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KRASNOSEL'SKII-TYPE PARAMETERS OF CONVEXITY SPACES

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ABSTRACT. The aim of this paper is to give several properties of the starshapedness number – the first Krasnosel'skii-type parameter of a convexity space. Here we also introduce a new combinatorial characteristic of a convexity space connected with starshaped sets and examine relationships between the two considered parameters. Special attention is paid to both parameters in convex product and sum spaces.

1. INTRODUCTION

A *convexity space* is a pair $(X, \mathcal{C})$, where X is a nonempty set and $\mathcal{C}$ is a family of subsets of X closed under arbitrary intersections and containing X and the empty set $\emptyset$. Members of $\mathcal{C}$ are called *$\mathcal{C}$-convex sets* or simply *convex sets.* The *convex hull* of a set $S \subset X$ is defined in the usual way as

$$\mathcal{C}(S) = \bigcap\{A \in \mathcal{C} \ : \ S \subset A\}.$$

The classical example of a convexity space is $(\mathbb{R}^n, \mathrm{conv})$, where conv denotes the family of ordinary convex sets in $\mathbb{R}^n$. For more examples of convexity spaces and systematic treatment of the subject we refer the reader to [6], [13] and [14].

We say that a *point $y \in S$ is seen from $x \in S$ via S* if $\mathcal{C}(x,y) \subset S$. The *star* of $x \in S$, denoted by $\mathrm{st}(x,S)$, is the set of all points in S which are seen from x via S. Thus

$$\mathrm{st}(x,S) = \{y \in S \ : \ \mathcal{C}(x,y) \subset S\}.$$

A set $S \subset X$ is called *$\mathcal{C}$-starshaped* or simply *starshaped* if there exists a point $x \in S$ such that $\mathrm{st}(x,S) = S$. The *kernel* of S is the set

$$\ker(S) = \{x \in S \ : \ \mathrm{st}(x,S) = S\}$$

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or equivalently

$$\ker(S) = \bigcap\{\mathrm{st}(x,S) \ : \ x \in S\}.$$

A first theorem giving necessary and sufficient conditions for starshapedness is due to Krasnosel'skii [11]. Although the following differs from the formulation in [11] it is commonly known as

Krasnosel'skii Theorem. *A compact set $M \subset \mathbb{R}^n$ is starshaped if and only if for every $n+1$ points $x_1, \ldots, x_{n+1} \in M$ there is a point $y \in M$ which sees them via M.*

The main problem in the investigations of starshaped sets in different settings is to give a new Krasnosel'skii-type theorem, see [1, 2, 5, 9, 10] and others. All known Krasnosel'skii-type theorems deal with sets satisfying some topological conditions. Of course $(X, \mathcal{C})$ can be endowed with a topology, see [14], and then an analogue of Krasnosel'skii's theorem can be formulated, see [5]. However those topological conditions in question are mainly needed to have a variant of the following equality

$$\ker S = \bigcap\{\mathrm{conv}(\mathrm{st}(x,S)) \ : \ x \in S\},$$

which can be also described in the very simple language of any convexity space. Moreover the concept of starshapedness is naturally connected with convexity and therefore we intend to describe Krasnosel'skii-type characteristics only by means of the tools accessible in convexity spaces. In [10] we noticed that the accessible in any convexity space notions of K-sets and K-families replace some topological conditions and enable us to give more general versions of some Krasnosel'skii-type theorems in $\mathbb{R}^N$, see also [7].

We will say that S is a K-*set* in $(X, \mathcal{C})$ if it satisfies the equality

$$\bigcap\{\mathrm{st}(x,S) \ : \ x \in S\} = \bigcap\{\mathcal{C}(\mathrm{st}(x,S)) \ : \ x \in S\}.$$

Obviously any convex set is a K-set. A finite family $\mathcal{F}$ of subsets of X is said to be a K-*family* in $(X, \mathcal{C})$ if $\cup\mathcal{G}$ is a K-set for each subfamily $\mathcal{G}$ of $\mathcal{F}$. It is clear that any subfamily $\mathcal{G}$ of a K-family $\mathcal{F}$ is a K-family itself and any element of $\mathcal{F}$ is a K-set. An example of a K-family is any finite collection of closed sets in $(\mathbb{R}^N, \mathrm{conv})$, see [8].

2. THE STARSHAPEDNESS NUMBER

In [10] we introduced the starshapedness number – the first combinatorial characteristic of a general convexity space in terms of starshaped sets – and next we proved a Krasnosel'skii-type theorem in a convexity space. In this section we will study some properties of the starshapedness number. Let us start with recalling the following definition.

The *starshapedness number* of a convexity space $(X, \mathcal{C})$ is the smallest nonnegative integer s (if such exists) with the property: any K-family $\mathcal{F}$ in $(X, \mathcal{C})$ has a starshaped union provided that each s-element subfamily of $\mathcal{F}$ has a starshaped union. The starshapedness number of $(X, \mathcal{C})$ can be also defined, see [10], as the smallest nonnegative integer s such that for any $(s+1)$-element K-family $\mathcal{F} = \{S_1, \ldots, S_{s+1}\}$ the following implication is true

$$\ker(\cup\mathcal{F}_n^{\wedge}) \neq \emptyset \quad \text{for} \quad n = 1, \ldots, s+1 \Longrightarrow \ker(\cup\mathcal{F}) \neq \emptyset,$$

where $\mathcal{F}_n^\wedge$ here and further on denotes the family $\mathcal{F} \setminus \{S_n\}$. Clearly $s \geq 0$ and $s = 0$ if and only if any K-set in $(X, \mathcal{C})$ is starshaped. Recall that the starshapedness number of $(\mathbb{R}^N$, conv) is equal to $N+1$.

Below we give several properties of the starshapedness number.

Property 1. *Suppose that $(X, \mathcal{C})$ has starshapedness number $s \geq 2$. Then in $\mathcal{C}$ there are convex sets $Q_1, \ldots, Q_s$ such that*

$$\bigcap\{Q_i \ : \ i \neq j\} \neq \emptyset \quad \text{for} \quad j = 1, \ldots, s \quad \text{and} \quad \bigcap\{Q_i \ : \ i = 1, \ldots, s\} = \emptyset.$$

Proof. From the definition of the starshapedness number $s \geq 2$ it follows that in $(X, \mathcal{C})$ there exists a K-family $\mathcal{F} = \{M_1, \ldots, M_s\}$ such that

$$\ker(\cup \mathcal{F}_i^\wedge) \neq \emptyset \quad \text{for} \quad i = 1, \ldots, s \quad \text{and} \quad \ker(\cup \mathcal{F}) = \emptyset.$$

Consider the following convex sets

$$Q_i = \bigcap\{\mathcal{C}(\mathrm{st}(x, \cup \mathcal{F})) \ : \ x \in M_i\}, \quad i = 1, \ldots, s.$$

We will show that Q_i's are the required sets. Indeed, from the fact that $\cup \mathcal{F}_j^\wedge$ is a K-set we have

$$\begin{aligned} \ker(\cup \mathcal{F}_j^\wedge) &= \bigcap\{\mathcal{C}(\mathrm{st}(x, \cup \mathcal{F}_j^\wedge)) \ : \ x \in \cup \mathcal{F}_j^\wedge\} \\ &\subset \bigcap\{\mathcal{C}(\mathrm{st}(x, \cup \mathcal{F})) \ : \ x \in \cup \mathcal{F}_j^\wedge\} \\ &= \bigcap\{\cap\{\mathcal{C}(\mathrm{st}(x, \cup \mathcal{F})) \ : \ x \in M_i\} \ : \ i \neq j\} \\ &= \bigcap\{Q_i \ : \ i \neq j\}. \end{aligned}$$

Therefore $\bigcap\{Q_i \ : \ i \neq j\} \neq \emptyset$ for $j = 1, \ldots, s$.

Since $\cup \mathcal{F}$ is also a K-set we similarly have

$$\begin{aligned} \ker(\cup \mathcal{F}) &= \bigcap\{\cap\{\mathcal{C}(\mathrm{st}(x, \cup \mathcal{F})) \ : \ x \in M_i\} \ : \ i = 1, \ldots, s\} \\ &= \bigcap\{Q_i \ : \ i = 1, \ldots, s\}. \end{aligned}$$

Hence $\bigcap\{Q_i \ : \ i = 1, \ldots, s\} = \emptyset$ and the proof is complete. ∎

The next property is an immediate corollary from Property 1.

Property 2. *If $\bigcap(\mathcal{C} \setminus \{\emptyset\}) \neq \emptyset$, then $(X, \mathcal{C})$ has starshapedness number $s = 1$.*

Property 3. *Suppose that $(X, \mathcal{C})$ is a convexity space. If in $\mathcal{C}$ there are two disjoint convex sets Q_1 and Q_2 such that $\mathrm{st}(x_i, Q_1 \cup Q_2) = Q_i$ for some points $x_i \in Q_i$, then for starshapedness number s of $(X, \mathcal{C})$ we have $s \geq 2$.*

Proof. Let Q_1 and Q_2 be disjoint convex sets and let $x_1 \in Q_1$ and $x_2 \in Q_2$ be points such that $\mathrm{st}(x_i, Q_1 \cup Q_2) = Q_i$, $i = 1, 2$. Let us check that $\mathcal{F} = \{Q_1, Q_2\}$ is a K-family. Obviously it is enough to show that the set $Q_1 \cup Q_2$ is a K-set. The following holds

$$\begin{aligned}
\ker(Q_1 \cup Q_2) &= \bigcap\{\mathrm{st}(x, Q_1 \cup Q_2) \ : \ x \in Q_1 \cup Q_2\} \\
&\subset \bigcap\{\mathcal{C}(\mathrm{st}(x, Q_1 \cup Q_2)) \ : \ x \in Q_1 \cup Q_2\} \\
&= \bigcap\{\mathcal{C}(\mathrm{st}(x, Q_1 \cup Q_2)) \ : \ x \in Q_1\} \cap \bigcap\{\mathcal{C}(\mathrm{st}(x, Q_1 \cup Q_2)) \ : \ x \in Q_2\} \\
&\subset \mathcal{C}(\mathrm{st}(x_1, Q_1 \cup Q_2)) \cap \mathcal{C}(\mathrm{st}(x_2, Q_1 \cup Q_2)) \\
&= Q_1 \cap Q_2 = \emptyset.
\end{aligned}$$

The above shows that $Q_1 \cup Q_2$ is indeed a K-set with empty kernel. Hence we have

$$\ker(Q_i) = Q_i \neq \emptyset,\ i = 1,2 \quad \text{but} \quad \ker(Q_1 \cup Q_2) = \emptyset.$$

This implies $s \geq 2$ and completes the proof. ▮

3. THE KRASNOSEL'SKII PARAMETER

We will say that $(X, \mathcal{C})$ has *Krasnosel'skii parameter* k, if k is the smallest nonnegative integer (if such exists) such that any K-set $S \subset X$ is starshaped provided that for any k points $x_1, \ldots, x_k$ in S there is a point $y \in S$ such that $\mathcal{C}(x_i, y) \subset S$ for $i = 1, \ldots, k$. From the definition it follows that any convexity space $(X, \mathcal{C})$ with finite X has Krasnosel'skii parameter k satisfying the following inequality $k \leq |X| - 1$. The upper bound can be lowered if $(X, \mathcal{C})$ has the Helly number.

We say that a convexity space $(X, \mathcal{C})$ has *Helly number* h if h is the smallest integer (if such exists) such that the intersection of any finite collection $\mathcal{F}$ of convex sets is nonempty provided that the intersection of each h-element subcollection of $\mathcal{F}$ is nonempty.

We have the following Krasnosel'skii-type theorem.

Theorem 1. *Suppose that X is finite. If h is the Helly number of $(X, \mathcal{C})$, then the Krasnosel'skii parameter k of $(X, \mathcal{C})$ satisfies $k \leq h$.*

Proof. The finiteness condition of X implies that both the Helly number and the Krasnosel'skii parameter exist. Let S be a K-set in $(X, \mathcal{C})$ such that any its h points are seen from a common point via S. Consider the family

$$\mathcal{F} = \{\mathcal{C}(\mathrm{st}(x, S)) \ : \ x \in S\}.$$

Of course $\mathcal{F}$ is a finite family of convex sets. Take arbitrary points $x_1, \ldots, x_h$ from S. There exists $y \in S$ such that

$$y \in \bigcap\{\mathrm{st}(x_i, S) \ : \ i = 1, \ldots, h\} \subset \bigcap\{\mathcal{C}(\mathrm{st}(x_i, S)) \ : \ i = 1, \ldots, h\}.$$

It means that any h elements of $\mathcal{F}$ have a common point. Hence $\cap\mathcal{F} \neq \emptyset$. This ends the proof since the equality $\cap\mathcal{F} = \ker(S)$ implies that S is starshaped. ▮

Theorem 2. *Let $(X, \mathcal{C})$ be a convexity space having Krasnosel'skii parameter k. Then $(X, \mathcal{C})$ also has starshapedness number s and $s \leq k$.*

Proof. Take a K-family $\mathcal{F} = \{S_1, \ldots, S_{k+1}\}$ such that

$$\ker(\cup \mathcal{F}_n^\wedge) \neq \emptyset \quad \text{for } n = 1, \ldots, k+1.$$

We will show that the existence of the Krasnosel'skii parameter k implies that $\ker(\cup \mathcal{F}) \neq \emptyset$.

The set $\cup \mathcal{F}$ is a K-set. Take arbitrary k points $x_1, \ldots, x_k$ from $\cup \mathcal{F}$. The points lie in $\cup \mathcal{F}_j^\wedge$ for some $1 \leq j \leq k+1$. Since $\ker(\cup \mathcal{F}_j^\wedge) \neq \emptyset$ there exists a point q such that

$$\mathcal{C}(q, x_i) \subset \cup \mathcal{F}_j^\wedge \subset \cup \mathcal{F} \quad \text{for } i = 1, \ldots, k.$$

This shows that any k points in $\cup \mathcal{F}$ are seen from a common point via $\cup \mathcal{F}$ which implies that $\cup \mathcal{F}$ is starshaped and the proof is complete. ∎

In connection with Theorem 2 let us remark that the existence of the starshapedness number does not imply the existence of the Krasnosel'skii parameter. For example $(\mathbb{R}^2, \text{conv})$ has starshapedness number 3 but no Krasnosel'skii parameter. To this end take

$$S = \{q = (x, y) \, : \, x \geq 0 \quad \text{and} \quad y \leq [x+1] - x\},$$

where $[x]$ denotes the greatest integer not exceeding x. S is clearly a K-set in $(\mathbb{R}^2, \text{conv})$. It is easily verified that for each $n \in \mathbb{N}$ and arbitrary points $q_1, \ldots, q_n$ from S there exists a point in S which sees them via S, but S is not a starshaped set. Consequently, Krasnosel'skii parameter of $(\mathbb{R}^2, \text{conv})$ does not exist.

4. KRASNOSEL'SKII-TYPE PARAMETERS OF CONVEX PRODUCT AND SUM SPACES

In the investigations of convexity spaces the convex product and sum spaces take a part, see [5, 12, 13, 14]. In this section we will examine Krasnosel'skii-type parameters in convex product and sum spaces.

Let $(X_i, \mathcal{C}_i)$, $i = 1, \ldots, n$, be convexity spaces. The pair $(\prod_{i=1}^n X_i, \mathcal{C}_\Pi)$, where

$$\mathcal{C}_\Pi = \prod_{i=1}^n \mathcal{C}_i = \left\{ \prod_{i=1}^n A_i \, : \, A_i \in \mathcal{C}_i \right\}$$

is a convexity space and is called the *convex product space*. For arbitrary set $S \subset \prod_{i=1}^n X_i$ by $\pi_i S$ we denote the projection of S into X_i. The product convex hull of a set $S \subset \prod_{i=1}^n X_i$ is given by

$$\mathcal{C}_\Pi(S) = \prod_{i=1}^n \mathcal{C}_i(\pi_i S).$$

Theorem 3. *Suppose that $(X_i, \mathcal{C}_i)$ has Helly number h_i, $i = 1, \ldots, n$. Then the convex product space $(\prod_{i=1}^n X_i, \mathcal{C}_\Pi)$ has starshapedness number s satisfying*

$$\max\{s_1, \ldots, s_n\} \leq s \leq \max\{h_1, \ldots, h_n\},$$

where s_i is a starshapedness numbers of $(X_i, \mathcal{C}_i)$, i=1,...,n.

Proof. From the results of [12] it follows that the Helly number h of $(\prod_{i=1}^{n} X_i, \mathcal{C}_\Pi)$ exists and satisfies $h = \max\{h_1, \ldots, h_n\}$. This in conjunction with Theorem 4.1 from [10] firstly implies the existence of a starshapedness number s of $(\prod_{i=1}^{n} X_i, \mathcal{C}_\Pi)$ and secondly establishes the upper bound for s.

Now we will establish the lower bound. Again by Theorem 4.1 from [10] we have the existence of starshapedness numbers s_i, $i = 1, \ldots, n$. Suppose that $s_1 = \max\{s_1, \ldots, s_n\}$. If $s_1 \leq 1$ then obviously $s \geq s_1$. Therefore we can assume that $s_1 \geq 2$. So in $(X_1, \mathcal{C}_1)$ there exists a K-family $\mathcal{G} = \{M_1, \ldots, M_{s_1}\}$ such that

$$\ker(\cup \mathcal{G}_m^\wedge) \neq \emptyset \quad \text{for } m = 1, \ldots, s_1 \qquad \text{and} \qquad \ker(\cup \mathcal{G}) = \emptyset.$$

It is easy to check that

$$\mathcal{F} = \{M_i \times \prod_{j=2}^{n} X_j \;:\; i = 1, \ldots, s_1\}$$

is a K-family in $(\prod_{i=1}^{n} X_i, \mathcal{C}_\Pi)$ with the following properties:

$$\ker(\cup \mathcal{F}_m^\wedge) = \ker(\cup \mathcal{G}_m^\wedge) \times \prod_{i=2}^{n} X_i \neq \emptyset \quad \text{for } m = 1, \ldots, s_1$$

and

$$\ker(\cup \mathcal{F}) = \ker(\cup \mathcal{G}) \times \prod_{i=2}^{n} X_i = \emptyset.$$

This shows that $s \geq s_1 = \max\{s_1, \ldots, s_n\}$ and ends the proof. ▮

Now we are going to show that the assumption of the existence of Helly numbers in Theorem 3 cannot be dropped. Namely we will show that the product of convexity spaces with finite starshapedness numbers need not have a starshapedness number. To this end we will use a modification of the example given in [5].

Example 1. Let $\mathbb{N}$ denote the set of natural numbers and $Y = \{1, 2\}$. Let us consider the product space

$$(X, \mathcal{C}_\Pi) = (\mathbb{N} \times \mathbb{N} \times Y, 2^{\mathbb{N}} \otimes 2^{\mathbb{N}} \otimes 2^{Y}).$$

Define

$$M_0 = \{(i, n+2, 2) \;:\; 1 \leq i \leq n+1\},$$
$$M_t = \{(i, t, 1) \;:\; 1 \leq i \leq n+2, i \neq t\}, \quad t = 1, \ldots, n+1,$$
$$M_{n+2} = \{(i, n+2, 1) \;:\; 1 \leq i \leq n+2\}.$$

Take a family

$$\mathcal{F} = \{S_1, \ldots, S_{n+1}\},$$

where

$$S_j = M_j \cup M_0 \cup M_{n+2}, \quad 1 \leq j \leq n+1.$$

Now we will check that $\mathcal{F}$ is a K-family. Denote by J a nonempty subset of the set $\{1, \ldots, n+1\}$. We will show that the set

$$C_J = \cup\{S_j \;:\; j \in J\}$$

is a K-set.

Every set C_J contains the points $x_i = (i, n+2, 1)$, $1 \le i \le n+2$, and $x_0 = (1, n+2, 2)$. It is easy to check that

$$\mathrm{st}(x_0, C_J) \cap \mathrm{st}(x_{n+2}, C_J) = M_{n+2} \setminus \{x_{n+2}\} \tag{1}$$

and

$$\mathcal{C}_\Pi(\mathrm{st}(x_0, C_J)) \cap \mathcal{C}_\Pi(\mathrm{st}(x_{n+2}, C_J)) = M_{n+2} \setminus \{x_{n+2}\}. \tag{2}$$

We also have

$$\mathrm{st}(x_j, C_J) = C_J \setminus M_j, \quad \text{for} \quad j \in J \tag{3}$$

and $\mathrm{st}(x_i, C_J) = C_J$ if $i \notin J$. Moreover, it is easy to check that

$$\bigcap\{\mathrm{st}(y, C_J) \ : \ y \in M_j\} = \bigcup\{M_i \ : \ i \in J \cup \{n+2\}\} \setminus \{(j, t, 1) \ : \ 1 \le t \le n+2\} \tag{4}$$

and for $y_j = (j+1, j, 1) \in M_j$

$$x_j \notin \mathcal{C}_\Pi(\mathrm{st}(y_j, C_J)). \tag{5}$$

Taking (1), (3) and (4) into account we have

$$\ker(C_J) = \bigcap\{\mathrm{st}(y, C_J) \ : \ y \in C_J\} = M_{n+2} \setminus \{x_j \ : \ j \in J \cup \{n+2\}\}. \tag{6}$$

Obviously

$$M_{n+2} \setminus \{x_j \ : \ j \in J \cup \{n+2\}\} \subset \bigcap\{\mathcal{C}_\Pi(\mathrm{st}(y, C_J)) \ : \ y \in C_J\}.$$

On the other hand, with the help of (2), (4) and (5) one can check that

$$\begin{aligned}\bigcap\{\mathcal{C}_\Pi(\mathrm{st}(y, C_J)) \ : \ y \in C_J\} \ &\subset \ \mathcal{C}_\Pi(\mathrm{st}(x_0, C_J)) \cap \mathcal{C}_\Pi(\mathrm{st}(x_{n+2}, C_J)) \\ &\cap \ \bigcap\{\mathcal{C}_\Pi(\mathrm{st}(y_j, C_J)) \ : \ j \in J\} \\ &\subset \ M_{n+2} \setminus \{x_j \ : \ j \in J \cup \{n+2\}\}\end{aligned}$$

which means that every C_J is indeed a K-set and consequently $\mathcal{F}$ is a K-family in $(X, \mathcal{C}_\Pi)$.

Let us notice that from (6) it follows that

$$x_j \in \ker(\cup\mathcal{F}_j^\wedge) \quad \text{for} \quad j = 1, \ldots, n+1$$

but

$$\ker(\cup\mathcal{F}) = \emptyset.$$

In view of this we see that $s \ge n+1$ for every $n \in \mathbb{N}$, which means that no starshapedness number of $(X, \mathcal{C}_\Pi)$ exists. Let us add that both spaces $(\mathbb{N}, 2^{\mathbb{N}})$ and $(Y, 2^Y)$ have starshapedness number zero. ∎

Using similar arguments to those in Theorem 1 and Theorem 3 one can check that the following theorem is true.

Theorem 4. *Suppose that $(X_i, \mathcal{C}_i)$ with finite X_i is a convexity space having Helly number h_i, $i = 1, 2$. Then the convex product space $(X_1 \times X_2, \mathcal{C}_1 \otimes \mathcal{C}_2)$ has Krasnosel'skii parameter k satisfying*

$$\max\{k_1, k_2\} \le k \le \max\{h_1, h_2\},$$

where k_i is a Krasnosel'skii parameter of $(X_i, \mathcal{C}_i)$. ∎

Let $(X_1, \mathcal{C}_1)$ and $(X_2, \mathcal{C}_2)$, $X_1 \cap X_2 = \emptyset$, be convexity spaces. The pair $(X_1 \cup X_2, \mathcal{C}_U)$, where

$$\mathcal{C}_U = \{A \cup B \;:\; A \in \mathcal{C}_1, B \in \mathcal{C}_2\}$$

is a convexity space and is called the *convex sum space*. Here we have

$$\mathcal{C}_U(S) = \mathcal{C}_1(S \cap X_1) \cup \mathcal{C}_2(S \cap X_2).$$

Theorem 5. *Let $(X_1, \mathcal{C}_1)$ and $(X_2, \mathcal{C}_2)$, $X_1 \cap X_2 = \emptyset$, be convexity spaces with Krasnosel'skii parameters k_1 and k_2, respectively. Then for the Krasnosel'skii parameter k of $(X_1 \cup X_2, \mathcal{C}_U)$ we have $k = k_1 + k_2$.*

Proof. First let us notice that a set S is a K-set in $(X_1 \cup X_2, \mathcal{C}_U)$ if and only if $S = S_1 \cup S_2$, where S_i is a K-set in $(X_i, \mathcal{C}_i)$, $i = 1, 2$.

Take a K-set S in $(X_1 \cup X_2, \mathcal{C}_U)$ such that any its $k_1 + k_2$ points are seen from a common point in S via S. We will show that S is starshaped. The assumption that any $k_1 + k_2$ points from S are seen from a common point in S via S implies that either any k_1 points from S_1 are seen from a common point in S_1 via S_1, or any k_2 points from S_2 are seen from a common point in S_2 via S_2. From the definition of the Krasnosel'skii parameter it follows that either $\ker(S_1) \neq \emptyset$, or $\ker(S_2) \neq \emptyset$. This, in virtue of the equality

$$\ker(S) = \ker(S_1) \cup \ker(S_2)$$

ends the proof. ∎

One can easily extend the definition of the sum space to any finite collection of convexity spaces. Then in Theorem 5 instead of two spaces we can consider any finite family of convexity spaces with Krasnosel'skii parameters $k_1, \ldots, k_n$. Standard argument reveals that the Krasnosel'skii parameter k of $(\bigcup_{i=1}^n X_i, \mathcal{C}_U)$ satisfies then $k = \sum_{i=1}^n k_i$.

5. TWO REMARKS CONCERNING K-SETS

Since the notion of K-sets is crucial for Krasnosel'skii-type parameters of convexity spaces we end the paper with two remarks about K-sets. The first one deals with K-sets in the product space. Clearly, products of K-sets are also K-sets. However the projection of a K-set in $(X_1 \times X_2, \mathcal{C}_\Pi)$ need not be a K-set in $(X_i, \mathcal{C}_i)$ which is illustrated by the following example.

Example 2. Consider the product space $(\mathbb{R}^2, \mathrm{clconv} \otimes \mathrm{clconv})$, where clconv denotes the family of closed convex sets (closed intervals) in $\mathbb{R}$. Take $S = \{(x, y) \;:\; x^2 + y^2 < 1\}$. First we will show that S is a K-set in $(X_1 \times X_2, \mathcal{C}_\Pi)$. Of course we have $(0,0) \in \ker(S)$. On the other hand, as is easy to check, we get

$$\begin{aligned} \ker(S) &= \bigcap\{\mathrm{st}(x, S) \;:\; x \in S\} \subset \bigcap\{\mathrm{st}(q_i, S) \;:\; i \in \mathbb{N}\} \cap \bigcap\{\mathrm{st}(p_i, S) \;:\; i \in \mathbb{N}\} \\ &= \{(0,0)\}, \end{aligned}$$

where $q_i = (1 - \frac{1}{i}, 0)$ and $p_i = (0, 1 - \frac{1}{i})$.

Similarly we have $(0,0) \in \bigcap\{\mathcal{C}_\Pi(\mathrm{st}(x,S)) : x \in S\}$ and

$$\begin{aligned}\bigcap\{\mathcal{C}_\Pi(\mathrm{st}(x,S)) : x \in S\} &\subset \bigcap\{\mathcal{C}_\Pi(\mathrm{st}(q_i,S)) : i \in \mathbb{N}\} \cap \bigcap\{\mathcal{C}_\Pi(\mathrm{st}(p_i,S)) : i \in \mathbb{N}\}\\ &= \{(0,0)\}\end{aligned}$$

so S is indeed a K-set. However, for $\pi_1 S = \{x : -1 < x < 1\}$ we get

$$\ker(\pi_1 S) = \pi_1 S \quad \text{but} \quad \bigcap\{\mathcal{C}_1(\mathrm{st}(x,\pi_1 S))\} = \{x : -1 \leq x \leq 1\} \neq \pi_1 S.$$

Hence $\pi_1 S$ is not a K-set in $(\mathbb{R}, \mathrm{clconv})$. ∎

The notion of visibility cell plays a part in starshaped sets, see [3, 4]. For each nonempty subset $S \subset X$ and each point $x \in S$ the *visibility cell* of x relative to S is the set

$$\mathrm{vis}(x,S) = \{y \in \mathrm{st}(x,S) : \mathrm{st}(x,S) \subset \mathrm{st}(y,S)\}.$$

In [4] it is shown that for any nonempty set $S \subset X$ the equality $\ker(S) = \bigcap\{\mathrm{vis}(x,S) : x \in S\}$ holds. It turns out that in the case of K-sets we also have the following

Property 4. *If S is a nonempty* K*-set in $(X,\mathcal{C})$, then*

$$\ker(S) = \bigcap\{\mathcal{C}(\mathrm{vis}(x,S)) : x \in S\}.$$

Proof. It is clear that $\mathrm{vis}(x,S) \subset \mathrm{st}(x,S)$. Hence we get

$$\begin{aligned}\ker(S) &= \bigcap\{\mathrm{vis}(x,S) : x \in S\} \subset \bigcap\{\mathcal{C}(\mathrm{vis}(x,S)) : x \in S\}\\ &\subset \bigcap\{\mathcal{C}(\mathrm{st}(x,S)) : x \in S\} = \bigcap\{\mathrm{st}(x,S) : x \in S\}\\ &= \ker(S)\end{aligned}$$

and the property is established. ∎

From the proof of Property 4 it follows that for any K-set $S \subset X$ we have

$$\bigcap\{\mathcal{C}(\mathrm{vis}(x,S)) : x \in S\} = \bigcap\{\mathcal{C}(\mathrm{st}(x,S)) : x \in S\}.$$

Let us notice that the equality can fail for sets which are not K-sets. This can be shown as follows. In $(\mathbb{R}^2, \mathrm{conv})$ take $S = \{q = (x,y) : 0 < x^2 + y^2 < 1\}$. For $q = (x,y) \neq (0,0)$ consider two half-lines $l_q^+ = \{qt : t > 0\}$ and $l_q^- = \{-qt : t \geq 0\}$. One can observe that for $q \in S$ we have

$$\mathrm{st}(q,S) = S \setminus l_q^- \quad , \quad \mathrm{vis}(q,S) = S \cap l_q^+$$

and

$$\mathrm{conv}(\mathrm{st}(q,S)) = S \quad , \quad \mathrm{conv}(\mathrm{vis}(q,S)) = S \cap l_q^+.$$

The above implies

$$\bigcap\{\mathrm{conv}(\mathrm{st}(q,S)) : q \in S\} = S$$

but

$$\bigcap\{\mathrm{conv}(\mathrm{vis}(q,S)) : q \in S\} = \emptyset.$$

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