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FUNCTIONAL EQUATIONS IN UTILITY AND GAME THEORY

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ABSTRACT:
There is a collection of functional equations that naturally arise when studying utility and game theory. The aim of this paper is to introduce as a survey such kind of equations, showing that they play an important role in order to find a relationship between those theories.

1. INTRODUCTION.

In the analysis of several questions related to utility and game theory some functional equations that cannot be considered classical (i.e.: differential equations, partial differential equations, integral equations, and finite-difference equations) arise in a natural way. Such equations, consequently, deserve a particular study. It is also noticeable that those equations link both contexts: As we shall comment here, they allow us to interprete some problems coming from utility theory by using game theory instead, and viceversa, because problems about optimization

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of preferences have a natural translation into problems of existence of equilibrium points in a two-person zero-sum game. The underlying game will be ruled by a payoff function which is closely related to the preferences or utilities considered, and in addition such payoff function will satisfy a functional equation of the aforementioned kind. The correspondence between utility and games, quite simple in nature, is not well-known in the literature: On the one hand, one can find results about optimization of preferences, and, on the other hand, there are results concerning the solution of two-person zero-sum games, but it is not usual to find a translation from one of such contexts into the other. Moreover, in classical texts that deal with such kind of functional equations, as Eichhorn [1978], Aczél [1987] or Castillo and Ruiz [1992], and also in the more advanced ones Smítal [1988] and Aczél and Dhombres [1989], despite there are mentions to applications in Economics and Social Sciences related to utility functions, almost nothing is said as regards the use of such equations in game theory. In this paper, it is our intention to fill this gap. It is also necessary to say that the relationship between utility and games may be suitably used to the effective computation of maximals for a given utility function: While in game theory it is well-known that through linear programming techniques some equilibrium points can be located, (see, e.g., Prawda [1987], p. 742), it is not so frequent to find a similar frame when dealing with utility functions or preference relations.

2. PRELIMINARIES.

The classical context of utility theory deals with a nonempty set X on which a binary relation $\mathcal{R}$ has been defined, looking for conditions that guarantee the existence of a *representation* from $(X, \mathcal{R})$ into the real line $\mathbb{R}$ endowed with the natural order "$\geq$", given by a real-valued function $u : X \to \mathbb{R}$ such that $x\mathcal{R}y \iff u(x) \geq u(y) \quad (x, y \in X)$. When such a representation exists, "u" is said to be a *utility function* for $(X, \mathcal{R})$. The mere existence of a utility function forces the binary relation "$\mathcal{R}$" to have restrictive properties, namely, it must be a *total preorder* (i.e.: transitive and total). Plainly, not all the binary relations that one could take into consideration when looking for applications into Economics or Social Choice will be total preorders. Therefore, it is usual to deal with other kinds of numerical representations in order to analyze such possible binary relations defined on a set X. For instance, given a pair (x, y) with $x\mathcal{R}y$, one can think about the existence of a suitable *bivariant function* $F : X \times X \to \mathbb{R}$ giving rise to a "representation" as $x\mathcal{R}y \iff F(x, y) \geq 0 \quad (x, y \in X)$. Representations of this kind are much more general that those obtained by utility functions for (representable) total preorders, as we may observe by just defining $F(x, y) = u(x) - u(y) \ (x, y \in X)$, where $u : X \to \mathbb{R}$ is a utility function for $(X, \mathcal{R})$. Notice also that the formula $F(x, y) = u(x) - u(y) \ (x, y \in X)$ corresponds to a functional equation whose solutions are the bivariant functions $F : X \times X \to \mathbb{R}$ such that $F(x, y)$ can be decomposed as the difference of the values that a function u of only one variable takes on two different points x and y of its domain. However, it seems more adequate to express the above equation $F(x, y) = u(x) - u(y)$ in an intrinsic equivalent manner, depending only on F. Such formulation exists and is given by the so called *Sincov's functional equation:* $F(x, y) + F(y, z) = F(x, z) \quad (x, y, z \in$

X). (See Castillo and Ruiz [1992], Ch. 4).

Thinking from a different starting-point, one could define a simple two-person zero-sum game through a binary relation $\mathcal{R}$ stated on a set X: Each player takes an element of X, so that when x is the strategy of the first player, and y the one of the second player, the payoff function F is constructed to verify $F(x,y) \geq 0 \iff x\mathcal{R}y$. So we see that we reobtain the idea of representability of the structure $(X, \mathcal{R})$ by means a bivariant function. Remember that an *equilibrium point* of a two-person zero-sum game is a pair $(x^*, y^*) \in X \times X$ such that $F(x, y^*) \leq F(x^*, y^*) \leq F(x^*, y)$, for every $x, y \in X$. Hence we have that when $F(x,y)$ satisfies Sincov's functional equation, and, consequently, there exists $u : X \to \mathbb{R}$ such that $F(x,y) = u(x) - u(y)$ $(x, y \in X)$, the occurrence of an equilibrium point (x^*, y^*) corresponds to a situation in which both x^* and y^* are maxima in X of the utility function u.

(For further information on elementary concepts coming from utility and game theory consult Burger [1963], Takayama [1985], Binmore [1994], or Bridges and Mehta [1995]).

3. FUNCTIONAL EQUATIONS RELATED TO UTILITY THEORY, AND THEIR INTERPRETATION THROUGH GAMES.

Once we have introduced Sincov's equation, we can state the following easy result:

PROPOSITION 1. :
(i) Every solution of Sincov's equation on a nonempty set X, generates a total preorder "$\mathcal{R}$" on X, such that the structure $(X, \mathcal{R})$ is representable by a utility function.
(ii) Given a nonempty set X endowed with a total preorder $\mathcal{R}$, the existence of a utility function for $(X, \mathcal{R})$ is equivalent to the existence of a solution $F : X \times X \to \mathbb{R}$ of Sincov's equation, such that $x\mathcal{R}y \iff F(x,y) \geq 0$ $(x, y \in X)$.

PROOF: To prove (i), let R be given by $x\mathcal{R}y \iff F(x,y) \geq 0$ $(x, y \in X)$. Part (ii) follows now immediately: Given a utility function u we obtain F as $F(x,y) = u(x) - u(y)$ $(x, y \in X)$. Conversely, if F is known, fix $x_0 \in X$ and define $u(x) = F(x, x_0)$ $(x \in X)$. ■

As we pointed out in the previous section 2, the consideration of suitable functional equations and bivariant maps may help us to find numerical representations of orderings more general than the total preorders. We will find possible characterizations of a wider family of orderings, preferences, or binary relations. These ideas were already implicitely given in Shafer [1974]. In that paper, Shafer called *preference* to any total binary relation "$\mathcal{R}$" defined on a nonempty set X. Then he realized that the mere definition of a preference on a nonempty set X corresponds to the presence of a solution of the *functional equation of skew-symmetry,* that is, a bivariant map $F : X \times X \to \mathbb{R}$ such that $F(x,y) + F(y,x) = 0$ $(x, y \in X)$. In this direction, Shafer proved the following result.

PROPOSITION 2. : *Let X be a nonempty set. Then, defining a preference "$\mathcal{R}$" on X is equivalent to finding a bivariant map $F : X \times X \to \mathbb{R}$ that is a solution of the functional equation of skew-symmetry.*

PROOF: If the preference "$\mathcal{R}$" is given, and we define $F(x,y) = 0 \iff x\mathcal{R}y$, $y\mathcal{R}x$; $F(x,y) = 1 \iff \neg(y\mathcal{R}x)$; $F(y,x) = -F(x,y)$, $(x,y \in X)$, the so defined bivariant map is skew-symmetric. On the other hand, if $F : X \times X \to \mathbb{R}$ is skew-symmetric, the binary relation "$\mathcal{R}$" given by $x\mathcal{R}y \iff F(x,y) \geq 0$ is plainly total, so it is a preference. ■

The fact of being the functional equation of skew-symmetry the starting point of the consideration of functional equations to deal with preferences, also has immediate implications as regards game theory: Just think about a two-person zero-sum game whose strategy sets coincide and with a payoff funcion F such that $F(x,y) = -F(y,x)$. This corresponds to a game in which if the players exchange their strategies one another, the payoff is exactly the contrary. (So, in particular, the identity of the players, or the turn in which they play, have no influence in the payoff). Thus we get the functional equation of skew-symmetry. Obviously from the point of view of the first player a two-person zero-sum game with the same strategy set $X \neq \emptyset$ for both players, defines a preference $\mathcal{R}$ on X, by just understanding $x\mathcal{R}y$ if and only if the first player wins ($F(x,y) \geq 0$, $F : X \times X \to \mathbb{R}$ being the payoff function for the first player) provided that his strategy is $x \in X$ and $y \in X$ is the strategy of the second player.

Following with our analysis of preferences that are not necessarily given by total preorders we can directly start from binary relations that come from suitable bivariant maps that solve some functional equations. Coming again to Sincov's functional equation, whose solutions are bivariant maps $F : X \times X \to \mathbb{R}$ such that $F(x,y)+F(y,z) = F(x,z)$ $(x,y,z \in X)$, the following generalizations are natural:
(i) $F(x,y) + F(y,z) = F(x,z) + F(y,y)$ $(x,y,z \in X)$,
(ii) $F(x,y) + F(y,z) = F(x,z) + K$ $(x,y,z \in X)$, $K \in \mathbb{R}$ being a constant.

Certainly, Sincov's equation is a particular case of the equation in (ii), taking $K = 0$. It also appears under (i), because if F solves Sincov's equation, then $F(y,y) = 0$, for every $y \in X$.

Now we wonder whether a binary relation "$\mathcal{R}$" defined by $x\mathcal{R}y \iff F(x,y) \geq 0$ $(x,y \in X)$, where F is now a solution of such generalized Sincov's equation, corresponds to some kind of classical and well-known binary relation, of the various existing in the literature. This indeed happens under some technical restrictions, for which the Sincov's generalized equation (i) gives rise to interval-order structures (see Bridges [1983, 1985, 1986]), while the generalized equation (ii) is related to semiorders (see Luce [1956] or Gensemer [1987]).

Let us introduce now such classical concepts.

DEFINITION: Let X be a nonempty set endowed with a reflexive binary relation "$\mathcal{R}$". The structure $(X,\mathcal{R})$ is said to be an *interval-order* if for every $x,y,z,t \in X$ it happens that when $x\mathcal{R}y$ and $z\mathcal{R}t$, then either $x\mathcal{R}t$ or $z\mathcal{R}y$ must hold. The structure $(X,\mathcal{R})$ is a *semiorder* if it is an order-interval such that, in addition, for every $x,y,z,t \in X$, it happens that when $x\mathcal{R}y$ and $y\mathcal{R}z$ then either $x\mathcal{R}t$ or $t\mathcal{R}z$ must hold. An structure of interval-order is said to be *representable* if there exist real-valued functions $u,v : X \to \mathbb{R}$ such that $u(x) \leq v(x)$ for every $x \in X$ and also $x\mathcal{R}y \iff u(y) \leq v(x)$ $(x,y \in X)$. An structure of semiorder is *representable* if there exist a function $u : X \to \mathbb{R}$ and a constant $K \in [0,+\infty)$ such that calling

$v(x) = u(x) + K$ $(x \in X)$, the underlying interval-order structure $(X, \mathcal{R})$ is representable through the pair of functions $u, v : X \to \mathbb{R}$.

PROPOSITION 3. : *When X is countable, every structure of interval-order or semiorder $(X, \mathcal{R})$ is representable.*

PROOF: In what concerns semiorders, this fact may be seen in Scott and Suppes [1958]. As regards interval-orders, it appears in Bridges [1983]. A detailed discussion of these facts may also be seen in Subiza [1992]. ■

Assume that an interval-order or semiorder structure $(X, \mathcal{R})$ is representable through the corresponding functions $u, v : X \to \mathbb{R}$. Define the bivariant map $F : X \times X \to \mathbb{R}$ by $F(x,y) = v(x) - u(y)$ $(x, y \in X)$. Obviously we have that $x\mathcal{R}y \iff F(x,y) \geq 0$. Moreover, the corresponding map F satisfies a suitable functional equation, namely, in the case of interval-order we have that $F(x,y) + F(y,z) = F(x,z) + F(y,y)$ $(x,y,z \in X)$, and in the case of semiorder $F(x,y) + F(y,z) = F(x,z) + K$, or equivalently: $F(x,y) + F(y,z) = G(x,z)$ $(x,y,z \in X)$, for some bivariant map $G : X \times X \to \mathbb{R}$. Thus, in case of representability the structures of interval-order and semiorder give rise to the possibility of finding suitable solutions of appropiated functional equations. For the case of semiorders, several sufficient conditions of representability are known (see Gensemer [1987]). For the case of interval-orders, some sufficient condition of representability is also known, as may be seen in Fishburn [1970a], Chateauneuf [1987] or Ch. 6 in Bridges and Mehta [1995]. Finally in Oloriz et al. [1997] a characterization of the representability of interval-orders has recently been achieved.

In other recent paper (Rodríguez-Palmero [1996]) possible relationships between acyclic binary relations and bivariant maps that are solution of functional equations, that generalize in a further step the ones that appeared for interval-orders and semiorders, have also been studied.

The classical concept of acyclic binary relation is defined as follows:

DEFINITION: Let X be a nonempty set and "$\mathcal{R}$" a binary relation on X. Then "$\mathcal{R}$" is said to be *acyclic* if for any $n \in \mathbb{N}$ it holds that, for every $x_1, \ldots, x_n \in X$, if $x_1\mathcal{R}x_2$, $x_2\mathcal{R}x_3$, $\ldots$, $x_{n-1}\mathcal{R}x_n$, then never happens that $x_n\mathcal{R}x_1$.

An outlook to the bivariant maps $F : X \times X \to \mathbb{R}$ that are representative of an structure of semiorder or interval-order $(X, \mathcal{R})$ shows that we must consider functions $u, v : X \to \mathbb{R}$ such that $F(x,y) = v(x) - u(y)$ $(x, y \in X)$. Moreover, in the case of semiorder $v(x) = u(x) + K$. In both cases we have that $x\mathcal{R}y \iff v(x) \geq u(y) \iff -u(y) \geq -v(x) \iff u(x) - u(y) \geq u(x) - v(x)$. Now observe that the so obtained representations are given by expressions of the kind: $x\mathcal{R}y \iff u(x) - u(y) \geq G(x)$. Notice, in addition, that this kind of expression also appears when dealing with utility functions that represent total preorders. In sum, the case $G(x) = 0$ $(x \in X)$ corresponds to representable total preorders, the case $G(x) = -K$, $(K > 0)$ corresponds to semiorders, and the case when $G(x)$ is a function of only one variable $(x \in X)$ and taking values in $(-\infty, 0]$ corresponds to interval-orders. This panorama is completed in Rodríguez-Palmero [1996], where it has been proved, generalizing these ideas, that representations of the kind $x\mathcal{R}y \iff u(x) - u(y) > G(x,y)$ $(x, y \in X)$ appear for certain families of acyclic binary relations, $G : X \times X \to \mathbb{R}$ being now a bivariant map. The interesting question of finding suitable functional equations that either the bivariant map G or its associated $F : X \times X \to \mathbb{R}$ given by $F(x, y) = u(x) - u(y) - G(x, y)$ could verify, depending on the acyclic binary relations considered, remains as an open problem.

Now we may observe that the study of interval-orders furnishes bivariant maps F on a nonempty set X, such that there exist functions $u, v : X \to \mathbb{R}$ with $F(x, y) = v(x) - u(y)$ $(x, y \in X)$. If we forget the sign of v, we may say that F is decomposed or separated as the sum of two functions of only one variable, that is, $F(x, y) = G(x) + H(y)$ $(x, y \in X)$, for some functions $G, H : X \to \mathbb{R}$. This gives rise to the *functional equation of separability* (also known as *equation of additivity,* see, e.g., Tanguiane [1981] or Wakker [1993]).

Another simple functional equation arises by changing a sign in the equation of skew-symmetry, so obtaining, $F(x, y) = F(y, x)$, X being a nonempty set, $x, y \in X$ and $F : X \times X \to \mathbb{R}$ being a bivariant map. For evident reasons, this equation is called the *equation of symmetry.* Actually, it also appears in the mathematical analysis of preference: For instance, when dealing with a total ordering, the equation of symmetry is trivially satisfied after making a restriction of the preorder to any indifference class. Nevertheless, it is more interesting the interpretation of the equation of symmetry in a context of game theory, dealing with two-person games that now may or may not be of zero-sum. To put an example, consider a firm whose benefits depend on the money invested for two agents. The benefits are fifty-fifty distributed, independently of the investement made by each agent. If the contribution of the first agent amounts x units and the second agent contributes with y units, the benefit of any agent is $B(x, y)$. This corresponds to a two-person game with the same set of strategies X for each agent, X corresponding to the set of all possible investements that could be made for any agent, and $B : X \times X \to \mathbb{R}$ being the payoff function of no matter which agent, since they are plainly interchangeable. It is clear that $B(x, y) = B(y, x)$ $(x, y \in X)$. There are important particular cases of the symmetry equation. For instance, if on X a commutative binary operation, that will be denoted by "+", has been defined, then a functional equation as $B(x, y) = H(x + y)$ $(x, y \in X)$, $H : X \to \mathbb{R}$ being a suitable numerical function, corresponds to one of those particular cases, that could be used to interprete a situation in which the benefit depends only on the total amount of money that has been invested. In addition, the structure and properties of the funtion H give rise to several equations that are classical in the specialized literature. Let us see three such examples:

i) $H(x + y) = H(x) + H(y)$ *(Cauchy),*

ii) $H(x+y) = K(x)+L(y)$, where $K, L : X \to \mathbb{R}$ are functions on only one variable *(Pexider),*

iii) $H(x + y) = 2 \cdot H(x) \cdot H(y) - H(x - y)$ *(D' Alembert).*

(For further information consult Aczél and Dhombres [1989], Smítal [1988], or else Castillo and Ruiz [1992]).

4. OPTIMIZATION OF BINARY RELATIONS AND EQUILIBRIUM POINTS

IN TWO-PERSON ZERO-SUM GAMES.

A fundamental reason to introduce a binary relation of preference on a given set is the idea of taking elements that are in a sense maximal, that is, they are the best possible as regards a property that is represented by the relation of preference. So we will be looking for elements such that none is strictly preferred to. The search for this kind of elements is called *optimization of preferences,* that has been studied in some specifical work, using preference relations or utility functions with some concrete additional properties. (See, e.g., Bergstrom [1975], Walker [1977], Yannelis and Prabhakar [1983], Campbell and Walker [1990], Tian [1993], Peris and Subiza [1994], Llinares [1994], or the panoramic Chapter 7 in Border [1985]).

We shall use the following definition of maximal element as regards a binary relation $\mathcal{R}$ defined on a nonempty set X:

DEFINITION: Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. An element $x \in X$ is said to be *strict maximal* as regards $\mathcal{R}$ if there is no $z \neq x$ such that $z\mathcal{R}x$. It is called *weak maximal* (or, simply, *maximal)* if for every $z \in X$ with $z\mathcal{R}x$ it holds that $x\mathcal{R}z$.

A well-known particular case about the existence of maximal elements appears when considering acyclic binary relations. For example, in Sen [1970] was already proved the following fact:

PROPOSITION 4. : *Let $\mathcal{R}$ be an acyclic binary relation on a nonempty and finite set X. Then there exists at least one strict maximal element as regards $\mathcal{R}$.*

PROOF: Let $x_1 \in X$. If it is not strict maximal, then there exists $x_2 \neq x_1$ such that $x_2\mathcal{R}x_1$. If x_2, in its turn, is not strict maximal, either, then we can find an element $x_3 \neq x_2$ with $x_3\mathcal{R}x_2$. Since $\mathcal{R}$ is acyclic it follows that $x_1 \neq x_3$. Proceeding in the same way, we will finally get an element $x_k \in X$ that is strict maximal. (Observe that the process stops because X is finite.) ■

Also in Fishburn [1970b] acyclic preferences are considered, but in a context in which some topological condition on X is required. Under some topological condition it is proved that if $\mathcal{R}$ is an acyclic binary relation defined on X, then there exists a function $U : X \to \mathbb{R}$ such that $x\mathcal{R}y \implies U(x) > U(y) \quad (x, y \in X)$, where U upper semicontinuous. Observe that, in general, U is no longer a utility function. The implication $x\mathcal{R}y \implies U(x) > U(y) \quad (x, y \in X)$ goes, unless otherwise proved, only in one sense. A function U with the above property is called a *pseudoutility function* for $(X, \mathcal{R})$, that is said to be *pseudorepresentable.* In several other works in the literature this kind of pseudorepresentations have been considered for certain families of binary relations. (See, e. g., Peleg [1970]). Fishburn argues that since U is upper semicontinuous, it takes relative maxima on any compact subset of X. He uses this fact to prove that when X is compact, then the points in which U takes a maximum are strict maximal as regards $\mathcal{R}$. Thus we see again that the keys of Fishburn's result lean on conditions that are topological in nature.

This kind of conditions also appear in other results about optimization of binary relations and preferences. We present now one of these topological results:

PROPOSITION 5. : *Let $\mathcal{R}$ be a binary relation defined on a nonempty set X endowed with a topology τ. If X is τ-compact and for every $x \in X$ the set $\{y \in X : y\mathcal{R}x\}$ is τ-open, then there is at least one strict maximal for $\mathcal{R}$ under any of the following two conditions:*
i) $\mathcal{R}$ is irreflexive and transitive,
ii) $\mathcal{R}$ is acyclic.

PROOF: The result under condition i), appears in Rader [1972], p. 134. The reinforced result, obtained under condition ii), may be seen in Bergstrom [1975]. ■

As regards game theory, topological conditions either on the strategy sets or on the payoff functions, are also a key to get results about the existence of equilibrium points in two-person zero-sum games. This becomes natural, if we realize that some of those results could come from a translation between the contexts of games and binary relations.

Before including here some of the classical results encountered in the study of the existence of equilibrium points in two-person zero-sum games, let us briefly comment a possible interpretation of binary relations by means of games, and viceversa.

Let $\mathcal{R}$ be a binary relation defined on a nonempty set X. We construct a two-person zero-sum game, as follows: The set X will be the strategy set for both players. The payoff function, $F : X \times X \to \mathbb{R}$, is conditionned to satisfy $F(x,y) \geq 0 \iff x\mathcal{R}y$. For instance, we can always take $F(x,y) = 1$ if $x\mathcal{R}y$, and $F(x,y) < 0$ otherwise.

On the other hand, given a two-person zero-sum game with the same set of strategies X for both players, and $F : X \times X \to \mathbb{R}$ being the payoff function, we can consider a binary relation $\mathcal{R}$ on X given by $x\mathcal{R}y \iff F(x,y) \geq 0$. This last construction is perhaps too naive, but it provides good interpretations when F satisfies some additional property as, say, solving a generalized Sincov's equation. In this line, the following propositions appear:

PROPOSITION 6. : *Let $\mathcal{R}$ be a total binary relation defined on a nonempty set X. Then an element $x^* \in X$ is strict maximal as regards $\mathcal{R}$ if and only if (x^*, x^*) is an equilibrium point for the two-person zero-sum game associated to $\mathcal{R}$, whose payoff function is $F(x,y) = 1 \iff y\mathcal{R}x$; $F(x,y) < 0 \iff \neg(y\mathcal{R}x)$.*

PROOF: Given $x, y \in X$ it follows, since $\mathcal{R}$ is total and x^* is strict maximal, that $x^*\mathcal{R}x$ and $y\mathcal{R}x^* \iff y = x^*$. Thus $F(x,x^*) \leq F(x^*,x^*) \leq F(x^*,y)$. Therefore (x^*,x^*) is an equilibrium point. The converse is analogous. ■

PROPOSITION 7. : *Let Γ be a two-person zero-sum game whose strategy set $X \neq \emptyset$ is the same for both players, and the payoff function is given by a bivariant map $F : X \times X \to \mathbb{R}$ that solves the functional equation $F(x,y) + F(y,z) = F(x,z) + F(y,y)$ $(x,y,z \in X)$ and such that $F(a,a) \geq 0$ for every $a \in X$. Let $u, v : X \to \mathbb{R}$ be functions such that $F(x,y) = v(x) - u(y)$ $(x,y \in X)$. Then the following conditions are equivalent:*
i) (x^*,y^*) is an equilibrium point of the game Γ,

ii) x^* and y^* are maximal as regards the binary relation $\mathcal{R}$ defined on X by $x\mathcal{R}y \iff F(x,y) \geq 0$,
iii) x^* and y^* are points in which, respectively, the functions v and u attain their maxima.

PROOF: To prove "$i) \implies ii)$" notice that for all $z, t \in X$ it holds that $F(z,y^*) \leq F(x^*,y^*) \leq F(x^*,t)$. Hence $F(z,y^*) \leq F(x^*,y^*) = F(x^*,z) + F(z,y^*) - F(z,z)$, so $F(x^*,z) \geq F(z,z) \geq 0$. This implies $x^*\mathcal{R}z$ for every $z \in X$, so x^* is maximal as regards $\mathcal{R}$. Similarly y^* is also maximal. The implication "$ii) \implies iii)$" follows immediately, because $x\mathcal{R}y \iff u(y) \leq v(x)$. Finally, let us prove "$iii) \implies i)$": If $x^*, y^* \in X$ are points in which, respectively, v and u attain its maximum, it is plain that for every $z, t \in X$ it holds $F(z,y^*) = v(z) - u(y^*) \leq v(x^*) - u(y^*) = F(x^*,y^*) \leq v(x^*) - u(t) = F(x^*,t)$. So (x^*,y^*) is an equilibrium point. ■

We illustrate Proposition 7 with some easy examples.

Let Γ be a two-person zero-sum game whose strategy set $X = [0,1]$ is the same for both players, and the payoff function is:
(i) $F(x,y) = K$, being $K > 0$ a positive constant,
(ii) $F(x,y) = y - x$,
(iii) $F(x,y) = y^2 - x^2$,
(iv) $F(x,y) = y - x^2$.

In the first case, every (x,y) is an equilibrium point. In the last three cases, $(0,0)$ is the only equilibrium point. Cases (ii) and (iii) correspond to a Sincov's functional equation. In case (ii) we find the solution $F(x,y) = u(x) - u(y)$, with $u(x) = -x$. Case (iii) is similar, with $u(x) = -x^2$.

Case (i) could be interpreted to analyze a semiorder with $u(x) = 0$, and constant $K > 0$. Finally, case (iv) corresponds to a generalized Sincov's equation relative to an interval-order. Here a solution is $F(x,y) = v(x) - u(y)$ with $u(y) = -y$ and $v(x) = -x^2$, $(x, y \in [0,1])$.

To conclude, we present here, as announced, some classical result of topological stuff, about the existence of equilibrium points in two-person zero-sum games.

DEFINITION: Call *n-dimensional unit simplex* to the set of points of $\mathbb{R}^{n+1}$ given by $\Sigma_n = \{(x_1, x_2, \ldots, x_{n+1}) \ : \ x_i \geq 0 \ (i = 1, \ldots, n+1) \ , \ x_1 + \ldots + x_{n+1} = 1\}$.

Notice that the set Σ_n is closed, bounded and convex as regards the Euclidean topology in $\mathbb{R}^{n+1}$.

PROPOSITION 8. *: Every two-person zero-sum game Γ whose strategy sets X_1, X_2 are, respectively, the n and m-dimensional unit simplices, and whose payoff function is given by the restriction to $X_1 \times X_2$ of a bilinear form $A : \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$ has at least an equilibrium point.*

PROOF: This is a restatement of the well-known "minimax theorem" due to von Neumann. See Ch. 7 in Nikaido [1978]. ■

DEFINITION: A function $F : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is said to be *concave* as regards the first p variables, if for every $(x_1, \ldots, x_p, y_{p+1}, \ldots, y_q)$, $(z_1, \ldots, z_p, y_{p+1}, \ldots, y_q)$, and $\lambda \in [0,1]$, it holds that

$$F(\lambda \cdot x_1 + (1-\lambda) \cdot z_1, \ldots, \lambda \cdot x_p + (1-\lambda) \cdot z_p, y_{p+1}, \ldots, y_q) \geq$$

$$\geq \lambda \cdot (F(x_1, \ldots, x_p, y_{p+1}, \ldots, y_q) + (1-\lambda) \cdot F(z_1, \ldots, z_p, y_{p+1}, \ldots, y_q)).$$

In the same way, it is said to be *convex* as regards the q last variables, if for every $(x_1, \ldots, x_p, y_{p+1}, \ldots, y_q)$, $(x_1, \ldots, x_p, t_{p+1}, \ldots, y_q)$, and $\lambda \in [0,1]$, it holds that

$$F(x_1, \ldots, x_p, \lambda \cdot y_{p+1} + (1-\lambda) \cdot t_{p+1}, \ldots, \lambda \cdot y_q + (1-\lambda) \cdot t_q) \leq$$

$$\leq \lambda \cdot (F(x_1, \ldots, x_p, y_{p+1}, \ldots, y_q) + (1-\lambda) \cdot F(x_1, \ldots, x_p, t_{p+1}, \ldots, t_q)).$$

By the way, the cases of equality, that is:

$$F(\lambda \cdot x_1 + (1-\lambda) \cdot z_1, \ldots, \lambda \cdot x_p + (1-\lambda) \cdot z_p, y_{p+1}, \ldots, y_q) =$$

$$= \lambda \cdot (F(x_1, \ldots, x_p, y_{p+1}, \ldots, y_q) + (1-\lambda) \cdot F(z_1, \ldots, z_p, y_{p+1}, \ldots, y_q))$$

for all $(x_1, \ldots, x_p, y_{p+1}, \ldots, y_q)$, $(x_1, \ldots, x_p, t_{p+1}, \ldots, y_q)$, $\lambda \in [0,1]$, and

$$F(x_1, \ldots, x_p, \lambda \cdot y_{p+1} + (1-\lambda) \cdot t_{p+1}, \ldots, \lambda \cdot y_q + (1-\lambda) \cdot t_q) =$$

$$= \lambda \cdot (F(x_1, \ldots, x_p, y_{p+1}, \ldots, y_q) + (1-\lambda) \cdot F(x_1, \ldots, x_p, t_{p+1}, \ldots, t_q))$$

for all $(x_1, \ldots, x_p, y_{p+1}, \ldots, y_q)$, $(x_1, \ldots, x_p, t_{p+1}, \ldots, y_q)$, $\lambda \in [0,1]$, correspond to the aforementioned Cauchy's functional equation. (See Aczél [1987] for further details).

PROPOSITION 9. : *Every two-person zero-sum game Γ whose strategy sets X_1, X_2 are nonempty, closed, bounded and convex subsets of, respectively, $\mathbb{R}^p$ and $\mathbb{R}^q$, and whose payoff function, $A : X_1 \times X_2 \subseteq \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$, is continuous, concave as regards the first p variables, and convex as regards the q last ones, has at least an equilibrium point.*

PROOF: This is a particular case of Nikaido-Isoda theorem, that appears, for instance, on p. 32 in Burger [1963]. ■

An extension of Proposition 9 to the non-convex case has been achieved in Tala and Marchi [1996]. As a noticeable consequence, let us finally observe that when dealing with a two-person zero-sum game whose strategy set X is the same for both players, it is a compact and convex nonempty subset of $\mathbb{R}^n$, and the payoff function F satisfies the functional equation $F(x,y)+F(y,z) = F(x,z)+F(y,y)$ $(x,y,z \in X)$ with $F(a,a) \geq 0$ for every $a \in X$, then the condition of concavity-convexity for F is equivalent to say that the corresponding functions u, v be *concave*. (That is: $u(\lambda \cdot x + (1-\lambda) \cdot y) \geq \lambda \cdot u(x) + (1-\lambda) \cdot u(y)$ $(x, y \in X, \lambda \in [0,1])$, and similarly for v).

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