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GEODESICS AND INTERPOLATION

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Abstract. *Let a and b be two invertible positive elements in a C^*-algebra $A \subset L(H)$, and denote by $\| \|_a$ and $\| \|_b$ the corresponding quadratic norms on H induced by them. We show that the curve of norms obtained by the complex method of interpolation applied to $\| \|_a$ and $\| \|_b$ corresponds precisely with the (unique) geodesic curve joining a and b in the natural homogeneous reductive structure of the space of A^+ of invertible positive elements of A. We study some of the consequences of this correspondence between interpolation theory and the differential geometry of A^+.*

Introduction.

The purpose of this paper is to relate the geometry of the homogeneous space A^+ of positive invertible elements of a C*-algebra $A \subset L(H)$ (see [5]) with the diverse methods for interpolating quadratic norms in H. In [13] Semmes suggests this geometrical point of view. We prove that the curve obtained by Calderón's method of complex interpolation between $\|.\|_a =< a.,. >$ and $\|.\|_b =< b.,. >$ agrees with the unique geodesic curve $\gamma_{a,b}$ joining a with b in A^+, via the identification $a = \|.\|_a$. We also show that the length of curves given by the Finsler structure of A^+ is the same as the natural length of interpolation curves of norms given in [13].

The relation between differential geometry of positive operators and interpolation theory may give several applications in both areas.

An easy example of this interaction is given by the reiteration property of the complex method of interpolation. This property states that no new norms are obtained by iterating the process of interpolation. If one regards this property from the geometric point of view it is natural and trivial - recall [5] that in A^+ two elements are joined by a unique geodesic.

As an application in the other direction the characterization of geodesics by interpolation gives a trivial proof of the convexity properties of the geodesics in A^+ (see 2.5).

In section §1 we recall from [5] the geometrical properties of the space A^+ of positive invertible elements of a C^*-algebra $A \subset L(H)$, where H is a separable complex Hilbert space. Specifically the formulas for its geodesics: for a, b in A^+ the curve

$$\begin{aligned}\gamma_{a,b}(t) &= a^{1/2} e^{t \log(a^{-1/2} b a^{-1/2})} a^{1/2} \\ &= a^{1/2} (a^{-1/2} b a^{-1/2})^t a^{1/2}\end{aligned}$$

is the unique geodesic in A^+ joining a with b.

In section §2 we show that these curves coincide with the curves obtained when applying the complex method of Calderon to the norms given by its endpoints. We also prove that the lengths of the curves measured with the Finsler metric of A^+ and with the natural metric for norms coincide.

1. Geodesics in A^+.

Let A be a C*–algebra with identity contained in the space $L(H)$ of bounded operators on the Hilbert space H. Let A^s be the real subspace of selfadjoint elements of A, A^+ the set of all positive invertible elements of A and G the group of invertible elements of A. We describe briefly the differentiable structure of A^+ (for details, see [5]).

Because A^+ is an open subset of A^s, it carries a natural structure of (real) differentiable manifold and its tangent space $(TA^+)_a$ $(a \in A^+)$ can be identified to A^s. The group G acts on A^+ by $L_g a = (g^*)^{-1} a g^{-1}$ $(g \in G,\ a \in A^+)$. The action is transitive because every $a \in A^+$ can be expressed as $L_{a^{-1/2}} 1$. For a fixed $a \in A^+$, the map $\pi_a : G \to A^+$ defined by $\pi_a(g) = L_g a$ $(g \in G)$ is a principal fibre bundle with structural group $U_a = \{u \in G : \pi_a(u) = a\} = \{u \in G : u^* a u = a\}$. Set $W_1 = \{s \in G : as = s^* a\}$ and, in general, $W_g = g W_1$ for $g \in G$. An easy computation shows that $W_{gu} = W_g u$ for every $g \in G$, $u \in U_a$. Thus, by [14], $g \mapsto W_g$ is the distribution of horizontal spaces for a connection on the principal bundle $\pi_a : G \to A^+$ (the "canonical" connection). This connection induces a connection on the tangent bundle (TA^+), with covariant derivative for a tangent field X along the curve γ given by

$$\frac{DX}{dt} = \frac{dX}{dt} - \frac{1}{2}(\dot{\gamma}\gamma^{-1} X + X \gamma^{-1} \dot{\gamma}) .$$

The corresponding exponential is

$$\begin{aligned}\exp_a X &= e^{\frac{1}{2}Xa^{-1}} a e^{\frac{1}{2}a^{-1}X} \\ &= a^{\frac{1}{2}} e^{a^{-1/2}Xa^{-1/2}} a^{\frac{1}{2}} \qquad (a \in A^+ \,,\ X \in (TA^+)_a)\,.\end{aligned}$$

Observe that $\exp_a : (TA^+)_a \to A^+$ is a diffeomorphism, with inverse

$$x \mapsto a^{1/2} \log(a^{-1/2} x a^{-1/2}) a^{1/2}.$$

Furthermore, for a, b in A^+ the curve

$$\begin{aligned}\gamma_{a,b}(t) &= a^{1/2} e^{t \log(a^{-1/2} b a^{-1/2})} a^{1/2} \\ &= a^{1/2} (a^{-1/2} b a^{-1/2})^t a^{1/2}\end{aligned}$$

is the unique geodesic in A^+ joining a with b.

Assume now that A is faithfully represented in a Hilbert space $(H, \langle\ ,\ \rangle)$. For each $a \in A^+$ define an inner product in H by $\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle$ $(\xi, \eta \in H)$. On the other hand, each $X \in (TA^+)_a$ determines the sesquilinear form $B_X(\xi, \eta) = \langle X\xi, \eta \rangle$ $(\xi, \eta \in H)$. Note that if H_a denotes the Hilbert space $(H, \langle\ ,\ \rangle_a)$, then every $g \in G$ is an isometry $g : H_a \to H_{L_g a}$ and the norm of $B_X : H_a \times H_a \to \mathbf{C}$ is $\|a^{-1/2} X a^{-1/2}\|$. This defines a Finsler metric by $\|X\|_a = \|a^{-1/2} X a^{-1/2}\|$ $(X \in (TA^+)_a)$ and G acts isometrically on A^+ for the Finsler metric.

As usual, the *length* of a C^1 curve γ in A^+ is defined by

$$\ell(\gamma) = \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)} dt$$

and the *(geodesic) distance* between a, b in A^+ is

$$d(a,b) = \inf\{\ell(\gamma) : \gamma \text{ joins } a \text{ and } b\}\,.$$

On the other hand, given two quadratic norms in H

$$\|x\|_a^2 = < ax, x > \quad \text{and} \quad \|x\|_b^2 = < bx, x >$$

where $a, b \in A^+$, the curve of quadratic interpolation norms in the sense of Uhlmann has the form:

$$\|x\|_t^2 = < c_t(a,b)x, x > \qquad \text{where}$$

$$c_t(a,b) = (a+b)^{1/2}[(a+b)^{-1/2}a(a+b)^{-1/2}]^{1-t}[(a+b)^{-1/2}b(a+b)^{-1/2}]^t(a+b)^{1/2}$$

It is easy to see that these two curves agree. Indeed, if a and b commute, $c_t(a,b) = \gamma_{a,b}(t) = a^{1-t}b^t$. If one applies the action of G to a geodesic in A^+ one obtains another geodesic. On the other hand, for any pair $a, b \in A^+$, we have that $a_1 = (a+b)^{-1/2}a(a+b)^{-1/2}$ and $b_1 = (a+b)^{-1/2}b(a+b)^{-1/2}$ commute. Then we can deduce that

$$\begin{aligned}\gamma_{a,b}(t) &= (a+b)^{1/2}\gamma_{a_1,b_1}(t)(a+b)^{1/2}\\ &= (a+b)^{1/2}a_1^{1-t}b_1^t(a+b)^{1/2}\\ &= c_t(a,b)\ .\end{aligned}$$

So we have proved the following:

Proposition (1.1). *Given a and b in A^+, Uhlmann's quadratic interpolation curve between the norms $\|x\|_a^2 = <ax, x>$ and $\|x\|_b^2 = <bx, x>$ is the unique geodesic joining a and b in A^+.*

2. The complex method.

Semmes states in [13] that the complex interpolation method of Calderón, when applied to a pair of quadratic norms $\|\ \|_0$ and $\|\ \|_1$ on $\mathbf{C}^n$, produces a curve of quadratic norms $\|\ \|_t$ which solves the following differential equation: if $F_t(v) = \|v\|_t^2$ for a fixed v in $\mathbf{C}^n$

$$(1)\qquad \begin{cases} \frac{d^2F}{dt^2} = \sum_{j,k} F^{j\bar{k}}\,\frac{dF_j}{dt}\,\frac{dF_{\bar{k}}}{dt} \\ F(0) = \|v\|_0 \quad \text{and} \quad F(1) = \|v\|_1 \end{cases}$$

where $F_j = \frac{\partial F}{\partial v_j}$, $F_{\bar{k}} = \frac{\partial F}{\partial \bar{v}_k}$, $F_{j\bar{k}} = \frac{\partial^2 F}{\partial v_j \partial \bar{v}_k}$ and $F^{j\bar{k}}$ is the inverse matrix of $F^{j\bar{k}}$.

Since these norms are quadratic, there exist positive matrices a, b and $\gamma(t)$ such that $\forall v \in \mathbf{C}^n$, $\|v\|_0^2 = <av, v>$, $\|v\|_1^2 = <bv, v>$ and $\|v\|_t^2 = <\gamma(t)v, v>$. Simple calculations show that (1) is equivalent to

$$(2)\qquad \begin{cases} \frac{d^2\gamma}{dt^2} = \frac{d\gamma}{dt}\,\gamma^{-1}\,\frac{d\gamma}{dt} \\ \gamma(0) = a \quad \text{and} \quad \gamma(1) = b \end{cases}$$

This is precisely the equation that describes the geodesics of the homogeneous geometry of $M_n(\mathbf{C})^+$. Note that this equation makes sense also in A^+, for any C^*-algebra A. Therefore it would be interesting to know if in the case of infinite dimensional Hilbert spaces, the complex method of Calderón produces curves of quadratic norms which are solutions of (2). In fact, this is the case:

Proposition (2.1). *Given $a, b \in A^+$, let $\| \ \|_t$ be the norm obtained by complex interpolation of the norms $\|v\|_0^2 =< av, v >$ and $\|v\|_1^2 =< bv, v >$. Then $\forall v \in H$ and $\forall t \in$ [0,1],*

$$\|v\|_t^2 =< a^{1/2}\,(a^{-1/2}ba^{-1/2})^t\,a^{1/2}\,v,\,v >$$
$$=< \gamma_{a,b}(t)\,v,\,v >$$

In other words, complex interpolation is determined by the geodesics of A^+.

Proof. By the results obtained in the previous section the result would follow if we show that the quadratic and complex methods of interpolation coincide isometrically for Hilbert norms. We remark that while the equivalence of the norms is not an issue in our context the equality of the norms requires a proof.

By definition,

$$\|v\|_t^2 = \inf_{G(t)=0}\ \sup_{y\in\mathbf{R}}\ \{\,\|v+G(iy)\|_0^2,\ \|v+G(1+iy)\|_1^2\,\}$$

where the infimum is taken over all bounded continuous H-valued functions G defined on the strip $S = \{0 \le Re(z) \le 1\}$ which are analytic on the interior of S and vanish at t. Let $c = a^{-1/2}ba^{-1/2}$ and denote by $\| \ \|$ the usual norm of H.

$$\|v\|_t^2 = \inf_{G(t)=0}\ \sup_{y\in\mathbf{R}}\ \{\,\|a^{1/2}v + a^{1/2}G(iy)\|,\ \|c^{1/2}(a^{1/2}v + a^{1/2}G(1+iy))\|\,\}.$$

Given a competing analytic function G we see that $F(z) = a^{1/2}G(z)$ is another competing analytic function and moreover, taking infimum over all possible G is the same as taking it over the functions F. Therefore, we have that $\|v\|_t$ equals the norm of $a^{1/2}v$ interpolated between the usual norm of H and the norm defined by c. Consequently it is sufficient to prove our statement for these two norms. Indeed,

$$\|v\|_t^2 =< c^t a^{1/2}v, a^{1/2}v >$$
$$=< a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}v, v >$$

Let us asume without loss that $a = 1$ and therefore $b = c$. First we consider the case when the spectrum of c is finite. Let $\{r_1, ..., r_n\}$ be the spectrum of c. There exist selfadjoint projections $p_1, ..., p_n$ in A such that $p_i p_j = 0$ if $i \ne j$, $\sum_{j=1}^n p_j = 1$ and $c = \sum_{j=1}^n r_j p_j$. Each analytic function G consists of an n-tuple $(G_1, ..., G_n)$ with values on the respective ranges of the projections p_j. Consider those $v = \sum_{j=1}^n p_j v = \sum_{j=1}^n v_j$

such that $v_j \neq 0$, $1 \leq j \leq n$. It suffices to prove the equality for these v, since they are dense in H. Then

$$\|v\|_t^2 = \inf_{G_j(t)=0} \sup_{y \in \mathbf{R}} \left\{ \sum_{j=1}^{n} \|v_j + G_j(iy)\|^2, \sum_{j=1}^{n} r_j \|v_j + G_j(1+iy)\|^2 \right\}.$$

Observe that

$$\|v_j + G_j(z)\|^2 \geq \|v_j + < G_j(z), \frac{v_j}{\|v_j\|^2} > v_j\|^2$$

which follows by simply expanding both members and using the Cauchy-Schwarz inequality.

Let $g_j(z) =< G_j(z), \frac{v_j}{\|v_j\|^2} >$. Clearly, g_j is a scalar function bounded and continuous on S, analytic on the interior of S, and which vanishes al t. Taking infimum over the n-tuples of functions $h(z) = (h_1(z)v_1, ..., h_n(z)v_n)$ where the h_j are scalar, bounded, analytic and vanish at t, would produce a result bigger than $\|v\|_t$. On the other hand, the inequality shown above proves that it is also smaller. Therefore,

$$(1) \qquad \|v\|_t^2 = \inf_{h(t)=0} \sup_{y \in \mathbf{R}} \left\{ \sum_{j=1}^{n} \|v_j + h_j(iy)v_j\|^2, \sum_{j=1}^{n} r_j \|v_j + h_j(1+iy)v_j\|^2 \right\}.$$

Let $f_j(z) = h_j(z)\|v_j\|$. Using that $\|v_j + h_j(z)v_j\|^2 = |\ \|v_j\| + f_j(z)\ |^2$ and replacing in (1), we obtain that

$$\|v\|_t^2 = \|(\|v_1\|, \ldots, \|v_n\|)\|_t^2$$

where the right hand side denotes the square norm of the n-tuple $(\|v_1\|, \ldots, \|v_n\|) \in \mathbf{C}^n$ obtained by interpolation between the norms of $\mathbf{C}^n$ defined by the identity and the diagonal matrix with $r_1, \ldots, r_n$ on the diagonal. By the Stein-Weiss theorem (see [1]), we have

$$\|v\|_t^2 = \sum_{j=1}^{n} {r_j}^t \|v_j\|^2 =< c^t v, v > .$$

For the general case, it would suffice to use the continuity of the complex interpolation on the parameters (in this case 1 and c). We can also prove it directly by taking two sequences a_n and b_n of positive invertible operators on A, all of them with finite spectrum, such that $a_n \leq c \leq b_n$ and a_n and b_n converge (in norm) to c.

If we denote by $\|v\|_{t,a_n}$ and $\|v\|_{t,b_n}$ the respective interpolated norms, it is straightforward to verify that

$$< a_n^t v, v >= \|v\|_{t,a_n}^2 \leq \|v\|_t^2 \leq \|v\|_{t,b_n}^2 =< b_n^t v, v >$$

and taking limits, $\|v\|_t^2 = < c^t v, v >$, which completes the proof.

The Finsler structure in A^+ determines a metric in this space. We define, for $a, b \in A^+$, the distance between a and b as the infimum of the length of C^1 curves in A^+ joining these two points. Since in [5] it was shown that geodesics are minimal, this distance can be computed by

$$d_{A^+}(a,b) = \|\log(a^{-1/2}ba^{-1/2})\|$$

We can also regard a and b as quadratic norms on H. Let $\|v\|_a = < av, v >^{1/2}$ and $\|v\|_b = < bv, v >^{1/2}$. A natural distance between norms is the following one, due to Banach and Mazur: define

$$\delta(\|\cdot\|_a, \|\cdot\|_b) = \log \sup_{v \neq 0} \frac{\|v\|_a}{\|v\|_b} \qquad \text{and}$$

$$d(\|\cdot\|_a, \|\cdot\|_b) = \max(\delta(\|\cdot\|_a, \|\cdot\|_b), \delta(\|\cdot\|_b, \|\cdot\|_a))$$

Proposition (2.2). *Let $a, b \in A^+$, then*

$$d_{A^+}(a,b) \;=\; 2d(\|\cdot\|_a, \|\cdot\|_b)$$

Proof. Clearly, if $v \in H$, $\|v\|_a = \|a^{1/2}(v)\|$. So

$$\begin{aligned}\delta(\|\cdot\|_a, \|\cdot\|_b) &= \log \sup_{v \neq 0} \frac{\|a^{1/2}(v)\|}{\|b^{1/2}(v)\|} \\ &= \log \sup_{v \neq 0} \frac{\|a^{1/2}b^{-1/2}(v)\|}{\|v\|} \\ &= \log \|a^{1/2}b^{-1/2}\|\end{aligned}$$

Therefore, $d(\|\cdot\|_a, \|\cdot\|_b) = \max(\log(\|a^{1/2}b^{-1/2}\|), \log(\|b^{1/2}a^{-1/2}\|))$.

It is easy to see that, if $c \in A^+$, $\|\log(c)\| = \max(\log\|c\|, \log\|c^{-1}\|)$. But

$$\|a^{-1/2}ba^{-1/2}\| = \|(b^{1/2}a^{-1/2})^* b^{1/2}a^{-1/2}\| = \|b^{1/2}a^{-1/2}\|^2.$$

So

$$\begin{aligned}d_{A^+}(a,b) &= \|\log(a^{-1/2}ba^{-1/2})\| \\ &= \max(\log\|a^{-1/2}ba^{-1/2}\|, \log\|a^{1/2}b^{-1}a^{1/2}\|) \\ &= 2\max(\log\|(a^{1/2}b^{-1/2}\|, \log\|(b^{1/2}a^{-1/2}\|) \\ &= 2d(\|\cdot\|_a, \|\cdot\|_b).\end{aligned}$$

Remark (2.3). The factor 2 appears because we identified $a \in A^+$ with $\|x\|_a =< ax, x >^{1/2} = \|a^{1/2}x\|$. If we identify $a \in A^+$ with $\|x\|'_a = \|ax\|$, the last proposition holds without the factor 2.

Corollary (2.4). *The Calderón interpolation curve between two quadratic norms* $\|.\|_1$ *and* $\|.\|_2$ *has minimal length. That is, the length of that curve equals* $d(\|.\|_1, \|.\|_2)$.

Proof. If $\gamma(t)$ is the Calderón interpolation curve between $\|.\|_1$ and $\|.\|_2$, it's length = length(γ) is given by

$$\sup \sum d(\gamma(t_j), \gamma(t_{j-1}))$$

where the supremum is taken over all partitions $\{t_j\}$ of [0,1]. But we know that the Calderón curves are the geodesics of A^+ and they have minimal length. We also know that $d = \frac{1}{2} d_{A^+}$. Therefore, length(γ) = $d(\|.\|_1, \|.\|_2)$ (it is clear that both notions of length agree, using the fact that geodesics and Calderón curves have the reiteration property).

Remark (2.5). Using the Calderón's method it is easy to see that if $0 < a \leq c$ and $0 < b \leq d$, then

$$\gamma_{a,b}(t) \leq \gamma_{c,d}(t)$$

for all $t \in$ [0,1].

In particular, putting $b = d = 1$ one obtains Loewner's theorem:
if $0 \leq a \leq c$ and $t \in$ [0,1], then

$$a^t \leq c^t.$$

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