functions $\eta_{n,i,j}(x) \geq 0$ belonging to $C_0^\infty(I\!R)$ such that

$$\left(\sum_{j\geq 1} \eta_{n,i,j}(x)\right) \chi_{I_{n,i}}(x) = \chi_{I_{n,i}}(x), \qquad i > 1\ ; \tag{3.10}$$

and polynomials $P_{n,i,j}(f)$ of degree less than γ, explicity given by the formula

$$P_{n,i,j}(f)(x) = \sum_{k=0}^{\gamma-1} \left(\int f(s) e_k^{n,i,j}(s) \eta_{n,i,j}(s) \chi_{I_{n,i}}(s) ds\right) e_k^{n,i,j}(x)\ ,$$

where $\{e_k^{n,i,j}\}_{k=0}^{\gamma-1}$ is an orthonormal basis of the subspace of $L^2(\eta_{n,i,j}\chi_{I_{n,i}})$ generated by $1, x, \ldots, x^{\gamma-1}$. From their definition, it follows that the polynomials $P_{n,i,j}(f)$ satisfy

$$\int f(x)\, x^s\, \eta_{n,i,j}(x)\, \chi_{I_{n,i}}(x)\, dx =$$

$$= \int P_{n,i,j}(f)(x) x^s\, \eta_{n,i,j}(x)\, \chi_{I_{n,i}}(x)\, dx, \quad 0 \leq s < \gamma\,. \tag{3.11}$$

For an explicit definition of the functions $\eta_{n,i,j}$ see section 5 in **[3]**.

We recall that in the proof of Theorem (2.2) in [**3**], (see (5.1)), for every $i > 1$ and $j \geq 1$, we obtained the estimate

(3.12) $$\sup_{x \in \text{support}(\eta_{n,i,j})} |P_{n,i,j}(f)(x)| \leq c\, 2^n ,$$

where the constant c is independent of n and f.

Taking into account the notations introduced above, for each integer n, we consider the function $g_n(x)$ defined as

(3.13) $$g_n(x) = f(x)\chi_{c\Omega_n}(x) + \sum_{i>1}\sum_{j\geq 1} P_{n,i,j}(f)(x)\eta_{n,i,j}(x)\chi_{I_{n,i}}(x) ,$$

which satisfies

(3.14) $$|g_n(x)| \leq c\, 2^n \qquad \text{a.e. in } (x_{-\infty}, \infty) ,$$

where the constant c is independent of n and f.

Proof of Theorem (2.2). Let ℓ belong to $BMO_+(p,\gamma,w)$. For every bounded function f supported in an interval $I = (\alpha, \beta) \subset (x_{-\infty}, \infty)$ with $w(I) < \infty$, we have that

$$\int |\ell(x)|\ |f(x)|\ dx \leq \|f\|_\infty \int_I |\ell(x)\ dx < \infty .$$

Then, the linear functional $L(f) = \int \ell(x) f(x) dx$ is well defined on the dense set D (see Remark (1.6)). We want to show that L is a bounded functional and therefore that it can be extended to $H^p_{+,\gamma}(w)$. Since $f \in L^\infty$, if M is large enough, then the set Ω_M is empty and by (3.13), we have $g_M = f$. Thus,

(3.15) $$f(x) = \sum_{n=N}^{M-1} [g_{n+1}(x) - g_n(x)] + g_N(x) .$$

From the definition of g_n, if follows that its support is contained in the union $I \cup \Omega_n \subset (x_{-\infty}, \beta)$.

If $\ell \in BMO_+(p,\gamma,w)$ and $w((x_{-\infty}, \beta)) < \infty$, then ℓ is integrable on $(x_{-\infty}, \beta)$ and taking into account (3.14), we get

$$\int f\ell\ dx = \sum_{n=N}^{M-1} \int_{x_{-\infty}}^{\beta} (g_{n+1} - g_n)\ell\ dx + \int_{x_{-\infty}}^{\beta} g_N \ell\ dx .$$

For the last integral on the right hand side, we have

$$\left|\int g_N \ell\ dx\right| \leq c\, 2^N \int_{x_{-\infty}}^{\beta} |\ell|\ dx \leq c\, 2^N \|\ell\|_{BMO_+(p,\gamma,w)} w((x_{-\infty}, \beta))^{1/p} ,$$

which goes zero for N tending to $-\infty$.

Now, let us suppose $w((x_{-\infty},\beta)) = \infty$. Then, the hypothesis $w(I) = w((\alpha,\beta)) < \infty$ implies that $x_{-\infty} = -\infty < \alpha < \beta < +\infty$. By Lemma (1.4) if f belongs to $L^{\infty}_{\gamma}(I)$, then we have

$$f^*_+(x) \le c_\gamma \ \|f\|_\infty [M^+ \chi_I(x)]^{\gamma+1} .$$

On the other hand, it is easy to see that for $x < \beta$, the following inequalities

$$(3.16) \qquad \frac{1}{2}\ \frac{|I|}{\alpha - x + 2|I|} \le M^+ \chi_I(x) \le 4\ \frac{|I|}{\alpha - x + 2|I|} .$$

hold. Thus,

$$\begin{aligned}\Omega_n = \{x : 2^n < f^*_+(x)\} &\subset \{x : 2^n < c_\gamma \ \|f\|_\infty [M^+ \chi_I(x)]^{\gamma+1}\} \\ &\subset \left\{ x : x < \beta, 2^n < c_\gamma \ \|f\|_\infty \ 4^{\gamma+1} \left[\frac{|I|}{\alpha - x + 2|I|}\right]^{\gamma+1} \right\} = J_n .\end{aligned}$$

It can be verified without difficult that J_n is either the empty set or an interval with finite end points, where the upper end point is equal to β. Since $I = (\alpha,\beta)$, then $I \cup J_n = K_n$ is a bounded interval. Besides, if n is negative enough then $J_n \supset I$ and therefore $K_n = J_n$. In conclusion, g_n is supported in a bounded interval $K_n = (\delta,\beta)$ with $w(K_n) < \infty$. We shall estimate the w-measure of K_n for very negative values of n, i.e., when $K_n = J_n$. In virtue of the first inequality of (3.16) we have

$$J_n \subset \{x : 2^n < c_\gamma \ \|f\|_\infty \ 8^{\gamma+1}[M^+ \chi_I(x)]^{\gamma+1}\} .$$

By Chebyshev's inequality, if $s > 0$ then

$$w(J_n) \le c_\gamma \ \|f\|^s_\infty \ 2^{-ns} \int [M^+ \chi_I(x)]^{(\gamma+1)s} w(x) \ dx .$$

Since the weight w satisfies the hypotheses we can assume that $w \in A^+_r$, with $(\gamma+1)p > r > 1$. Let s be a real number such that $0 < s < p$ and $(\gamma+1)s = r > 1$. Then

$$\int [M^+ \chi_I(x)]^{(\gamma+1)s} w(x) \ dx \le c_{\gamma,s,w} \ w(I) ,$$

and thus, we obtain

$$(3.17) \qquad w(J_n) \le c_{\gamma,s,w} \ \|f\|^s_\infty \ 2^{-ns} \ w(I) .$$

It is easy to verify directly that $\int g_N(x)\, x^s\, dx = 0$ for $0 \le s < \gamma$. In fact, adding in (3.11) for $j \ge 1$ and $i > 1$, we have

$$\sum_{i>1}\sum_{j\ge1}\int f(x)\, x^s\, \eta_{n,i,j}(x)\, \chi_{I_{n,i}}(x)\, dx =$$

$$= \sum_{i>1}\sum_{j\ge1}\int P_{n,i,j}(f)(x)\, x^s\, \eta_{n,i,j}(x)\, \chi_{I_{n,i}}(x)\, dx\,, \qquad 0 \le s < \gamma\,. \tag{3.18}$$

In virtue of (3.12), since $|P_{n,i,j}(f)(x)\eta_{n,i,j}(x)| \le c\, 2^n$, $\bigcup_{i\ge1} I_{n,i} = \Omega_n$ (in this case: $I_{n,1} = \emptyset$) and $\Omega_n \subset K_n = (\delta, \beta)$, where δ is finite, then by Lebesgue's Dominated Convergence Theorem the right hand side of (3.18) is equal to

$$\int x^s \left[\sum_{i>1}\sum_{j\ge1} P_{n,i,j}(f)(x)\eta_{n,i,j}(x)\chi_{I_{n,i}}(x)\right] dx\,.$$

On the other hand, taking into account that f belongs to L^∞ and that its support is a bounded set, by (3.10) and the Lebesgue's Dominated Convergence Theorem, the left hand side of (3.18) is equal to

$$\begin{aligned}\sum_{i>1}\sum_{j\ge1}\int f(x)x^s\eta_{n,i,j}(x)\chi_{I_{n,i}}(x)\, dx &= \sum_{i>1}\int f(x)x^s\chi_{I_{n,i}}(x)\, dx \\ &= \int_{\Omega_n} f(x)x^s\, dx\,.\end{aligned}$$

Thus,

$$\int g_n(x)\, x^s\, dx = \int_{c\Omega_n} f(x)\, x^s\, dx + \int_{\Omega_n} f(x)\, x^s\, dx = \int f(x)\, x^s\, dx = 0$$

holds for $0 \le s < \gamma$.

Going back to (3.15), we have that if ℓ belongs to $BMO_+(p,\gamma,w)$ then

$$\int f(x)\ell(x)\, dx = \int_{K_N}\left[\sum_{n=N}^{M-1}[g_{n+1}(x) - g_n(x)] + g_N(x)\right]\ell(x)\, dx\,,$$

and, since ℓ is integrable on K_N, we get

$$\int f(x)\ell(x)\, dx = \sum_{n=N}^{M-1}\int_{K_N}[g_{n+1}(x) - g_n(x)]\ell(x)\, dx + \int_{K_N} g_N(x)\ell(x)\, dx\,.$$

If N is negative enough, then $K_N = J_N$ and, from the fact that g_N has null moments up to the order $\gamma - 1$, for the last integral on the right hand side we have

$$\begin{aligned}\left|\int g_N\ell\right| &= \left|\int_{J_N} g_N\ell\right| = \left|\int_{J_N} g_N[\ell - P_{J_N}(\ell)]dx\right| \\ &\le c\, 2^N\int_{J_N}|\ell(x) - P_{J_N}(\ell)(x)|dx \le c\, 2^N\|\ell\|_{BMO_+(p,\gamma,w)}w(J_N)^{1/p}\end{aligned}$$

which, in virtue of (3.17) is bounded by

$$c\, 2^{N(1-s/p)} \|f\|_\infty^{s/p}\, w(I)^{1/p}\ .$$

Since $s < p$ it follows that $1 - s/p > 0$ and then the last expression goes to zero when N tends to $-\infty$. This proves that $\lim_{N\to\infty} \int g_N \ell\, dx = 0$, also in this case. Therefore, we always have

$$\int f(x)\ell(x)\, dx = \sum_{-\infty}^{M-1} \int [g_{n+1} - g_n]\ell\, dx\ .$$

In the proof of Theorem (2.2), in section 5 of [3], it was shown that

$$g_{n+1}(x) - g_n(x) = \sum_{i>1} \widetilde{A}_{n,i}(x) + \widetilde{A}_{n,1}(x)\ ,$$

where the support of the function $\widetilde{A}_{n,i}$ are contained in the connected components $I_{n,i}$ of Ω_n, $\|\widetilde{A}_{n,i}\|_\infty \le c\, 2^n$ and, moreover, if $i > 1$ then $\int \widetilde{A}_{n,i}(x)x^s\, dx = 0$ holds for $0 \le s < \gamma$. Since Ω_n is contained in an interval with finite w-measure ($(x_{-\infty}, \beta)$ or J_n) and, by definition, ℓ is integrable on these intervals, the Lebesgue's Dominated Convergence Theorem and Remark (1.14) imply that

$$\begin{aligned}\left|\int (g_{n+1} - g_n)\ell\, dx\right| &= \left|\sum_{i\ge1} \int \widetilde{A}_{n,i}(x)\ell(x)\, dx\right| \\ &\le c\, 2^n \|\ell\|_{BMO_+(p,\gamma,w)} \sum_{i\ge1} w(I_{n,i})^{1/p} \\ &\le c\, 2^n \|\ell\|_{BMO_+(p,\gamma,w)} w(\Omega_n)^{1/p}\ .\end{aligned}$$

Then

$$\begin{aligned}\left|\int f\ell\, dx\right| &\le \sum_n \left|\int (g_{n+1} - g_n)\ell\, dx\right| \\ &\le c\, \|\ell\|_{BMO_+(p,\gamma,w)} \left[\sum_{-\infty}^{\infty} 2^{np} w(\Omega_n)\right]^{1/p} \\ &\le c\, \|\ell\|_{BMO_+(p,\gamma,w)}\, \|f\|_{H^p_{+,\gamma}(w)}\ ,\end{aligned}$$

as we wanted to show. ∎

Proof of Theorem (2.3). We define the application i as $i(L) = \ell$, where ℓ is the function associated to L in part (i) of Theorem (2.1), or $i(L) = \widetilde{\ell}$, where $\widetilde{\ell}$ is the class associated to L in part (ii) of the same theorem. Since D is a dense set in $[H^p_{+,\gamma}(w)]^*$, it follows that i is an inyective application. Taking into account Theorems (2.1) and (2.2) then Theorem (2.3) follows immediately. ∎

For every non-negative integer n and every real number $a > -2^{n+1}$, we define the interval

$$I_{n,a} = [-2^{n+1}, a] .$$

If $a = -2^n$ then, we denote by I_n the interval $I_{n,-2^n} = [-2^{n+1}, -2^n]$. Moreover, given a function ℓ belonging locally to $L^1(-\infty, a]$, we denote the orthogonal projections $P_{I_{n,a}}(\ell)$ and $P_{I_n}(\ell)$ by $P_{n,a}$ and P_n respectively.

Lemma 3.19. *Let $w \in A_q^+$, γ a positive integer and $0 < p \leq 1$ such that $(\gamma+1)p \geq q$ if $q > 1$ or $(\gamma+1)p > 1$ if $q = 1$. Assume $x_{-\infty} = -\infty$ and let a and n such that $|a| \leq 2^n$ and $w((-\infty, a)) < \infty$. If ℓ belongs to $BMOF_+(p,\gamma,w)$, then*

(i) for every $k > n$, we have

$$\sup_{x \in I_{n,a}} |P_{k+1}(x) - P_k(x)| \leq c_\gamma \, \|\ell\|_{BMOF_+(p,\gamma,w)} 2^{-k} \, w(I_{k+1,a})^{1/p}$$

and,

(ii) $\sup_{x \in I_{n,a}} |P_{n+1}(x) - P_{n,a}(x)| \leq c_\gamma \, \|\ell\|_{BMOF_+(p,\gamma,w)} 2^{-n} \, w(I_{n+1,a})^{1/p}$.

Proof. For $|a| \leq 2^n$ and $k > n$, we have

$$|I_{k+1,a}| \leq 5|I_k| \leq 5|I_{k+1}| .$$

Then, by Lemma (1.13) and since $I_k \cup I_{k+1} \subset I_{k+1,a}$, we get

$$\begin{aligned}
\sup_{x \in I_{n,a}} |P_{k+1}(x) - P_k(x)| &\leq \\
&\leq \sup_{x \in I_{k+1,a}} |P_{k+1}(x) - P_{k+1,a}(x)| + \sup_{x \in I_{k+1,a}} |P_{k+1,a}(x) - P_k(x)| \\
&\leq c_\gamma \, 2^{-k-1} \int_{I_{k+1}} |\ell - P_{k+1,a}| dx + c_\gamma \, 2^{-k} \int_{I_k} |\ell - P_{k+1,a}| dx \\
&\leq \frac{3}{2} c_\gamma \, 2^{-k} \int_{I_{k+1,a}} |\ell - P_{k+1,a}| dx .
\end{aligned}$$

Therefore

$$\sup_{x \in I_{n,a}} |P_{k+1}(x) - P_k(x)| \leq \frac{3}{2} c_\gamma \, 2^{-k} \, \|\ell\|_{BMOF_+(p,\gamma,w)} w(I_{k+1,a})^{1/p} ,$$

which is part (i) of the lemma. In order to prove part (ii), we observe that $|I_{n+1,a}| \leq (5/2)|I_{n+1}|$ and $|I_{n+1}| \leq 2|I_{n,a}|$, thus $|I_{n+1,a}| \leq 5|I_{n+1}|$ and $|I_{n+1,a}| \leq 5|I_{n,a}|$. Then, by Lemma (1.13) and since $I_{n+1} \cup I_{n,a} \subset I_{n+1,a}$,

$$\begin{aligned}
\sup_{x\in I_{n,a}} |P_{n+1}(x) - P_{n,a}(x)| \leq \\
&\leq c_\gamma |I_{n+1}|^{-1} \int_{I_{n+1}} |\ell - P_{n+1,a}| dx + c_\gamma |I_{n,a}|^{-1} \int_{I_{n,a}} |\ell - P_{n+1,a}| dx \\
&\leq \frac{3}{2} c_\gamma \, 2^{-n} \int_{I_{n+1,a}} |\ell - P_{n+1,a}| dx \\
&\leq \frac{3}{2} c_\gamma \, \|\ell\|_{BMOF_+(p,\gamma,w)} \, 2^{-n} \, w(I_{n+1,a})^{1/p} ,
\end{aligned}$$

as we wanted to show. ■

In order to prove Theorem (2.4) we need the following proposition.

Proposition 3.20. *Let $w \in A_q^+$, γ a positive integer and $0 < p \leq 1$ such that $(\gamma+1)p \geq q$ if $q > 1$ or $(\gamma+1)p > 1$ if $q = 1$. Assume $x_{-\infty} = -\infty$ and that there exists β satisfying $w((-\infty,\beta)) < \infty$. Then, given $\widetilde{\ell} \in BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma$ there exists a unique $\ell' \in \widetilde{\ell}$ belonging to $BMO_+(p,\gamma,w)$ such that*

$$\|\ell'\|_{BMO_+(p,\gamma,w)} \leq c_{p,\gamma} \, \|\ell\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma} .$$

Proof. Let $\ell \in BMOF_+(p,\gamma,w)$ and b such that $w((-\infty,b)) < \infty$. Choose m such that $|b| \leq 2^m$. By part (i) of Lemma (3.19), we have that for every $k > m$

$$\sup_{x\in I_{m,b}} |P_{k+1}(x) - P_k(x)| \leq c_\gamma \|\ell\|_{BMOF_+(p,\gamma,w)} \, 2^{-k} \, w(I_{k+1,b})^{1/p} .$$

Then given $i > j > m$

$$\begin{aligned}
\sup_{x\in I_{m,b}} |P_i(x) - P_j(x)| &\leq \sum_{k=j}^{i-1} \sup_{x\in I_{m,b}} |P_{k+1}(x) - P_k(x)| \\
&\leq c_\gamma \, \|\ell\|_{BMOF_+(p,\gamma,w)} \sum_{k=j}^{i-1} 2^{-k} \, w(I_{k+1,b})^{1/p} \\
&\leq c_\gamma \, \|\ell\|_{BMOF_+(p,\gamma,w)} \, w((-\infty,b))^{1/p} \, 2^{-j+1} .
\end{aligned}$$

This implies that $\{P_k\}_{k>m}$ is a Cauchy sequence in the Banach space of the continuous functions on $I_{m,b}$. Therefore there exists a polynomial $P \in \mathcal{P}_\gamma$ such that

$$\lim_{k\to\infty} \sup_{x\in I_{m,b}} |P(x) - P_k(x)| = 0 .$$

Thus,

$$\begin{aligned}\int_{-\infty}^{b}|\ell-P|dx &= \sum_{n=m+1}^{\infty}\int_{I_n}|\ell-P|dx+\int_{-2^{m+1}}^{b}|\ell-P|dx\\ &\le \sum_{n=m+1}^{\infty}\int_{I_n}|\ell-P_n|dx+\sum_{n=m+1}^{\infty}\int_{I_n}|P_n-P|dx\\ &\quad+\int_{I_{m,b}}|\ell-P_{m,b}|dx+\int_{I_{m,b}}|P_{m,b}-P|dx\\ &= I+II+III+IV\ .\end{aligned}$$

Let us estimate $I+III$. By definition of $BMOF_+(p,\gamma,w)$ and recalling that $0<p\le 1$, we have

$$\begin{aligned}I+III &\le \|\ell\|_{BMOF_+(p,\gamma,w)}\left(\sum_{n=m+1}^{\infty} w(I_n)^{1/p}+w(I_{m,b})^{1/p}\right)\\ &\le \|\ell\|_{BMOF_+(p,\gamma,w)}\ w((-\infty,b))^{1/p}\ .\end{aligned}$$

Next, we shall estimate II. We have

$$II\le \sum_{n=m+1}^{\infty}\sum_{k=n}^{\infty}\int_{I_n}|P_{k+1}-P_k|dx\ .$$

Using part (i) and (ii) of Lemma (3.19) with $a=-2^n$, we get

$$II\le c'_\gamma\ \|\ell\|_{BMOF_+(p,\gamma,w)}\sum_{n=m+1}^{\infty} 2^n\sum_{k=n}^{\infty}2^{-k}\ w(I_{k+1}\cup\ldots\cup I_n)^{1/p}\ .$$

Since $0<p\le 1$, the double series on the right hand side is bounded by

$$\begin{aligned}&\left[\sum_{n=m+1}^{\infty}2^{np}\sum_{k=n}^{\infty}2^{-kp}\sum_{j=n}^{k+1}w(I_j)\right]^{1/p}\\ &\le \left[\sum_{n=m+1}^{\infty}2^{np}\sum_{j=n}^{\infty}w(I_j)\sum_{k=j-1}^{\infty}2^{-kp}\right]^{1/p}\\ &\le c_p\left[\sum_{n=m+1}^{\infty}2^{np}\sum_{j=n}^{\infty}2^{-jp}w(I_j)\right]^{1/p}\\ &=c_p\left[\sum_{j=m+1}^{\infty}2^{-jp}w(I_j)\sum_{n=m+1}^{j}2^{np}\right]^{1/p}\\ &\le c'_p\left[\sum_{j=m+1}^{\infty}w(I_j)\right]^{1/p}\le c'_p w((-\infty,b))^{1/p}\ .\end{aligned}$$

Thus,

$$II \leq c_{p,\gamma} \ \|\ell\|_{BMOF_+(p,\gamma,w)} \ w((-\infty, b))^{1/p} \ .$$

Finally, let us estimate IV. We have

$$\begin{aligned}\int_{I_{m,b}} |P_{m,b} - P|dx &\leq \sum_{k=m+1}^{\infty} \int_{I_{m,b}} |P_{k+1} - P_k|dx + \int_{I_{m,b}} |P_{m+1} - P_{m,b}|dx \\ &= A + B \ .\end{aligned}$$

Using part (i) of Lemma (3.19) with $n = m$ and $a = b$, we get

$$\begin{aligned}A &\leq c_\gamma \ \|\ell\|_{BMOF_+(p,\gamma,w)} |I_{m,b}| \sum_{k=m+1}^{\infty} 2^{-k} w(I_{k+1,b})^{1/p} \\ &\leq c'_\gamma \ \|\ell\|_{BMOF_+(p,\gamma,w)} \ w((-\infty, b))^{1/p} \ ,\end{aligned}$$

and using part (ii) of Lemma (3.19), we have

$$\begin{aligned}B &\leq c_\gamma \ \|\ell\|_{BMOF_+(p,\gamma,w)} |I_{m,b}| 2^{-m} w(I_{m+1,b})^{1/p} \\ &\leq c'_\gamma \ \|\ell\|_{BMOF_+(p,\gamma,w)} \ w((-\infty, b))^{1/p} \ .\end{aligned}$$

Let us consider two different values of b, say b and b', and let P and P' the polynomials obtained above that satisfy

$$\int_{-\infty}^{b} |\ell - P|dx \leq c \ w((-\infty, b))^{1/p} < \infty$$

and

$$\int_{-\infty}^{b'} |\ell - P'|dx \leq c \ w((-\infty, b'))^{1/p} < \infty \ .$$

Then, if $\beta = \min(b, b')$, we have

$$\int_{-\infty}^{\beta} |P - P'|dx \leq \int_{-\infty}^{b} |\ell - P|dx + \int_{-\infty}^{b'} |\ell - P'|dx < \infty \ .$$

Thus, $P - P' \equiv 0$ showing that there exists a unique $P \in \mathcal{P}_\gamma$ satisfying

$$\int_{-\infty}^{b} |\ell - P|dx \leq c_{p,\gamma} \ \|\ell\|_{BMOF_+(p,\gamma,w)} \ w((-\infty, b))^{1/p} \ .$$

Taking $\ell' = \ell - P$ we find that $\ell' \in BMO_+(p, \gamma, w)$ and

$$\begin{aligned}\|\ell'\|_{BMO_+(p,\gamma,w)} &\leq c_{p,\gamma} \ \|\ell\|_{BMOF_+(p,\gamma,w)} \\ &= c_{p,\gamma} \ \|\widetilde{\ell}\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma}\end{aligned}$$

as we wanted to show. ∎

Proof of Theorem (2.4). If we have that $w((-\infty,\beta)) = \infty$ holds for every β then, since $D_1 = D$, Theorem (2.3) coincides with Theorem (2.4).

Now, let us assume that there exists β satisfying $w((-\infty,\beta)) < \infty$. If L belongs to $[H^p_{+,\gamma}(w)]^*$, by part (i) of Theorem (2.3), we have that $i(L) = \ell \in BMO_+(p,\gamma,w)$ and

$$L(f) = \int \ell(x) f(x)\, dx$$

holds for every $f \in D$. If f belongs to the dense set D_1, then

$$L(f) = \int \ell'(x) f(x)\, dx$$

holds for every $\ell' \in \tilde{\ell} \in BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma$, since $\ell - \ell' = P \in \mathcal{P}_\gamma$ and $\int f(x)P(x)\, dx = 0$. Then, we can define $\tilde{i}(L) = \tilde{\ell}$ and, in virtue of part (i) of Theorem (2.3) we obtain that

$$\|\tilde{\ell}\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma} = \|\ell\|_{BMOF_+(p,\gamma,w)} \leq \|\ell\|_{BMO_+(p,\gamma,w)} \leq c\|L\| .$$

Thus,

$$\|\tilde{i}(L)\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma} \leq c\|L\| . \tag{3.21}$$

By Proposition (3.20) given a class $\tilde{\ell} \in BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma$, there exists a unique representative ℓ' such that $\ell' \in BMO_+(p,\gamma,w)$ and,

$$\|\ell'\|_{BMO_+(p,\gamma,w)} \leq c_{p,\gamma}\, \|\tilde{\ell}\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma} . \tag{3.22}$$

Now, by part (i) of Theorem (2.2), the functional

$$L(f) = \int \ell'(x) f(x)\, dx ,$$

is well defined on the dense set D and,

$$\|L\| \leq c_{p,\gamma,w}\, \|\ell'\|_{BMO_+(p,\gamma,w)} . \tag{3.23}$$

Therefore, $i(L) = \ell'$ and, in consequence, $\tilde{i}(L) = \tilde{\ell'} = \tilde{\ell}$ showing that $\tilde{i}$ is a surjective application. Moreover, in virtue of (3.21), (3.23) and (3.22) we have that

$$\begin{aligned} c_1\|\tilde{i}(L)\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma} &\leq \|L\| \leq c_2\|\tilde{\ell'}\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma} \\ &= c_2\|\tilde{i}(L)\|_{BMOF_+(p,\gamma,w)/\mathcal{P}_\gamma} . \end{aligned}$$

This finishes the proof. ■

REFERENCES

[1] J. García-Cuerva, *"Weighted H^p spaces"*, Dissertationes Mathematicae **162**, Warszana, 1979.

[2] F.J. Martín-Reyes, *"New proof of weighted inequalities for the one-sided Hardy-Littlewood maximal functions"*, Proc. Amer. Math. Soc. **117** (1993), 691-698.

[3] L. de Rosa and C. Segovia, *"Weighted H^p spaces for one sided maximal functions"*, Contemporary Mathematics (AMS series) **189** (1995), 161-183.

[4] E. Sawyer, *"Weighted inequalities for the one-sided Hardy-Littlewood maximal functions"*, Trans. Amer. Math. Soc. **297** (1986), 53-61.

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