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Successive Approximations and Osgood's Theorem

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Abstract

The Picard's method for solving $y' = f(x,y)$, $y(x_0) = y_0$, is considered here for $|f(x,y_1) - f(x,y_2)| \leq \varphi(|y_1 - y_2|)$. It is shown that for rather general Osgood's functions φ , the difference of two successive approximations converges at exponentially decreasing rate. An application to parabolic partial differential equations is given as well.

Key Words: *Successive approximations. Theoretical approximation of solutions. Osgood's functions.*

1 Introduction

In a landmark paper W. Osgood (1898) introduced a condition weaker than the well known Cauchy-Lipschitz one that guarantees the uniqueness of the solution of the initial value problem for a first order equation. Later, A. Wintner (1946) showed that Osgood's uniqueness condition implies the convergence of the successive approximation on a sufficiently small interval. Later on, J. Lasalle (1949) extended Wintner's method to the case of the uniqueness condition for the solution given by P. Montel and M. Nagumo in 1926. Lasalle's method is essentially Wintner's but adapted to a different condition. One of the improvements introduced by Lasalle is the fact that the zero approximation can be chosen to be any one but sufficiently small in L^∞

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norm. Lasalle and Wintner's papers raise questions concerning the speed of the convergence of the successive approximations. It is the aim of this paper to answer these questions for the cases of rather general Osgood's functions. In fact, we show that in cases such as $\varphi(t) = t\left|\log t\right|^{\beta}$ $0 < \beta \leq 1$, the difference of two successive approximations $|y_{n+1} - y_n|$ converges at exponentially decreasing speed.

The result in this paper can be easily extended to the case of first order systems very much in the same way as Lasalle (1949). For such a generalization we refer the reader to the paper by Lasalle.

We will deal with the equation

$$y' = f(x, y)$$

with initial condition $y(x_0) = y_0$. f is a continuous function which verifies that $|f(x, y_1) - f(x, y_2)| \leq \varphi(|y_1 - y_2|)$, where $a \leq x \leq b$ and $c \leq y \leq d$. φ satisfies the modified Osgood's conditions below in 2.1. For the sake of simplicity we may assume that $-a \leq x \leq a$, $-c \leq y \leq c$ and $x_0 = 0$, $y_0 = 0$. We consider the successive approximations

$$y_{n+1}(x) = \int_0^x f(t, y_n(t))\, dt$$

and study the modality of convergence of

$$\omega_n(x) = |y_{n+1}(x) - y_n(x)|,$$

for $|x| \leq \delta$ and $\delta > 0$ suitably small. Within that range we will have

$$0 \leq \omega_n \leq Cr^n, \tag{1}$$

for $n \geq n_0$, for some n_0, $r, 0 < r < 1$, and C a positive constant. Consider the auxiliary iteration

$$z_{n+1}(x) = \int_0^x \varphi(z_n(t))\, dt,$$

in general $\omega_n \leq z_n$. We study the speed of convergence to zero of $z_n(x)$, which automatically will imply the result.

In the final section we briefly indicate some other applications to parabolic partial differential equations.

2 Osgood's functions

A real valued function φ is an Osgood's function if the following conditions are met:

1) φ is non-negative continuous and monotone on $(0, \delta)$, for some $\delta > 0$.
2) $\int_0^{\varepsilon} \frac{dt}{\varphi(t)} = \infty$, for any ε, $0 < \varepsilon \leq \delta$.

Remark 1 *1) and 2) above imply immediately that* $\lim_{x \to 0^+} \varphi(x) = 0$ *and that* φ *is non-decreasing. We will define* $\varphi(0) = 0$.

2.1 Modified Osgood's condition

Throughout this paper we shall be concerned with Osgood's functions of the type

$$\varphi(t) = t \int_t^b \frac{w(s)}{s} ds,$$

where $b > 0$ is a given constant. w is a non-negative continuous non-decreasing function on $[0, b]$, such that

$$\int_0^b \frac{w(s)}{s} ds = \infty.$$

Notice that $\varphi(0^+) = 0$. Occasionally we use the notation $\sigma(t) = \int_t^b \frac{w(s)}{s} ds$. From now on we will assume $b \leq 1$.

Lemma 1 *The function* φ *satisfies the following properties:*

(i) There is some $a, a > 0$, such that $\varphi(a) = a$.

(ii) $\lim_{t \to 0^+} \varphi'(t) > 1$.

There is some $\delta, 0 < \delta \leq a$, such that for $t, 0 < t \leq \delta$, it holds that

(iii) φ' and σ are decreasing,

(iv) $\sigma(t^n) \leq n\sigma\left(b^{\frac{n-1}{n}} t\right)$, for all $n = 1, 2, \ldots$

(v) the quotient $\frac{\sigma(t)}{\varphi'(t)} = \frac{\varphi(t)}{t\varphi(t)}$ is bounded, and

(vi) $\frac{\varphi^2(t)}{t} = t\sigma^2(t)$ is increasing.

Proof. (i) holds because $\varphi(0) = 0, \varphi(b) = 0$ and φ is concave. (ii) and (iii) follow from the representation

$$\varphi'(t) = \sigma(t) - w(t) = \int_t^b \frac{w(s)}{s} ds - w(t).$$

(iv) We show the case $b = 1$, the others follow by considering the function $\tilde{\sigma}(\tau) = \sigma(b\tau)$ and the change of variable $t = b\tau$. Let α, β, $0 < \alpha, \beta \le 1$, then

$$\sigma(\alpha\beta) = \int_{\alpha\beta}^{1} \frac{w(s)}{s} ds = \int_{\alpha\beta}^{\alpha} \frac{w(s)}{s} ds + \int_{\alpha}^{1} \frac{w(s)}{s} ds$$

$$\begin{aligned} &= \int_{\beta}^{1} \frac{w(\alpha s)}{s} ds + \int_{\alpha}^{1} \frac{w(s)}{s} ds \le \int_{\beta}^{1} \frac{w(s)}{s} ds + \int_{\alpha}^{1} \frac{w(s)}{s} ds \\ &= \sigma(\alpha) + \sigma(\beta). \end{aligned}$$

(v) $\frac{\sigma(t)}{\varphi'(t)} = \frac{\sigma(t)}{\sigma(t) - w(t)} \to 1$, as $t \to 0^+$.
(vi) $\frac{d}{dt}\left(t\sigma(t)^2\right) = \sigma(t)(\sigma(t) - 2w(t)) > 0$.

2.2 Most common modified Osgood's functions

It is straightforward to show that the following functions satisfy the modified Osgood's conditions:

$$\varphi_k(t) = t(\log(1/t))^{\beta_1}(\log\log(1/t))^{\beta_2}\dots(\log\log\log(1/t))^{\beta_k}$$

for some $0 \le \beta_i < 1$, $i = 1, \dots, k-1, 0 < \beta_k \le 1, k = 1, 2, \dots$. These are the most common modified Osgood's functions. Observe that for all of them, the constant b satisfies that $b \le 1$.

3 Auxiliary lemmata

From now on we will assume $b \le 1$. The following lemmata are the technical tools for proving that $\|\omega_n\| \le Cr^n$ for some r , $0 < r < 1$, C a positive constant. Assume that $0 < t \le \delta$, where $\delta > 0$ is as in Lemma 1.

Lemma 2 $\varphi(\lambda\varphi(x)^n) \leq n\frac{\lambda}{b}\frac{\varphi(x)^{n+1}}{x}$, *for any* $\lambda \geq 1, n = 1, 2, \ldots$.

Proof. From $\varphi(t) = t\sigma(t)$ and Lemma 1 (iii), (iv) it follows that

$$\begin{aligned}
\varphi(\lambda\varphi(x)^n) &= \lambda\varphi(x)^n\sigma(\lambda\varphi(x)^n) \\
&= \lambda\varphi(x)^n\sigma(\lambda x^n\sigma(x)^n) \\
&\leq \lambda\varphi(x)^n\sigma(x^n) \\
&\leq n\frac{\lambda}{b}\varphi(x)^n\sigma(x) \\
&= n\frac{\lambda}{b}\frac{\varphi(x)^{n+1}}{x}
\end{aligned}$$

Lemma 3 $\int_0^x \varphi(t)^n\,dt \leq \frac{1}{\varphi'(x)}\frac{\varphi(x)^{n+1}}{n+1} \leq \frac{\varphi(x)^{n+1}}{n+1}$, $n = 1, 2, \ldots$.

Proof. Lemma 1 (iii) and (ii) imply that

$$\int_0^x \varphi(t)^n\,dt \leq \int_0^x \varphi(t)^n\frac{\varphi'(t)}{\varphi'(x)}\,dt = \frac{1}{\varphi'(x)}\frac{\varphi(x)^{n+1}}{n+1} \leq \frac{\varphi(x)^{n+1}}{n+1}.$$

Lemma 4 *There exists a positive constant* C *such that* $\int_0^x \frac{\varphi(t)^n}{t}\,dt \leq \frac{C}{n-1}\varphi(x)^n$, $n = 1, 2, \ldots$.

Proof. Lemma 1 (v)-(vi) and Lemma 3 give

$$\begin{aligned}
\int_0^x \frac{\varphi(t)^n}{t}\,dt &= \int_0^x \frac{\varphi(t)^2}{t}\varphi(t)^{n-2}\,dt \\
&\leq \frac{\varphi(x)^2}{x}\int_0^x \varphi(t)^{n-2}\,dt \\
&\leq \frac{\varphi(x)^2}{x}\frac{1}{\varphi'(x)}\frac{\varphi(x)^{n-1}}{n-1} \\
&\leq \frac{C}{n-1}\varphi(x)^n.
\end{aligned}$$

Lemma 5 *Let* z_0 *be a non-negative function defined for* $|x| \leq \delta$, $z_0(x) \leq \eta$, δ *and* η *small enough. Suppose* $z_{n+1}(x) = \int_0^x \varphi(z_n(t))\,dt$, *then,*

$0 \leq z_1(x) \leq x$,

$0 \leq z_n(x) \leq C^{n-2}\varphi(x)^n$, $\quad n = 2, 3, \ldots,$ $C > 1$ a suitable constant.

Proof. φ is a continuous function with $\varphi(0) = 0$, so for $\eta > 0$ sufficiently small,

$$0 \leq z_1(x) \leq \int_0^x \varphi(\|z_0\|)\, dt \leq x.$$

Put $C = \frac{C'}{b}$, where C' is as in Lemma 4 and such that $C' > b$. For $n = 2$, we use Lemma 3 to get

$$0 \leq z_2(x) \leq \int_0^x \varphi(z_1(t))\, dt \leq \int_0^x \varphi(t)\, dt \leq \frac{1}{2}\varphi(x)^2 \leq \varphi(x)^2.$$

Now, by induction on n, for $n \geq 2$,

$$\begin{aligned} 0 &\leq z_{n+1}(x) = \int_0^x \varphi(z_n(t))\, dt \leq \int_0^x \varphi\left(C^{n-2}\varphi(t)^n\right) dt \\ &\leq n\frac{C^{n-2}}{b}\int_0^x \frac{\varphi(t)^{n+1}}{t}\, dt \\ &\leq n\frac{C^{n-2}}{b}\frac{C'}{n}\varphi(x)^{n+1} \\ &= C^{n-1}\varphi(x)^{n+1}, \end{aligned}$$

because of Lemmas 2 and 4.

4 Rate of convergence of the successive approximations

In order to analyze the rate of convergence of ω_n, observe that

$$\omega_{n+1}(x) \leq \int_0^x |f(t, y_{n+1}(t)) - f(t, y_n(t))|\, dt \leq \int_0^x \varphi(\omega_n(t))\, dt.$$

We will consider a different iteration. If we begin with small initial values, namely:

$$\omega_0(x) \le \eta, \qquad \text{for } |x| \le \delta,$$

we study the auxiliary iteration

$$z_{n+1}(x) = \int_0^x \varphi(z_n(t))\, dt,$$

where $z_0 = \eta$. It is clear that $\omega_1 \le z_1$ and, in general $\omega_n \le z_n$, which is a consequence of the monotonicity of φ. Then it will be enough to prove the estimate (1) for z_n. It will be worth to notice that

$$|\varphi(z_{n+1}) - \varphi(z_n)| \le \varphi(|z_{n+1} - z_n|)$$

because of the concavity of φ, plus $\varphi(0) = 0$. The next step will be to study the speed of convergence to zero of $z_n(x)$, $|x| \le \delta$ and to show that for $z_n(x)$ the estimate (1) is valid. Indeed, from Lemma 5, there exists a constant $C > 1$ such that $0 \le z_n(x) \le C^{n-2}\varphi(x)^n$. By the continuity of φ and the fact that $\varphi(0) = 0$, for any r, $0 < r < 1$, there is some $\delta', 0 < \delta' \le \delta$, for which $C\varphi(x) \le r$, for $|x| \le \delta'$.

Thus, we have proven the following theorem:

Theorem 6 *Let f be a continuous functions on the rectangle $R = [-c, c] \times [-d, d]$. Suppose that*

$$|f(x, y_1) - f(x, y_2)| \le \varphi(|y_1 - y_2|),$$

for $|x| \le c, |y_i| \le d, i = 1, 2$, where φ verifies the modified Osgood's conditions stated in section 2.1. Then, the successive approximations

$$y_{n+1}(x) = \int_0^x f(t, y_n(t))\, dt,$$

$|y_0| \le \eta, \eta > 0$ sufficiently small, satisfy

$$|y_{n+1}(x) - y_n(x)| \le Cr^n,$$

for some $r, 0 < r < 1$, some C large enough and $|x| \le \delta$, for some positive δ.

5 An application to partial differential equations

Consider the following parabolic partial differential equation

$$u_t = \Delta u + F(u,x,t), \qquad u(x,0) = 0.$$

Assume that F is a continuous function that satisfies

$$\|F(u_1,x,t) - F(u_2,x,t)\|_x \le \varphi(\|u_1 - u_2\|_x),$$

where $\| \|_x$ is an ad-hoc norm in the space variables and φ is a modified Osgood's function as in section 2.1.

Let $K(x,t)$ be the usual L^1 fundamental solution[1] and implement the iterations

$$u_{n+1}(x,t) = \int_0^t \int_{\mathbf{R}^m} K(x-y,t-\tau) F(u_n(y,\tau),y,\tau)\, dy\, d\tau,$$

u_0 is chosen to be C_0^∞ in $\mathbf{R}^m$ with small L^∞ norm. Then $\|u_{n+1} - u_n\|_x$, which is a function of t, satisfies that

$$\begin{aligned}
\|u_{n+1} - u_n\|_x &\le \int_0^t \left\| \int_{\mathbf{R}^m} K(x-y,t-\tau)\left(F(u_n,y,\tau) - F(u_{n-1},y,\tau)\right) dy \right\|_x d\tau \\
&\le \int_0^t \|F(u_n,.,\tau) - F(u_{n-1},.,\tau)\|_x\, d\tau \\
&\le \int_0^t \varphi(\|u_n - u_{n-1}\|_x)\, d\tau.
\end{aligned}$$

With a similar procedure as the one employed before, by choosing u_0 suitably small in the appropriate ad-hoc norm, it follows that

$$\|u_{n+1} - u_n\|_x \le C r^n,$$

for $0 \le t < \delta$, some $\delta > 0$ and some r, $0 < r < 1$.

[1] As it is well known, B. Frank Jones has constructed fundamental solutions that are not of the usual type.

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