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ON THE TAME DRAGON

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ABSTRACT. We prove that the boundary of the tame dragon is a Jordan curve J whose interior is a uniform domain. J is the union of six similar Jordan arcs. Each of these arcs is a self-similar set that satisfies the open set condition. J is an s-set with $s \cong 1.21076$. *Precisely, s= =2(log v)/log 2 where* $v=\sqrt[3]{1+\sqrt{26/27}}+\sqrt[3]{1-\sqrt{26/27}}$. *The disk F defined by J is the set of complex numbers that have a binary representation with integer part zero in the base* $\mu=-1/2+i\sqrt{7}/2$.

1. INTRODUCTION. Let $\mu \in \mathbf{C}$, $|\mu|>1$, $D=\{0,1\}$. $\alpha \in \mathbf{C}$ is said *representable* in base μ with ciphers D if there exists a set of digits, $\{a_j \in D\ ; j=M, M-1, M-2,\ldots\}$, such that $\alpha=\sum_{-\infty}^{M} a_j\mu^j$. We write $\alpha = a_M\ldots a_0.a_{-1}a_{-2}\ldots = (e.f)_\mu$ and call (e) the integral part of α and $(.f)$ the fractional part of α. Denote G the set of all representable numbers and define the set F of *fractional numbers* as those numbers in G with a representation such that $(e)=0$ and the set W of *integers* of the system as those with a representation such that $(.f)=0$. A number r will be called a *rational* of the numerical system (μ,D) if it has a finite positional representation, i.e., with $a_j=0$ for $j < J(r)$. U will denote the set of rationals of the system. F will also be denoted by F_0.

In what follows $\mu := -1/2+i\sqrt{7}/2$. L:=[1,μ] is the point-lattice defined by 1 and μ. It holds that W=L and that the Lebesgue measure of F, m(F), equals $|\mathrm{Im}\,\mu| = \sqrt{7}/2$. Besides, G=**C** and $0\in int(F)$, (cf. [Z]). μ satisfies the equation $x^2+x+2=0$ and $|\mu|=\sqrt{2}$. It is easy to see that

$$(1)\qquad |d_0+d_1\mu+d_2\mu^2| \le \sqrt{11} \qquad \text{if } -d_k \in D \text{ or } d_k \in D.$$

The present work completes the results of our paper [Z] providing a proof of its Th. 11. Most of the arguments used are similar to those given in our treatment of the Knuth dragon in [BP] except for particular details. Thus, **when a result is not followed by a proof or a reference we understand that an analogous proposition appears in [BP] and that its formal proof can be repeated almost verbatim in the present case.**

2. GRAPHS OF STATES. Given a representation of the complex number z, $z=\sum_{-\infty}^{L} p_j\mu^j$, and an integer k, we denote with $p(k)$:= the integral part of $z\mu^{-k}$ and call it the *state k of this representation.* If z has another representation $z=\sum_{-\infty}^{L} q_j\mu^j$ then the successive states verify:

(2) $$p(k-1)-q(k-1)=\mu[p(k)-q(k)]+(p_{k-1}-q_{k-1}).$$

We have $p(k-1)-q(k-1)=\sum_{j=1}^{\infty} d_j\mu^{-j}$ with $d_j \in\{0,\pm 1\}$ and by (1),

(3) $$\left|\sum_{j=1}^{\infty} d_j\mu^{-j}\right| \le \sqrt{11}\sum_{j=1}^{\infty}|\mu|^{-3j} = \frac{\sqrt{11}}{\sqrt{8}-1} < 2.$$

Therefore $p(k-1)-q(k-1)\in$ S:=$\{0,\pm 1,\pm\mu,\pm(\mu+1)\}$, (cf. Fig. 3). But $p(k)-q(k)$ also belongs to S and p_{k-1} and q_{k-1} belong to D. These coefficients can be chosen then in a few definite ways. In Fig. 1, the nodes of the graph Γ are the differences $p(k)-q(k)$ of the states $(p(k),q(k))$. The nomenclature we use in that diagram is inspired in that of Gilbert ([G1], [G2]). Specifically, $|qp|$ and $q|p$ mean that $p(k)-q(k)=0$ and 1 respectively and $\frac{|p}{q|}$ and $\frac{p|}{|q}$ that $p(k)-q(k)=\mu+1$ and μ respectively. According to (2), the vector $\begin{vmatrix} p_{k-1} \\ q_{k-1} \end{vmatrix}$ beside the arrow yields the transition to the state $(p(k-1),q(k-1))$.

THEOREM 1. *Each number with two different representations is associated to an infinite string in the graph Γ that starts in a node of the graph. Conversely, each such an infinite string is associated to a number $z\in F$ with more than one representation that* *is uniquely determined if p(0)=0, q(0)∈S\{0}. Numbers of the form* $w\mu^m$, $w \in W$, $m \in Z$, *i.e., the rational numbers, have only one representation* ■

Γ

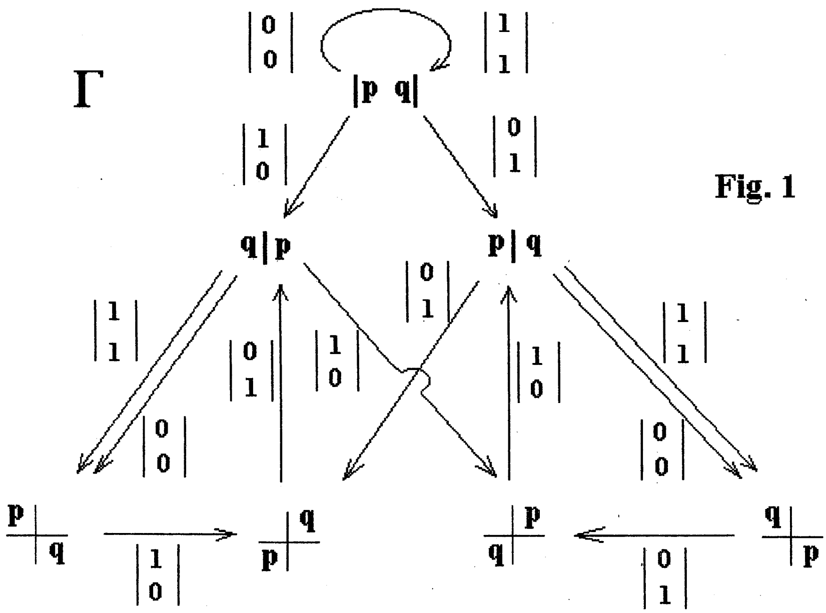

Fig. 1

In particular 0 has only one representation. Let us define $F_g := F + g, g \in W = L$. Then $F \cap F_g \neq \varnothing$ if and only if $g \in S$. In the diagram of the graph τ in Fig. 2 we used the following notation: $\frac{r|p}{q}$ means p(k)–q(k)=1+μ, r(k)–q(k)=μ, p(k)–r(k)=1 and $\frac{p}{q|r}$ means p(k)–q(k)=1+μ, r(k)–q(k)=1, p(k)–r(k)= μ.

THEOREM 2. *Let z be a number with three different representations and p(0)=0≠ ≠q(0)≠r(0)≠0. Then p(0), q(0) and r(0) are related as in one of the nodes of the graph τ and the succesive ciphers of these representations can be read following the graph from the columns beside the arrows.*

Each infinite string of τ that starts in one of the nodes defines a unique complex number z∈F if p(0)=0. The ciphers of the three representations of z are the entries in the columns beside the arrows. There is no number with four representations.

If z has three representations then z'=w+z, w∈W, has also three representations. z and z' are associated to the same infinite string of τ. These numbers are ultimately periodic with period 100 *or* 110 ■

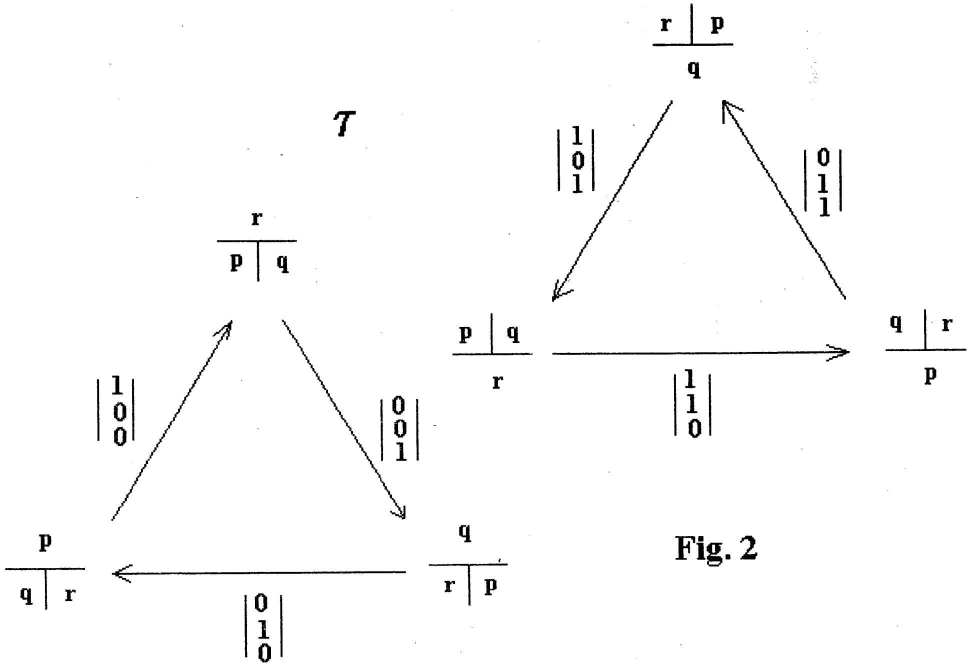

Fig. 2

3. THE COMPACT SET F. The contractions $\Phi_0(z)=z/\mu$ and $\Phi_1(z)=(z+1)/\mu$ could be used to define the set F since $F=\Phi_0(F)\cup\Phi_1(F)$. F is a disk, as will be shown, whose boundary is a Jordan curve that looks like the curve exhibited in Fig. 5. We obtain from (3) that |F|:=diam F <2. The family $\{F_w : w\in W\}$ is a *tessellation of the plane* in the sense that $R^2=\bigcup\{F_g : g\in W\}$ and that any two different sets of the family have an intersection of Lebesgue measure zero. This fact will be established in §6 but was already proved in [Z], Th. 10. Fig. 4 shows the set F_0 (=F) and its (exactly) six neighbors (cf. Th. 1). *F* will denote* the set of rational numbers of (μ,D) in *F*.

THEOREM 3. *i) Let* $g\in W$ *and k be a nonnegative integer such that* $|g|\le(\sqrt{2})^k+3$. *Then, g has a positional representation with no more than k*+8 *ciphers.*

ii) If $z\in\mathbf{C}$ *and* $|z|<(\sqrt{2})^{-8}$ *then* $z\in F$.

iii) $F^* \subset F^o = int(F)$ *and* $F = \overline{F^*} = \overline{F^o} = cl(int\,(F))$.

iv) Let $g \in S \setminus \{0\}$. *Then* $z \in F \cap F_g$ *if and only if z is associated to an infinite string of the graph* Γ *that starts at the node corresponding to the type of the state* $(0,g)$.

v) Assume $g \in S/\{0\}$ *and* $z \in F \cap F_g$ *Then, neither radix representation of z has more than* 3 *consecutive equal ciphers after the point.*

vi) If $0 \neq g \in W$ *then* $F^o \cap F_g = \varnothing$. ■

8 is the maximum number of ciphers necessary to represent the integers of the numerical system of modulus not greater than 4 (cf. Table 1). As a matter of fact, $4=(11100100)_\mu$ and $4+\mu=(11100110)_\mu$ are the only ones of these that need 8 ciphers to be represented.

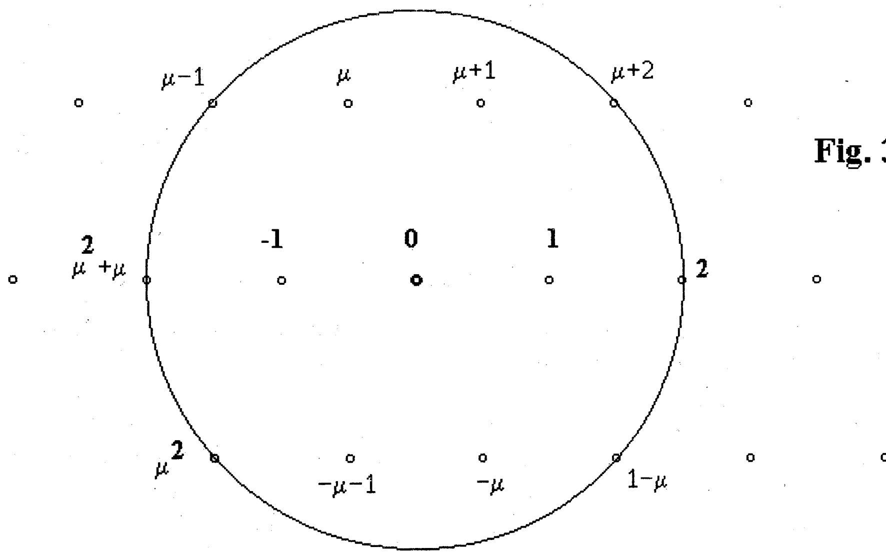

Fig. 3

$\mu = -1/2 + i\sqrt{7}/2$ $\mu-1$=111001 $\mu+2$=11100 -2=110 -1=111 2=1010

$\mu^2 + \mu + 2 = 0$ $-\mu-2$=100 $-\mu-1$=101 $-\mu$=1110 $1-\mu$=1111

[Th. 3, i) implies that 0 has a unique positional representation in (μ, D), (cf.[Z],§2), a fact that we deduced from Th. 1.]

4. THE BOUNDARY OF F. Most of the times in F_g we shall replace g by its representation in the numerical system (μ,D). It will be clear from the context the meaning of the subindex. So, $F_{-1} = F_{111}, F_\mu = F_{10}, F_{-\mu} = F_{1110}$ and $F_{\mu+1} = F_{11}, F_{-1-\mu} = F_{101}$.

DEFINITION 1. *$J:=\partial F$; A, C, B, A^, C^, B^ are the intersections of F with g+F where g is, respectively, 1, $\mu+1$, μ, -1, $-\mu-1$, $-\mu$.*

We obtain from theorems 1 and 2 that: $J=A \cup B \cup C \cup A\hat{} \cup C\hat{} \cup B\hat{}$, (cf. Fig. 5).

Fig. 4

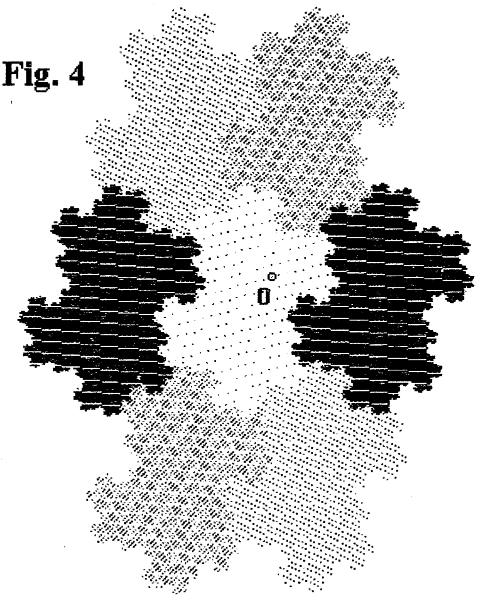

Fig. 5

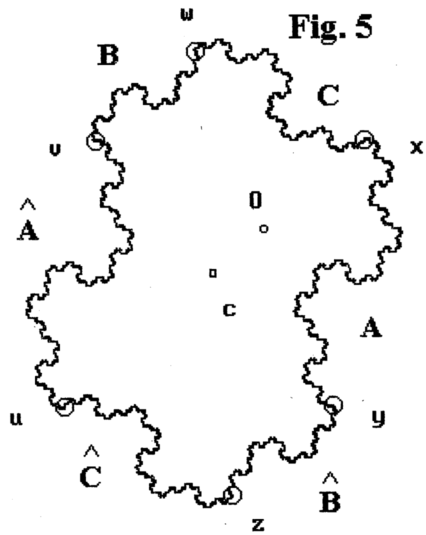

TABLE 1

Table 1 shows the positional representation in base μ of numbers $w \in W$ of modulus not greater than four. The integer at the right is the square of the modulus of w.

$-4-2\mu=101000$	16	$-1-2\mu=1110011$	7	$1+3\mu=10111$	16
$-4-\mu=101010$	14	$-1-\mu=101$	2	$2-2\mu=1010110$	16
$-4=111100$	16	$-1=111$	1	$2-\mu=1000$	8
$-3-2\mu=101001$	11	$-1+\mu=111001$	4	$2=1010$	4
$-3-\mu=101011$	8	$-1+2\mu=111011$	11	$2+\mu=11100$	4
$-3=111101$	9	$-2\mu=1100$	8	$2+2\mu=11110$	8
$-3+\mu=111111$	14	$-\mu=1110$	2	$2+3\mu=10000$	16
$-2-3\mu=1110000$	16	$\mu=10$	2	$3-\mu=1001$	14
$-2-2\mu=1110010$	8	$2\mu=10100$	8	$3=1011$	9
$-2-\mu=100$	4	$1-2\mu=1101$	11	$3+\mu=11101$	8
$-2=110$	4	$1-\mu=1111$	4	$3+2\mu=11111$	11
$-2+\mu=111000$	8	$1=1$	1	$4=11100100$	16
$-2+2\mu=111010$	16	$1+\mu=11$	2	$4+\mu=11100110$	14
$-1-3\mu=1110001$	16	$1+2\mu=10101$	7	$4+2\mu=11000$	16

THEOREM 4. *i)* $B=\Phi_0(C\hat{}\,)$, *i.e.,* $z\in B \Leftrightarrow \mu z\in C\hat{}$

ii) $C=\Phi_0(A\hat{}\,)$, *i.e.,* $z\in C \Leftrightarrow \mu z\in A\hat{}$

iii) $A=\Phi_0(B\cup C\cup(B+1))=\Phi_0(B\cup C)\cup\Phi_1(B)$, *i.e.* $z\in A \Leftrightarrow \mu z\in B\cup C\cup(B+1)$

iv) $C\hat{}=\Phi_1(A)$, *i.e.,* $C\hat{}=C-(1+\mu)$

v) $B\hat{}=\Phi_1(C)$, *i.e.,* $B\hat{}=B-\mu$

vi) Call $H=F_1\cap F_{1111}$ *where* $1-\mu=(1111)_\mu$. *Then,* $z\in A\hat{} \Leftrightarrow \mu z\in B\hat{}\cup H\cup(C\hat{}+1)$, *i.e.,*

$A\hat{}=\Phi_0(B\hat{}\cup H\cup(C\hat{}+1))=\Phi_0(B\hat{}\cup H)\cup\Phi_1(C\hat{}\,)$.■

PROOF. We prove iii) and vi). The statements i) and ii) are easier to prove than iii) and they imply iv) and v).

iii) Assume $z=0.p_{-1}\ldots=1.q_{-1}\ldots\in A$. Then, following one step the three branches that start in the node p | q in the graph Γ we have the following possibilities: z= 0.0…= 1.0…, z = 0.0…= 1.1…, z = 0.1…= 1.1… .Therefore, μz = 0. …= 10. … ∈B or μz = 0. …= =11. … ∈C or μz = 1. …= 11. … , i.e., μz–1 = 0. … = 10. … ∈B. Assume now that w∈B∪C∪(B+1). If w ∈B then w/μ = 0.0…= 1.0… ∈A, if w∈C then w/μ = 0.0… = = 1.1…∈A and if w ∈B+1, w = 1. …= 11. …, that is, w/μ = 0. …= 1.1… ∈A.

vi) $z\in A\hat{} \Leftrightarrow z=0.p_{-1}\ldots=111.q_{-1}\ldots$ corresponds to the state q | p. Thus, using the graph Γ, if z= 0.0…= 111.0… then μz= 0. …= 1110. … ∈B^, if z= 0.1…= 111.1… then μz= 1. …= 1111. … ∈H and if z= 0.1…= 111.0…, μz= 1. …= 1110. … ∈C^+1.

It can be shown as before that w ∈B^∪H∪(C^+1) ⇒ w/μ ∈A^, QED.

5. CONSPICUOUS POINTS OF F AND J. The next theorem provides the positional representations and values of some distinguished points of F and J=∂F. For example: $1/\mu=(-1-i\sqrt{7})/4$ =0.1.

THEOREM 5. *A period will be represented by* $\overline{\ldots}$. *It holds that*

$\{x\}=F_0\cap F_1\cap F_{1+\mu}$ $\quad x=0.\overline{001}=1.\overline{010}=11.\overline{100}$ = (3+i√7)/8

$\{y\}=F_0\cap F_1\cap F_{-\mu}$ $\quad y=0.\overline{101}=1.\overline{110}=1110.\overline{011}$ = (1–i√7)/4

$\{z\}=F_0\cap F_{-\mu}\cap F_{-1-\mu}$ $\quad z=0.\overline{100}=1110.\overline{010}=101.\overline{001}$ = (–1–i3√7)/8

$\{u\} = F_0 \cap F_{-1} \cap F_{-1-\mu}$ $u = 0.\overline{110} = 111.\overline{101} = 101.\overline{011} = (-3-i\sqrt{7})/4$

$\{v\} = F_0 \cap F_{-1} \cap F_{\mu}$ $v = 0.\overline{010} = 111.\overline{001} = 10.\overline{100} = (-5+i\sqrt{7})/8$

$\{w\} = F_0 \cap F_{\mu} \cap F_{1+\mu}$ $w = 0.\overline{011} = 11.\overline{110} = 10.\overline{101} = (-1+i\sqrt{7})/4$

$c := -x/2 = -(3+i\sqrt{7})/16 \in F$ $2c = 0.\overline{1}$

$P := (x+y)/2 = x.y = -\mu/2(1-\mu) \in A$ $2P = 1.\overline{1}$ ■

PROOF. If a point $y=0.r_{-1}\ldots=1.p_{-1}\ldots=1110.q_{-1}\ldots$ then its set of states corresponds to the node $\frac{r|p}{q}$ in the graph τ (cf. Fig. 2). In consequence, the p-representation is periodic of period 110 beginning immediately after the point, the r-representation has period 101 and the q-representation has period 011. Such a point is unique. The positional representations of x, z, u, v and w are obtained in an analogous way.

We have: $0.\overline{1} = 1/(\mu-1) = -(3+i\sqrt{7})/8$ and $0.\overline{1}+0.\overline{001}=0.\overline{112}$, $0.\overline{1}-0.\overline{001}=0.\overline{110}$. Since $\mu^2+\mu+2=0$ we get $(112)_\mu=0$. Therefore, $0.\overline{1}+x=0$ and $0.\overline{1}-x=u$. That is, $0.\overline{1} = 2c =$ $= -x = x+u$ and from this the values of x and u are obtained and also that $x=1/(1-\mu)$. The calculations of the values of y, z, v and w are easier. Finally we observe that $-\mu/2 =$ $= (1-i\sqrt{7})/4 = y$. In consequence, $P = (x+y)/2 = 1/2(1-\mu) - \mu/4 = \mu/2(\mu-1) = (\mu/2).0.\overline{1} =$ $= (1.\overline{1})/2$. Recall that $x = 1/(1-\mu)$. Since $\mu^2+\mu+2=0$ we get $y = -\mu/2 = 1/(1+\mu)$. Then,

(4) $$P = [-\mu/2][1/(1-\mu)] = 1/(1-\mu^2) = (P-1)/\mu^2.$$

It will be shown later using this formula that $P \in A \subset J$, QED.

THEOREM 6. *c is the center of symmetry of F.* ■

PROOF. Define $s = W(z) := 2c-z$. If $z = 0.p_{-1}p_{-2}\ldots$ then $s = 0.(1-p_{-1})(1-p_{-2})\ldots$ and $z \in F \Leftrightarrow s \in F$, QED.

It is easy to check that

(5) $A\hat{} = W(A)$ $B\hat{} = W(B)$ $C\hat{} = W(C)$.

(For example, if $b = 10.p_{-1}p_{-2}\ldots$ then $0.\overline{1}-b = 1110.(1-p_{-1})(1-p_{-2})$, i.e., $b \in B \Rightarrow$ $2c-b \in B\hat{}$). The proof of Th. 6 uses essentially the graph τ and shows also, since there is no point with four representations (Th. 2), that the following relations hold:

(6) $A\cap(B\cup A^\wedge\cup C^\wedge) = C\cap(A^\wedge\cup C^\wedge\cup B^\wedge) = B\cap(C^\wedge\cup B^\wedge\cup A) = \varnothing$

(7) $A\cap C=\{x\}$, $C\cap B=\{w\}$, $B\cap A^\wedge=\{v\}$, $A^\wedge\cap C^\wedge=\{u\}$, $C^\wedge\cap B^\wedge=\{z\}$, $B^\wedge\cap A=\{y\}$.

We *denote* with *dim K* the Hausdorff dimension of the set *K*.

THEOREM 7. *J is a closed simple curve and dim J = dim A.*■

PROOF. We prove in paragraph 8 that A is a simple arc. Because of formulae (5) and Th. 4, A is similar to B, C, B^, A^, and C^. Thus, the thesis follows from (6), (7) and the fact, proved in § 4, that $J = A \cup B \cup C \cup A^\wedge \cup C^\wedge \cup B^\wedge$, QED.

6. THE SET A. The object of this section is to prove the next result:

THEOREM 8. *i) A is the invariant set of the following family of similarities*

(8) $$\sigma_1(z) = \frac{z+1}{\mu^3} \qquad \sigma_2(z) = \frac{z-1}{\mu^2} \qquad \sigma_3(z) = \frac{z+1+\mu^2}{\mu^3}$$

ii) $\sigma_1(x) = x \quad \sigma_2(P) = P \quad \sigma_3(y) = y \quad \sigma_1(y) = \sigma_2(y) = \frac{1}{2} \quad \sigma_2(x) = \sigma_3(x) = \frac{y}{2}$

iii) Let α *be the similarity dimension of A. Then* $\alpha = 2\frac{\log(v)}{\log(2)}$ *where* v *is the real root of* $v^3 - v - 2 = 0$ *and* $v = \sqrt[3]{1+\sqrt{26/27}} + \sqrt[3]{1-\sqrt{26/27}}$. $\alpha \cong 1.21076$ *and* $v \cong 1.52138$.

iv) $\sigma_1(A) \cap \sigma_3(A) = \varnothing$

v) $\sigma_1(A) \cap \sigma_2(A) = \{\sigma_1(y)\} = \{\sigma_2(y)\} \qquad \sigma_2(A) \cap \sigma_3(A) = \{\sigma_2(x)\} = \{\sigma_3(x)\}$.■

PROOF. i) implies iii) since v is the only real root of $x^3 - x - 2 = 0$. The precise expression for v is obtained from Cardano's formula. ii) exhibits the fixed points of the contractions and follows from (4) and easy calculations. Let us see i).

Because of Theorem 4 we have, $B=\Phi_0(C^\wedge)=\Phi_0(C-1-\mu)=\Phi_0(C)-1-1/\mu=$
$=\Phi_0(\Phi_0(A^\wedge))-1-1/\mu=\Phi_0(\Phi_0(A)-1/\mu)-1-1/\mu=\Phi_0^2(A)-(1+\mu+\mu^2)/\mu^2=\Phi_0^2(A)+1/\mu^2$

Taking into account formulae (8), we obtain,

(9) $\Phi_0(B) = \Phi_0^3(A) + 1/\mu^3 = \sigma_1(A) \qquad \Phi_0(C) = \Phi_0^2(A) - 1/\mu^2 = \sigma_2(A)$

$$\Phi_0(B+1) = \Phi_0^3(A) + (1+\mu^2)/\mu^3 = \sigma_3(A)$$

It follows from Th. 4 iii) that A equals the union of the three sets in (8). Therefore,

$A = \bigcup_1^3 \sigma_j(A)$.

To prove iv), we must know the action of the σ's on the positional representation of a number z. This is shown in (10). There ~ means a sequence of binary digits that does not change after the transformation.

$$(10)\quad \sigma_1\begin{cases}\mathbf{0.\sim} \longrightarrow \mathbf{0.001\sim}\\ \mathbf{1.\sim} \longrightarrow \mathbf{1.010\sim}\end{cases}\qquad \sigma_2\begin{cases}\mathbf{0.\sim} \longrightarrow \mathbf{1.11\sim}\\ \mathbf{1.\sim} \longrightarrow \mathbf{0.00\sim}\end{cases}\qquad \sigma_3\begin{cases}\mathbf{0.\sim} \longrightarrow \mathbf{0.101\sim}\\ \mathbf{1.\sim} \longrightarrow \mathbf{1.110\sim}\end{cases}$$

Assume a, b∈A and σ_1 (a)=σ_3 (b)=z. Then, z must be equal to 0.001 ...and to 1.010 ... because of the action of σ_1 on a and also z must be equal to 0.101... and to 1.110... because of the action of σ_3 on b. Multiplying by μ^3 one obtains a number that shows clearly four representations, a contradiction.

Let us prove the first formula in v). If z∈σ_1 (A)∩σ_2 (A) then z = 0.001...= 1.010...= 1.11... and μ^2z = 0.1...= 101. ...= 111. ..... Therefore, $\mu^2 z \in F\cap F_{-1}\cap F_{-1-\mu}$.

Because of Th. 5, z = u/μ^2. Thus, z = σ_2 (y) = σ_1 (y), QED.

We note here that it follows from (10) that

(11) $$\sigma_j(F_0\cup F_1)\subseteq F_0\cup F_1 \qquad \forall j\in\{1,2,3\}.$$

7. THE HAUSDORFF DIMENSION OF A. The aim of this paragraph is to prove that A is a self-similar s-set, *s*=*dim*(*A*)=the Hausdorff dimension of A. In view of Hutchinson's theorem it suffices to show that the family of similarities (8) satisfies the open set condition.

We extend our earlier *notation* as follows: $F_{C...D.E...K}$:={*z*: *z*=*C*...*D*.*E*...*K*~} where ~ is any sequence of binary digits. Besides, if $\gamma=(i_1,\dots,i_r)$, we write σ_γ instead of $\sigma_{i_r}\circ\dots\circ\sigma_{i_1}$.

DEFINITION 2. *f*:=$F^o_{0.01111}$= *int*{*z*: *z*=0.01111~} and

$$V:=\bigcup_{r=1}^{\infty}\bigcup\{\sigma_{i_r}\circ\dots\circ\sigma_{i_1}(f): i_j\in\{1,2,3\}\}=\bigcup_{\gamma}\sigma_\gamma(f).$$

THEOREM 9. *i) A is a self-similar set and 0<H^s (A)<∞, s=similarity dimension of A*

ii) $V=V^o\subset F_0\cup F_1$ $\qquad \sigma_i(V)\subset V \qquad \sigma_i(V)\cap\sigma_j(V)=\varnothing$ *if* $i\neq j$.■

PROOF. ii) implies i). Let us prove ii). (11) yields the first inclusion. It is obvious that V is open and that the second inclusion holds. To prove the third statement in ii) we need some auxiliary propositions.

Proposition 1. *Assume that* $\sigma_\gamma(f) = F^o_\alpha$. *Then,* $\alpha = \sigma_\gamma(0.01111)$. ■

Proof. It is an immediate consequence of formulae (10), qed.

Proposition 2. *Let* $\gamma = (i_1,...i_r)$ *and* $\sigma_\gamma(F^o_{0.01111}) = F^o_\alpha$ *where* $\alpha = a_0.a_{-1}...a_{-k}01111$ *then* $a_0 \in \{0,1\}$ *and* $\{a_{-1},...,a_{-k}\}$ *does not contain four consecutive ciphers* 1.■

Proof. We use repeatedly (10) for the proof by induction on r. For r=1 it is true. Suppose the statement is true for σ_γ but not for $\sigma_d \circ \sigma_\gamma$ with some d∈{1,2,3}. That is, if $\sigma_\gamma(0.1111)=a_0.a_{-1}a_{-2}...a_{-k}01111$ and $\sigma_d(a_0.a_{-1}...a_{-k}01111)=b_0.b_{-1}...b_{-j}01111$ then $\{a_{-1},...,a_{-k}\}$ does not contain four consecutives 1's, but $\{b_{-1},...,b_{-j}\}$ does. The only way for this to happen is that $a_0.a_{-1}a_{-2}... = 0.11\sim$ (cf. (10)). However, neither *f* nor any outcome of the applications σ_j have such a beginning, a contradiction, qed.

Proposition 3. *Assume* $\alpha = a_0.a_{-1}...a_{-j}$, $\beta = b_0.b_{-1}...b_{-k}$, $k \geq j$ *and* $F^o_\alpha \cap F^o_\beta \neq \emptyset$. *Then* $a_i = b_i$ *for i* = 0,−1,…,−j. *Moreover,* $k > j \Rightarrow F^o_\alpha \supset F^o_\beta$ *properly.*■

Proof. It follows immediately from vi) of Theorem 3, qed.

Proposition 4. *Assume that* $\gamma = (i_1,...,i_r)$ *and* $\delta = (j_1,...,j_s)$. *If* $\sigma_\delta(f) = \sigma_\gamma(f) = F^o_\alpha$ *then* $\gamma = \delta$, *i.e., s=r and* $\forall k$: $i_k = j_k$. ■

Proof. Suppose that $s \geq r$ and let $\chi = (i_1,...,i_k)$ where k< r is such that $i_1 = j_1,..,i_k = j_k$ and $\sigma_\chi(0.01111) = c_0.c_{-1}...c_{-m}$, (if χ is empty then σ_χ = identity map).

Assume that $\sigma_t(c_0.c_{-1}...c_{-m}) = ...\text{cd}\,c_{-1}...c_{-m}$ for $t = i_{k+1}$ and $\sigma_t(c_0.c_{-1}...c_{-m}) =$ $= ...\text{CD}\,c_{-1}...c_{-m}$ for $t = j_{k+1}$. If $i_{k+1} = 2$ and $j_{k+1} = 1$ or 3, then, because of (10), cd=00 or 11 and CD=01 or 10. This is a contradiction, since α is uniquely determined (Prop. 3). Assume next that $i_{k+1}=1$ and $j_{k+1}=3$. If $\sigma_t(c_0.c_{-1}...c_{-m}) = ...\text{bcd}\,c_{-1}...c_{-m}$ in the first case and $\sigma_t(c_0.c_{-1}...c_{-m}) = ...\text{BCD}\,c_{-1}...c_{-m}$ in the second case then bcd=001 or 010 and BCD=101 or 110, again a contradiction. In consequence, $i_{k+1} = j_{k+1}$. This implies that $\gamma = (j_1,\ldots,j_r)$. Taking into account that the applications σ_j are contractions, we conclude that s = r and then, $\gamma = \delta$, qed.

Proposition 5. *Assume* $\alpha = a_0.a_{-1}\ldots a_{-j}$, $\beta = b_0.b_{-1}\ldots b_{-k}$, $k \geq j$, $\delta = (j_1,\ldots,j_s)$ *and* $\gamma = (i_1,\ldots,i_r)$. *If* $\sigma_\gamma(f) = F_\alpha^o$, $\sigma_\delta(f) = F_\beta^o$ *and* $\sigma_\delta(f) \cap \sigma_\gamma(f) \neq \varnothing$ *then* $\delta = \gamma$.■

Proof. From proposition 3 we obtain $a_i = b_i$ for i = 0,−1,…,−j. Taking into account that the last ciphers of α and β are 01111, k > j leads to a contradiction with Proposition 2. Therefore, α = β. Because of proposition 4, δ = γ, qed.

To finish the proof observe that if i≠j then $\sigma_i(V) = \bigcup_{r=1}^{\infty} \bigcup \{ \sigma_i(\sigma_{i_r} \circ \ldots \circ \sigma_{i_1}(f)) \}$ and $\sigma_j(V) = \bigcup_{r=1}^{\infty} \bigcup \{ \sigma_j(\sigma_{i_r} \circ \ldots \circ \sigma_{i_1}(f)) \}$ are unions of sets pairwise disjoint in view of proposition 5. Therefore, $\sigma_i(V) \cap \sigma_j(V) = \varnothing$, QED.

Corollary. *For any p∈A and any ball B(p;ε), it holds that* $H^s(B(p;\varepsilon) \cap A) > 0$.

8. THE SIMPLE ARC A. The application S(z):=2P − z = $1.\bar{1}$ − z is such that S(A)⊂A and since $S^2(A) \subset S(A)$, S(A)=A. In fact, if $z = 0.p_{-1}p_{-2}\ldots = 1.q_{-1}q_{-2}\ldots$ then S(z) = $= 1.(1-p_{-1})(1-p_{-2})\ldots = 0.(1-q_{-1})(1-q_{-2})\ldots \in A$. P is the *center of symmetry* of A.

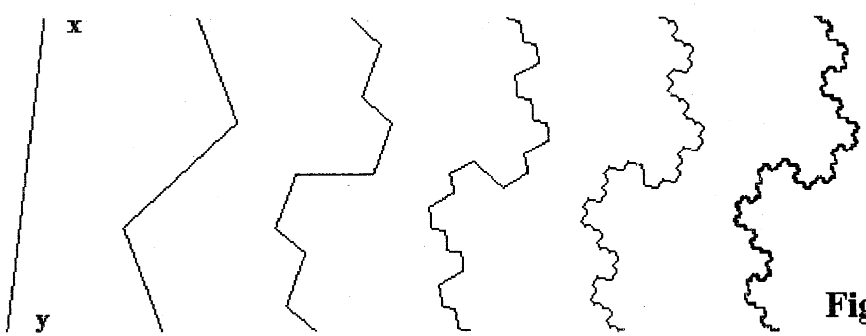

Fig. 6

The set A and the first five steps of its construction.

DEFINITION 3. $\tau_0(z)=\sigma_1(z)=\dfrac{z+1}{\mu^3}$ $\tau_1(z)=\sigma_2(S(z))=-\dfrac{z}{\mu^2}+\dfrac{1}{4}$

$$\tau_2(z)=\sigma_3(z)=\frac{z+1+\mu^2}{\mu^3} \quad ■$$

Let B=B(0;2)={z: |z|<2}. We obtain i) of the following auxiliary result from Theorem 8. ii) is easy to check (recall that |F|< 2 and that 0 ∈ *int*(F)).

Lemma 1. *i)* $A=\tau_0(A)\cup\tau_1(A)\cup\tau_2(A)$, $\tau_0(A)\cap\tau_2(A)=\emptyset$, $\tau_0(A)\cap\tau_1(A)=\{\tau_0(y)\}=$ $=\{\tau_1(x)\}$, $\tau_1(A)\cap\tau_2(A)=\{\tau_1(y)\}=\{\tau_2(x)\}$.

ii) $\forall i$: $\tau_i(B)\subset B\supset F$.■

We use in the next lemma the same notation for composition of applications that was introduced in section 7 before the proof of Th. 9.

Lemma 2. *If* z_1 *and* z_2 *belong to B,* $\alpha=(a_N,\dots,a_1)$, $a_i\in\{0,1,2\}$ *and N is a positive integer, then*

i) $|z_1-z_2|(\sqrt{2})^{-3N}\le|\tau_\alpha(z_1)-\tau_\alpha(z_2)|\le|z_1-z_2|(\sqrt{2})^{-2N}$

ii) $|\tau_1\tau_0^{N-1}(z_1)-\tau_0\tau_2^{N-1}(z_2)|\le 8(\sqrt{2})^{1-3N}$ $|\tau_2\tau_0^{N-1}(z_1)-\tau_1\tau_2^{N-1}(z_2)|\le 8(\sqrt{2})^{1-3N}$

iii) *for h*= 0,1, $\tau_{h+1}\tau_0^{N-1}(x)=\tau_h\tau_2^{N-1}(y)$.■

Proof. The proofs of i) and ii) are completely similar to those of i) and ii), respectively, of Proposition 4, [BP]. iii) follows from Theorem 8 and Definition 3, qed.

THEOREM 10. *A is a simple arc with initial point x and terminal point y.*■

PROOF. Assume t∈[0,1]. Let us define *f*: [0,1]→A by

(12) $$t=\sum_1^\infty a_j 3^{-j} \rightarrow f(t)=lim_{n\to\infty}\tau_\alpha(0) \text{ where } \alpha=(a_n,\dots,a_1).$$

f is a well defined, injective and continuous application. The proof of an analogous fact in § 3.2 [BP], precisely the proof of Th. 4 of that paper, can be repeated verbatim, QED.

9. F IS A QUASI-DISK. Theorem 10 is the result we needed to assure that J, the boundary of F, is a Jordan curve. However, more can be said about this homeomorphic copy of a circle. It is a *quasi-circle* or what is the same, F^o is a *uniform domain* (in relation with this notion we refer to [L]). To see that J is a quasi-circle it is enough to prove the next theorem. Its statement is called the Ahlfors' condition and it can be taken as a definition of quasi-circle.

THEOREM 11. *There exists* $K > 0$ *such that for any pair z, w* $\in J$, *it holds that*

(13) $$\inf\{|\overline{zw}|, |\overline{wz}|\} \leq K|z-w|$$

where $\overline{zw}$ *is the arc in J, positively oriented, with initial point z and terminal point w.*■

PROOF. This property is shared with the Knuth's dragon. So, to show that the diameter of the arc zw is bounded by K | z–w | one can repeat the demonstration of an analogous result in [BP]. That proof requires the next two Lemmas.

Lemma 3. *There exists* $K > 0$ *such that if* $0 \leq t_1 < t \leq t_2 \leq 1$ *then*

(14) $$|f(t) - f(t_1)| \leq K|f(t_2) - f(t_1)| \;.■$$

Proof. From the definitions of the τ's we get

(15) $$\tau_1\tau_0(z) = \tau_0\tau_1(z) + \eta \qquad \eta = \frac{-\mu}{\mu+6} = \frac{1}{\mu^4} \qquad \tau_0\tau_2(z) = \tau_0^2(z) + \eta$$

Assume $t \in [0, \frac{1}{9} + \frac{1}{27}]$. Then, $t = (0.00\sim)_3$ and $t + 2/9 = (0.02\sim)_3$ or $t = (0.010\sim)_3$ and $t + 2/9 = (0.100\sim)_3$. In the second case, $t = 1/9 + s/9$, $t + 2/9 = 1/3 + s/9$. Here, $s = 0.0\sim$.

We have, by the definition of *f*, $f(t) = (\tau_0 \circ \tau_1)(f(s))$, $f(t+2/9) = (\tau_1 \circ \tau_0)(f(s))$. By (15),

(16) $$f(t + \frac{2}{9}) = \tau_0\tau_1 f(s) + \eta = f(t) + \eta$$

The same formula can be obtained from (15) in the first case. (16) implies that the subarc of A defined by $t \in [\frac{1}{3} - \frac{1}{9}, \frac{1}{3} + \frac{1}{27}]$ is a translation of the subarc of A defined by $t \in [0, \frac{1}{9} + \frac{1}{27}]$. It is possible at this point to follow the same line of proof of the Proposition 5, [BP], § 4, to obtain the desired inequality (14), qed.

To verify that (14) is satisfied around any of the corners x, y, z, u, v, w of J, it is convenient to find a similarity transformation that applies a neighborhood of the corner under consideration into A. Because of the symmetry of the set J, it is sufficient to examine the points x, w and v.

DEFINITION 4. $\Theta(z) = \dfrac{z}{\mu^2} + (1.11)_\mu$ $\qquad \varphi(z) = \Theta(\dfrac{z}{\mu}) = \dfrac{z}{\mu^3} + (1.11)_\mu$ ■

Lemma 4. $\varphi(B \cup C) \subset A$ $\qquad \Theta(A \cup C) \subset A$ $\qquad \varphi(B \cup A\hat{}\,) \subset A$ ■

Proof. We check only the third relation. Recall that $B \cup A\hat{} = F \cap (F_{10} \cup F_{111})$. But $\varphi(0.\sim)=$ $=1.110\sim$ belongs to F_1, $\varphi(10.\sim)= 0.010\sim + 1.11= 0.000\sim \in F$ and $\varphi(111.\sim)=$ $= 0.111\sim + 1.11= 0.101\sim \in F$. That is, $\varphi(B \cup A\hat{}\,) \subset F_0 \cap F_1 = A$, qed.

Now we are able to prove our present Theorem 11 repeating verbatim the proof of Theorem 7, [BP], § 4.2, QED.

COROLLARY. *i) there exists a* $\delta>0$ *such that given* $t_1, t_2 \in [0,1]$, $t_2 > t_1$, *there is a similarity u such that* $u(f([t_1,t_2]) \subset A$ *and* $|u(f(t_1)) - u(f(t_2))| \geq \delta$.

ii) there exist a, b and $r>0$ *such that for any set* $\Sigma \subset J$ *with* $0 < |\Sigma| \leq r$ *there is a similarity* Λ: $\Sigma \to A$ *such that* $|\Lambda(\Sigma)| \geq \delta$ *and for X, Y* $\in \Sigma$, *it holds that*

(17) $$a|X-Y| \leq |\Sigma| \cdot |\Lambda(X) - \Lambda(Y)| \leq b|X-Y|$$ ■

(Cf. [BP], pgs. 27 and 28.)

10. ON THE SELF-SIMILARITY OF A. Let us introduce property **P**.

DEFINITION 5. A *has property* **P** *if there exists* $\Delta > 0$ *such that for any* $x \in A$ *and any ball* $B(x;r)$ *with* $r < \Delta$ *there exist* $y \in A$ *and a similarity* Y *with contraction ratio equal to one such that* $B(y;r) \cap A \subset \tau_j(A)$ *for some* $j \in \{0,1,2\}$ *and* $Y(B(y;r) \cap A) = B(x;r) \cap A$.■

That is, the affine isometry Y^{-1} sends $B(x;r) \cap A$ onto a copy completely included in one of the sets $\tau_k(A)$.

THEOREM 12. A *has property* **P**.■

We shall not enter into the details because the proof is the same as that given in [BP] § 6. There is a misprint in that proof; the definition of δ should read:

$$\delta = (\tfrac{1}{2}) \inf\{dist(f(1/9), f([1/9+1/27,1]), dist(f(1/3), A \setminus f([1/3-1/9, 1/3+1/27]))\}.$$

Theorem 12 implies that A is a $2^{3/2}$*-quasi-self-similar set of standard size* Δ/2 in the sense of McLaughlin. This is shown in [BPP], § 6.2, where a discussion of these concepts is included.

11. THE CONVEX HULL OF ∂F. J shares many properties with the boundary of the Knuth dragon, ∂K, though has a smaller Hausdorff dimension. It seems not so wild as ∂K but its convex hull is much more complex. We proved in [BP] that co(K) is an octogon.

THEOREM 13. *The convex set* co(*J*) = co(*F*) *is not a polygon.*■

PROOF. We observe first that arg(μ)=ψπ, ψ *irrational*. In fact, $\mu^2 = -\mu - 2$ and by induction one can prove that $\mu^{2k} = a\mu + b$ with $a = a(k)$ an odd integer and $b = b(k)$ an even integer. So, μ^{2k} is not real for any positive integer k. Therefore, ψ is not a rational number. Thus, $\left\{\mu^j / |\mu|^j : j \in Z\right\}$ is a family of pairwise different unit vectors.

Let L be a support line to F at the point u=$0.a_{-1}...a_{-j}...$, parallel to μ^{-j}. Then L = $=\left\{z{:}\operatorname{Im}(z.\mu^j) = \operatorname{Im}(u.\mu^j)\right\}$. If $U = u + \dfrac{\varepsilon}{\mu^j} = 0.a_{-1}a_{-2}...(1-a_{-j})...$, where ε=1 if a_{-j}=0 and ε=−1 if a_{-j}=1, then $\operatorname{Im}(U.\mu^j) = \operatorname{Im}(u.\mu^j)$ and $U \in$ L. Therefore, L is a support line to F also at U and the segment $\overline{uU} \subset \partial(\text{co(F)}) \subset \text{co(F)}$, QED.

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