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A STAIRCASE ANALOGUE OF THE PRISON YARD PROBLEM FOR ORTHOGONAL POLYGONS

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ABSTRACT. Let S be an orthogonal polygon bounded by a simple closed curve with n vertices. If $4 \leq n \leq 7$, then S is orthogonally convex. If $8 \leq n$, then S is expressible as a union of $[\frac{n-4}{4}]$ sets, each starshaped via staircases. Similarly, for $4 \leq n$, cl($\sim S$) is expressible as a union of $[\frac{n+4}{4}]$ such starshaped sets. These results yield a staircase version of the "prison yard" problem, for $\frac{n}{2}$ guards suffice to see the whole plane via staircase paths, with each path in S or in cl($\sim S$). Finally, analogous results provide decompositions of S and cl($\sim S$) into orthogonally convex sets.

1. INTRODUCTION. We begin with some definitions. Let S be a nonempty set in the plane. Point x in S is a *point of local convexity* of S if and only if there is a neighborhood N of x such that $N \cap S$ is convex. If S fails to be locally convex at q in S, then q is a *point of local nonconvexity* (lnc point) of S. Set S is called *orthogonal* if and only if S is a closed, connected set whose boundary is a finite union of segments (edges) and rays, each parallel to one of the coordinate axes. An edge e of S is a *locally convex edge* if and only if both endpoints of e are points of local convexity of S. Similarly, edge e is a *dent edge* if and only if both endpoints are lnc points of $S \cap H$, for H an appropriate closed halfplane determined by the line of e. For λ a simple polygonal path in the plane whose edges $[v_{i-1}, v_i] = v_{i-1}v_i$, $1 \leq i \leq n$, are parallel to the coordinate axes, λ is called a *staircase path* if and only if the associated vectors $[v_{i-1}, v_i]$ alternate between one (and only one) vertical direction and one (and only one) horizontal direction. For points x and y in S, we say x *sees* y (x is *visible* from y) via staircase paths if and only if there is a staircase path in S containing both x and y. The subset of S seen by x via staircase paths is the *visibility set* of x, and S is *starshaped via*

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staircase paths if and only if for some point p of S, the visibility set of p is exactly S. Finally, set S is called *horizontally convex* if and only if for each x, y in S with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. *Vertically convex* is defined analogously. We say set S is *orthogonally convex* if and only if S is an orthogonal set which is both horizontally convex and vertically convex.

There are many interesting results in convexity which involve the idea of visibility via straight line segments. Among these are a collection of guard problems, discussed at length in [10]. One example is the art gallery problem, which asks how many guards are required so that each point of a polygon A (the art gallery) is visible via a straight segment in A from at least one of the guards. (See Klee [8], Chvátal [3].)

A second example, the prison yard problem, asks a similar question but stipulates that the guards be placed at vertices of polygon A and that they protect both the interior of A (the prison itself) and the exterior of A (the corresponding yard). (See Füredi and Kleitman [6].) Typically, the number of guards required is given in terms of the number of vertices of A. Here we attempt to adapt these problems to orthogonal sets, replacing the concept of visibility via segments with the notion of visibility via staircase paths.

Some related work on orthogonal polygons appears in [2]. Moreover, results in [1] show that dent edges for orthogonal polygons behave much like lnc points for arbitrary closed connected sets in the plane. Here we extend this idea, using the dent edges of an orthogonal polygon S to decompose S into a union of sets which are starshaped via staircases. Further, just as a finite collection of lnc points may be used to decompose a closed connected set into a union of convex sets ([7]), the dent edges help to decompose orthogonal polygon S into a union of orthogonally convex sets. Since the locally convex edges for S are exactly the dent edges for $\mathrm{cl}(\sim S)$, the results yield some predictable analogues for the complement of S as well. Finally, the results for S and its complement are combined to obtain a staircase analogue of the prison yard problem, again in terms of the number of vertices of the associated polygon.

Throughout the paper, $\mathrm{cl}\, S$ and $\mathrm{bdry}\, S$ will denote the closure and boundary, respectively, for set S. The reader is referred to Valentine [11], to Lay [9], to Danzer, Grünbaum, Klee [4] and to Eckhoff [5] for a discussion of visibility via straight line segments and associated starshaped sets.

2. A STAIRCASE ANALOGUE OF THE PRISON YARD PROBLEM.

In [6], Füredi and Kleitman prove that if P is a nonconvex simple polygon with n vertices, $\left[\frac{n}{2}\right]$ guards suffice to cover both the interior and the exterior of P. We will obtain a similar result for orthogonal polygons, using staircase paths.

The following definition will be helpful.

Definition. Let S be an orthogonal polygon bounded by a simple closed curve, and let $s_1, \ldots, s_n$ be the vertices of S, ordered in a clockwise or counterclockwise direction along bdry S. Similarly, define orthogonal polygon S' and vertices $s'_1, \ldots, s'_n$. We say S and S' *have the same edge arrangement* if and only if, for an appropriate labeling of their vertices, S and S' have the same lnc points. That is, s_i is an lnc point for S if and only if s'_i is an lnc point for S'.

Theorem 1. *Let k and m be integers, $0 \le m \le k$. Let S be an orthogonal polygon whose boundary is a simple closed curve with n vertices, $n \ge 4$. If S has k dent edges, grouped into m collections of consecutive edges, then S has at least $k+2m+4$ nondent edges. The bound $k+2m+4$ is best possible. Moreover, exactly $k+4$ of the nondent edges are locally convex edges.*

Proof. We proceed by induction. If $k = 0$, then S is orthogonally convex, and it is easy to see that S has at least 4 edges, exactly 4 of which are locally convex. Similarly, if $k = 1$, clearly S has at least 7 nondent edges, exactly 5 of which are locally convex. To establish the result for general k and m, $k \ge 2$, $k \ge m \ge 1$, assume that the theorem is true for natural numbers less than k. Furthermore, for this k assume that the result has been proved for natural numbers less than m (if any exist). Finally, for k and m, suppose that the result holds for permissible natural numbers less than n (if any exist). Let S be an orthogonal polygon satisfying our hypothesis for k, m, and n.

The vertices S may be labeled either in clockwise or in counterclockwise direction along bdry S by $v_0, v_1, \ldots, v_n$. We assert that for an appropriate choice of v_0 and for an appropriate order, v_0v_1 is a nondent edge, v_1v_2 is dent, and v_0v_1 is no longer than v_2v_3: Suppose that edge v_1v_2 is a dent edge of S. Certainly one of the edges v_0v_1, v_2v_3 is no longer than the other, so without loss of generality assume v_0v_1 is no longer than v_2v_3. If v_0v_1 is not a dent edge, then our assertion is satisfied. If v_0v_1 is dent, consider the remaining edge v_nv_0 at v_0. Observe that it is shorter than v_1v_2. (See Figure 1.) If v_nv_0 is dent, continue. Obviously not all edges of S can be dent, so in finitely many steps we reach a (first) edge not a dent edge. Renumber the vertices $w_1, w_2, \ldots, w_n$ so that w_0w_1 is not dent and w_1w_2 is dent. Observe that w_0w_1 is shorter than w_2w_3. Therefore, we may assume that our original labeling $v_0, v_1, \ldots, v_n$ produces the required properties.

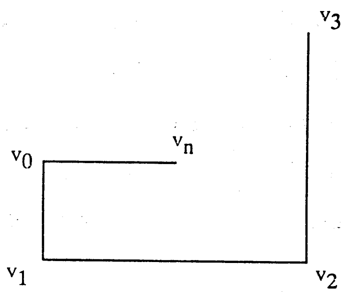

Figure 1.

For future reference, observe that since v_0v_1 is not dent, one of its endpoints cannot be an lnc point for S. Since v_1v_2 is dent, v_1 is an lnc point. Thus v_0 is not an lnc point, and v_nv_0 cannot be a dent edge. Also observe that, relative to our ordering, v_1v_2 will be the first edge in one of the m collections of consecutive dent edges. Let A be the rectangle determined by vertices v_0, v_1, v_2, and let z be the fourth vertex of A, $z \in (v_2, v_3]$. We may assume that $(\text{int}\, A) \cup (v_0, z)$ is disjoint from S, for otherwise, by adjusting lengths of appropriate edges of S, we could obtain an orthogonal polygon having the same edge arrangement as S and having the required property. There are several cases to consider.

Case 1. If neither v_3 nor v_n is an lnc point for S, proceed as follows. Since v_3 is not an lnc point, we may assume that $z = v_3$. (See Figure 2.) Now consider the orthogonal polygon $T = S \cup A$. Observe that $\text{bdry}\, T$ is a simple closed curve, that edges v_0v_1, v_1v_2, v_2v_3 for S are not edges for T, and that edges v_nv_0, v_3v_4 for S are just subsets of edge v_nv_4 for T. Further, edge v_nv_0 will be nondent for S (since its endpoints are not lnc points) and edges v_3v_4 for S and v_nv_4 for T both will be nondent (since neither v_3 nor v_n is an lnc point). Remaining edges will not be affected.

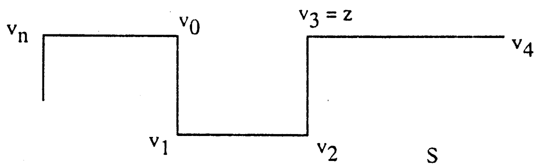

Figure 2.

Since in passing from S to T we lose dent edge v_1v_2 and do not acquire any new dent edges, T has $k-1$ dent edges. Moreover, since neither v_0 nor v_3 is an lnc point for S, edge v_1v_2 alone comprises one of the m groups of consecutive edges for S. Thus T has only $m-1$ groups of consecutive edges. Observe that since $k \geq 2$, T has $k-1 \geq 1$ dent edges, and $m-1 \geq 1$. We may apply our induction hypothesis to T to conclude that T has at least $(k-1)+2(m-1)+4$ nondent edges. When we return from T to S, we gain dent edge v_1v_2 and nondent edges v_0v_1, v_2v_3. We lose nondent edge v_nv_4 but gain nondents v_nv_0 and v_3v_4. Hence S has k dent edges and at least

$$(k-1)+2(m-1)+4+3 = k+2m+4$$

nondent edges.

Also by our induction hypothesis, since T has $(k-1)$ dent edges, T has exactly $(k-1)+4$ locally convex edges. Edges v_nv_4 for T and v_3v_4 for S are both locally convex or both not locally convex, according to whether or not v_4 is a point of local convexity. Hence in returning from T to S, there is a net gain of exactly one locally convex edge, contributed by v_nv_0, so S has exactly $k+4$ such edges. This finishes the proof for Case 1.

Case 2. If one of v_3 or v_n is not an lnc point for S, assume that $z \neq v_3$ and hence z is strictly between v_2 and v_3. (See Figure 3.) Again consider the orthogonal polygon $T = S \cup A$, which is bounded by a simple closed curve. Observe that edges v_0v_1, v_1v_2 for S are not edges for T, and edge v_nv_0 for S is just a subset of edge v_nz

for T. Edge v_2v_3 for S will be replaced by edge zv_3 for T. Moreover, z will be an lnc point for S, so edges v_2v_3 for S and zv_3 for T will be both dent or both nondent, depending on whether or not v_3 is an lnc point. Remaining edges are unaffected. There are two subcases, determined by the classification of v_n.

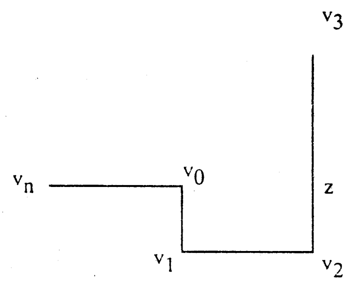

Figure 3.

Case 2a. If v_n is not an lnc point (and v_3 is an lnc point), then v_nz is not a dent edge for T. (See Figure 4.) Hence in passing from S to T, we lose dent edge v_1v_2, swap dent edge v_2v_3 for dent zv_3, and do not acquire any new dents, so T has $k-1$ dent edges. Moreover, since v_0 is not an lnc point for S but v_3 is an lnc point, v_1v_2 and v_2v_3 are first and second dent edges in a sequence of consecutive dents for S. In T, edge zv_3 will be first in the corresponding sequence of dents for T, so T (like S) will have m collections of consecutive dents. Applying our induction hypothesis, T will have at least $(k-1)+2m+4$ nondent edges. When we return from T to S, we replace nondent v_nz for T with nondent v_nv_0 for S and replace dent zv_3 for T with dent v_2v_3 for S. We gain nondent v_0v_1 and dent v_1v_2. Hence S has k dent edges and at least

$$(k-1)+2m+4+1=k+2m+4$$

nondent edges.

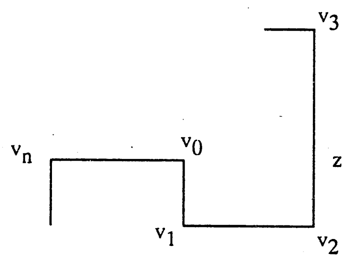

Figure 4.

Also by induction, T has exactly $(k-1)+4$ locally convex edges. Returning from T to S we lose no locally convex edges and gain the locally convex edge $v_n v_0$, giving S exactly $k+4$ such edges.

Case 2b. If v_n is an lnc point, then $v_n z$ will be a dent edge for T. Since $v_n v_0$ is not dent for S, in passing from S to T we lose dent edge $v_1 v_2$ and gain dent edge $v_n z$, leaving T with k dent edges. Unfortunately there are yet two more possibilities, determined by the classification of vertex v_3.

In case v_3 is an lnc point, then $v_2 v_3$ is a dent edge for S, so $v_1 v_2$ and $v_2 v_3$ belong to the same collection of consecutive dents in S. (See Figure 5.) In T, edge $z v_3$ leads the corresponding sequence, so T has m collections of consecutive dents. However, T has two fewer edges than S, so by our induction hypothesis, the theorem must hold for T. Thus T will have at least $k+2m+4$ nondent edges. Returning from T to S, we lose dent $v_n z$ and gain nondent $v_n v_0$. We gain nondent $v_0 v_1$ and dent $v_1 v_2$. We lose dent $z v_3$ and gain dent $v_2 v_3$. Thus S has k dents and at least

$$k+2m+4+2=k+2m+6$$

nondents. (This is 2 more than we needed.)

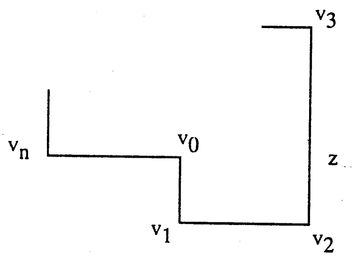

Figure 5.

Also, T has exactly k dent edges and, by induction, exactly $k+4$ locally convex edges. There is no change in locally convex edges when we pass from T to S, so S has exactly $k+4$ such edges as well.

Finally, in case v_3 is not an lnc point, then v_2v_3 is nondent for S, so v_1v_2 alone comprises one of the m sets of consecutive nondent edges in S. (See Figure 6.) In T, zv_3 is nondent, so T has only $m-1$ sets of consecutive nondent edges. Applying our induction hypothesis, T has at least $k+2(m-1)+4$ nondent edges. When we return from T to S, we lose dent v_nz, gain nondent v_nv_0, gain nondent v_0v_1, and gain dent v_1v_2. We lose nondent zv_3 to gain nondent v_2v_3. Hence S has k dents and at least

$$k+2(m-1)+4+2=k+2m+4$$

nondents.

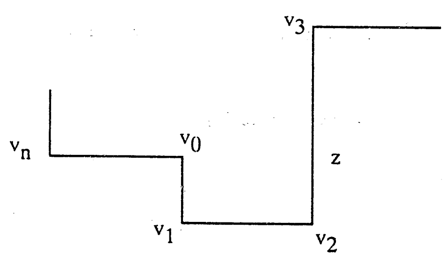

Figure 6.

Also by induction, T has exactly $k+4$ locally convex edges. There is no change in these edges when we return from T to S, so S has exactly $k+4$ such edges, also. This finishes the argument in Case 2. By induction, the theorem must hold for all suitable k and m.

It is easy to see that the result in Theorem 1 is best when $k=m=0$. To see that it is best for $k,\ m \geq 1$, consider the following example.

Example 1. For $k \geq 1$, $k \geq m \geq 1$, let $k_1, \ldots, k_m$ be m natural numbers whose sum is k. Construct orthogonal polygon S having k dent edges and $k+2m+4$ nondent edges as follows: Begin with nondent edge e as base. Above e (and following e) place edges in this sequence: 3 nondents, k_1 dents, k_1+2 nondents, k_2 dents, k_2+2 nondents, $\ldots$, k_m dents, k_m+2 nondents. This produces $\sum_{i=1}^{m} k_i = k$ dent edges and $4+\sum_{i=1}^{m}(k_i+2) = 4+k+2m$ nondent edges. Notice that exactly $k+4$ of the nondent edges are locally convex.

Figure 7 illustrates the construction for $k=8$, $m=3$, using $k_1=3$, $k_2=4$, $k_3=1$.

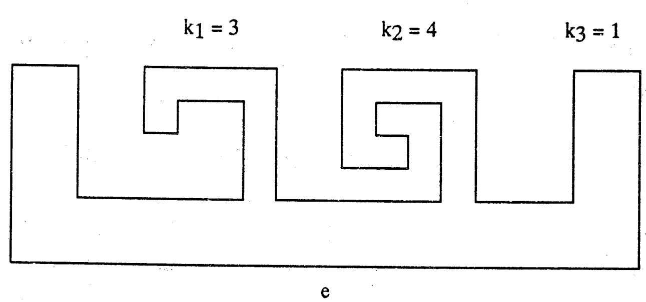

Figure 7.

Theorem 2. *Let S be an orthogonal polygon whose boundary is a simple closed curve with n vertices, $n \geq 4$. If $4 \leq n \leq 7$, then S is orthogonally convex and hence starshaped via staircases. If $n \geq 8$, then S is expressible as a union of $\left[\frac{n-4}{4}\right]$ (or possibly fewer) sets, each starshaped via staircases.*

Proof. We assume that S has k dent edges, grouped into m collections of consecutive edges, $0 \leq m \leq k$. If $k=0$ then by [1, Lemma 1], S is orthogonally convex and hence starshaped via staircases.

For the remainder of the proof, assume that $m \geq 1$. Then S has at least $k+2m+4$ nondent edges, so $n \geq k + (k + 2m + 4) \geq 8$. By [1, Theorem 2], for each point x of S there is at least one dent edge D of S such that x sees (via staircase paths in S) every point of D. Hence if we choose one point p of S from each dent edge, the corresponding visibility sets S_p will satisfy the theorem.

For $1 \leq i \leq m$, let G_i denote the corresponding collection of consecutive dent edges of S, where G_i contains k_i edges. In case each k_i is odd or each k_i is even, only minor notational changes are needed in the argument, so for simplicity, suppose that both odd and even k_i's appear. For convenience of notation, assume the G_i sets have been labeled so that $k_1, \ldots, k_l$ are odd and $k_{l+1}, \ldots, k_m$ are even, for some fixed l, $1 \leq l \leq m-1$. For each k_i, select a set of alternating endpoints from the corresponding dent edges so that for each edge, exactly one endpoint is chosen. Clearly $\frac{k_i+1}{2}$ points suffice when k_i is odd and $\frac{k_i}{2}$ points suffice when k_i is even. In all, we select

$$\sum_{i=1}^{l} \frac{k_i+1}{2} + \sum_{i=l+1}^{m} \frac{k_i}{2} = \frac{1}{2}\left(\sum_{i=1}^{m} k_i + l\right) = \frac{k+l}{2} \leq \frac{k+m}{2}$$

points. (If all k_i's are odd, we select $\frac{k+m}{2}$ points, and if all k_i's are even, we select $\frac{k}{2}$ points. However,

$$n \geq k + (k + 2m + 4) = 2k + 2m + 4$$

so $\frac{n-4}{4} \geq \frac{k+m}{2}$. We have chosen a set P of at most $\left[\frac{n-4}{4}\right]$ points. Since each dent edge contains a member of P, the corresponding collection of visibility sets $\{S_p : p \text{ in } P\}$ will satisfy the theorem.

It is interesting to note that Theorems 1 and 2 above may be adapted to produce the following results.

Theorem 3. *Let j and t be integers, $1 \leq t \leq j$. Let S be an orthogonal polygon whose boundary is a simple closed curve with n vertices, $n > 4$. If S has j locally convex edges, grouped into t collections of consecutive edges, then S has at least $j + 2t - 4$ non locally convex edges. The bound $j + 2t - 4$ is best possible.*

Proof. It is easy to see that every orthogonal polygon bounded by a simple closed curve has at least 4 locally convex edges. Moreover, such an orthogonal polygon has exactly 4 such edges if and only if it is orthogonally convex. For $j = 4$ and $n > 4$, it is not hard to show that S has at least $2t$ non locally convex edges, so the formula holds.

For $j \geq 5$, set S must be nonconvex and therefore must have at least one dent edge. We will apply the argument in Theorem 1. Observe that most of that argument depends on the features of the orthogonal curve λ which defines set S, not on the fact that S is the bounded region determined by λ. If we let U be the closed unbounded region determined by λ, then dent edges for U correspond to locally convex edges for S, and locally convex edges for U correspond to dent edges for S. This duality allows us to apply the inductive proof in Theorem 1 to region U. The only changes will be in the opening paragraph, when we begin the induction. If $j = 5$ and $t = 1$, it is easy to see that set S has at least 3 non locally convex edges. The rest of the argument follows the argument in Theorem 1, with -4 replacing $+4$ in the formula.

It is easy to find examples to show that the result in Theorem 3 is best for $j = 4$. For $j \geq 5$, S has at least one dent edge, and the result is best by Example 1 of this paper.

Theorem 4. *Let S be an orthogonal polygon whose boundary is a simple closed curve with n vertices, $n \geq 4$, and let $U = \mathrm{cl}(\sim S)$. Then U is expressible as a union of $\left[\frac{n+4}{4}\right]$ (or possibly fewer) sets, each starshaped via staircases. In case S is orthogonally convex. U is a union of 2 such starshaped sets.*

Proof. To begin, we assert that for x in U, x sees via staircases in U all points of some locally convex edge of S: Let V be a rectangular region whose interior contains $S \cup \{x\}$, and let $W = \mathrm{cl}(V \sim S)$. Then W is an orthogonal polygon, so by [1, Theorem 2], x sees via staircase paths in W all points of some dent edge of W. However, the dent edges of W are exactly the locally convex edges of S, so the assertion is established.

To prove Theorem 4 when S is orthogonally convex, recall that S has 4 locally convex edges, say e_1, e_2, e_3, e_4, labeled in a clockwise direction along bdry S. Choose one endpoint p_1 of e_1 and one endpoint p_3 of e_3. It is easy to show that U is the union of the corresponding visibility sets Se_1, Se_3.

For S not orthogonally convex, $n \geq 8$. We assume that S has j locally convex edges, grouped into t collections of consecutive edges, $1 \leq t \leq j$. Following the argument in Theorem 2, we choose a set of alternating endpoints from each collection, obtaining a set of at most $\frac{j+t}{2}$ points, with one point chosen from each locally convex edge. By Theorem 3, $n \geq j + (j + 2t - 4)$, so $\frac{n+4}{4} \geq \frac{j+t}{2}$. We have a set of at most $\left[\frac{n+4}{4}\right]$ points, and by our preliminary assertion above, the corresponding visibility sets satisfy the theorem.

Theorems 2 and 4 yield the following staircase analogue of the "prison yard" problem.

Theorem 5. *Let S be an orthogonal polygon whose boundary is a simple closea curve λ with n vertices. Guards, placed at vertices of S, can see points of the plane via staircase paths, with each path either in S or in* $\mathrm{cl}(\sim S)$. *When S is orthogonally convex, 2 guards suffice to see all points of the plane. In general, no more than $f(n)$ guards suffice, where*

$$f(n) = \begin{cases} \frac{n}{2} & \textit{when 4 divides } n \\ \frac{n-2}{2} & \textit{otherwise}. \end{cases}$$

Proof. Observe that n is always even and $n \geq 4$. If S is orthogonally convex, then (by Theorem 4) 2 guards suffice for $\mathrm{cl}(\sim S)$. Clearly, either of these can guard S as well, so we need only 2 guards in all. If S is not convex, then $n \geq 8$. By Theorem 2, $\left[\frac{n-4}{4}\right]$ guards suffice for S, while by Theorem 4, $\left[\frac{n+4}{4}\right]$ guards suffice for $\mathrm{cl}(\sim S)$. In case 4 divides n, this gives $\frac{n-4}{4} + \frac{n+4}{4} = \frac{n}{2}$ guards in all. Otherwise, this gives $\frac{n-6}{4} + \frac{n+2}{4} = \frac{n-2}{2}$ guards in all. Hence in general we need no more than $f(n)$ guards, and the theorem is proved.

3. A DECOMPOSITION INTO ORTHOGONALLY CONVEX SETS.

In [7], Guay and Kay prove that for S closed and connected and Q its corresponding set of lnc points, if $S \sim Q$ is connected and Q has exactly k members, then S is expressible as a union of $k+1$ convex sets. Here we use the dent edges of orthogonal polygon S to obtain an analogous result.

Theorem 6. *Let S be an orthogonal polygon whose boundary is a simple closed curve. If S has k dent edges, $k \geq 0$, then S is expressible as a union of $k+1$ (or possibly fewer) orthogonally convex polygons. The bound $k+1$ is best possible.*

Proof. We proceed by induction on k. If $k = 0$ then S is orthogonally convex by [1, Lemma 1]. To establish the result for general k, assume that the result has been established for whole numbers less than k, where $k \geq 1$. Let S be an orthogonal polygon satisfying our hypothesis and having k dent edges. Assume that the vertices of S are labeled $v_0, \ldots, v_n$ in clockwise direction along bdry S, with v_1v_2 a dent edge of S. Let w be the boundary point of S closest to v_1 such that $v_1 \in (w, v_2)$. It is easy to see that the segment $[w, v_2]$ separates S into two orthogonal polygons: Certainly bdry S consists of two curves λ_1 and λ_2, where λ_1 follows bdry S (in a clockwise direction) from v_1 to w and λ_2 follows bdry S (in the same direction) from w to v_1. Then $\lambda_1 \cup [w, v_1]$ is a simple closed curve bounding orthogonal polygon S_1, while $\lambda_2 \cup [v_1, w]$ is a simple closed curve bounding orthogonal polygon S_2. Clearly $S_1 \cup S_2 = S$.

For $i = 1, 2$, edge e of S_i will be a dent edge for S_i if and only if either e is a dent edge for S or e contains v_1v_2 as well as one other dent edge for S. (See Figure 8.) Letting k_i represent the number of dent edges of S_i, $i = 1, 2,$, it follows that $k_1 + k_2 = k - 1$. By applying our induction hypothesis to each set S_i, S_i is a union of $k_i + 1$ orthogonally convex polygons, $i = 1, 2$. Hence S is a union of $(k_1 + 1) + (k_2 + 1) = k + 1$ such polygons, finishing the induction and completing the proof of the theorem.

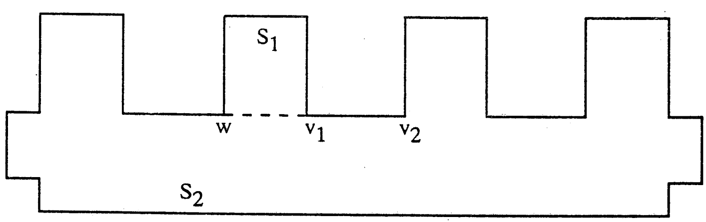

Figure 8.

To see that the result is best, simply modify the set in Figure 8 to a set with k dent edges, $k \geq 0$.

Corollary. *Let S be an orthogonal polygon whose boundary is a simple closed curve with n vertices, $n \geq 6$. Then S is a union of $\frac{n-4}{2}$ (or possibly fewer) orthogonally convex polygons. In case no two dent edges of S are consecutive, then S is a union of $\frac{n}{4}$ orthogonally convex polygons, and $\frac{n}{4}$ is best.*

Proof. If S has no dent edges, then S is orthogonally convex. If S has k dent edges, $k \geq 1$, then by Theorem 1, $n \geq k + (k+6) = 2k+6$. Hence $\frac{n-4}{2} \geq k+1$, and by Theorem 6, S is a union of $\frac{n-4}{2}$ (or possibly fewer) orthogonally convex polygons. In case no two dent edges of S are consecutive, then by Theorem 1, $n \geq k + (k + 2k+4) = 4k+4$. Hence $\frac{n}{4} \geq k+1$, and S is a union of $\frac{n}{4}$ orthogonally convex sets. Again, the set in Figure 8 may be modified to show that $\frac{n}{4}$ is best. (To begin, remove the extreme east and west rectangles.)

Theorem 7 provides an analogue of this result for $\mathrm{cl}(\sim S)$.

Theorem 7. *Let S be an orthogonal polygon whose boundary is a simple closed curve. If S has j locally convex edges, $j \geq 4$, then $U = \mathrm{cl}(\sim S)$ is expressible as a union of j (or possibly fewer) orthogonally convex sets. The bound j is best possible.*

Proof. Assume that bdry S is ordered in a clockwise direction, and let R_1, R_2, R_3, R_4 be rays, directed north, east, south, west, respectively, which follow this order and whose corresponding lines support set S, with $R_i \cap S$ a locally convex edge of S for each i, $1 \leq i \leq 4$. Then $\left(\bigcup_{i=1}^4 R_i\right) \cup (\text{bdry } S)$ divides U into four closed sets U_i, $1 \leq i \leq 4$, each bounded by two consecutive rays and a subset of bdry S. (See Figure 9.)

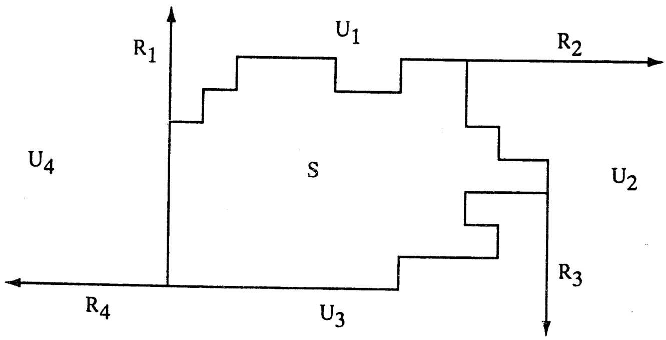

Figure 9.

The remaining $j-4$ locally convex edges of S become dent edges for these corresponding sets U_i. Letting j_i denote the number of dent edges of U_i, $1 \leq i \leq 4$, the argument in Theorem 6 may be adapted appropriately to show that U_i is a union of j_i+1 orthogonally convex sets. Hence U is a union of $\sum_{i=1}^{4}(j_i+1) = \sum_{i=1}^{4} j_i + 4 = (j-4)+4 = j$ (or possibly fewer) orthogonally convex sets.

To see that the bound j is best, modify the set in Figure 8 to a set with j locally convex edges, $j \geq 4$.

Corollary. *Let S be an orthogonal polygon whose boundary is a simple closed curve with n vertices. Then $U = (\sim S)$ is a union of $\frac{n+2}{2}$ (or possibly fewer) orthogonally convex sets. In case no two locally convex edges of S are consecutive, then U is a union of $\frac{n+4}{4}$ orthogonally convex sets, and $\frac{n+4}{4}$ is best.*

Proof. The argument is like the proof of the Corollary to Theorem 6. It uses Theorems 3 and 7 and (once again) the example in Figure 8.

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