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Symetrization of quasi linear parabolic problems

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dedicated to the memory of Julio Bouillet

Abstract: We present a new method to obtain symetrization results for parabolic equation in divergential form. This method applies for quite general quasilinear boundary value problems.

1. Introduction.

In this paper we extend results of symetrization of parabolic problems first obtained in the linear case by C. Bandle [Ba] under smooth assumptions. We first recall these results to explain our extensions. We refer the reader to [D] for a general presentation of the symetrization of parabolic problem.
Consider the problem

$$(P_0) \qquad \begin{cases} u_t = \sum(a_{ij}u_{x_i})_{x_j} & \text{on} \quad Q =]0,T[\times\Omega \\ u = 0 \quad \text{on} \quad]0,T[\times\Gamma & ,u(0,.) = u_0 \quad \text{on} \quad \Omega \end{cases}$$

when Ω is a bounded open set in $\mathbb{R}^N$ with boundary Γ , $u_0 \in L^1(\Omega)$, $a_{ij} = a_{ji} \in L^\infty(Q)$ satisfy the ellipticity condition

$$\sum a_{ij}\xi_i\xi_j \geq |\, \xi \,|^2 \quad \text{on} \;\; Q \times \mathbb{R}^N$$

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This problem is compared to the heat equation

$$\underset{\sim}{P_0}\begin{cases} v_t = \Delta v \quad \text{on} \;\;]0,T[\times \underset{\sim}{\Omega} \\ v = 0 \quad \text{on} \;\;]0,T[\times\partial\underset{\sim}{\Omega}\;,\; v(0,.) = \underset{\sim}{u_0} \quad \text{on} \;\; \underset{\sim}{\Omega} \end{cases}$$

where $\underset{\sim}{\Omega}$ is the ball $B(0,R)$ in $\mathbb{R}^N$ such that $|\underset{\sim}{\Omega}| = |\Omega|$ (1) and for $u \in L^1(\Omega)$, $\underset{\sim}{u}$ is the spherical rearrangement of u defined on $\underset{\sim}{\Omega}$ by

$$\begin{cases} \underset{\sim}{u}(x) = g(|x|) \quad \text{with} \;\; g :]0,R[\to \mathbb{R}^+ \;\text{non-decreasing} \\ \text{such that} \quad |\{\underset{\sim}{u} > k\}| = |\{|u| > k\}| \quad \text{for any } k \geq 0 \end{cases}$$

Then the following has been proved:

Theorem 0. ([Ba] , [MR]). Let u be the (weak) solution of (P_0) and v be the solution of $(\underset{\sim}{P_0})$.
Then

$$(1) \qquad \int_{B(0,r)} \underset{\sim}{u(t)}(x)dx \leq \int_{B(0,r)} v(t)(x)dx \quad \forall (t,r) \in]0,T[\times]0,R[$$

Notice that in the parabolic case, one cannot compare directly the rearrangement of $u(t)$ with $v(t)$ as it is the case for the elliptic problem

$$(E) \qquad -\sum (a_{ij}u_{x_i})_{x_j} = f \quad \text{on} \;\; \Omega\;, \quad u = 0 \quad \text{on} \;\; \Gamma$$

In the elliptic case indeed, if u is the solution of (E), then (see [Ba]) $\underset{\sim}{u} \leq v$ on $\underset{\sim}{\Omega}$, where v is the solution of

$$(\underset{\sim}{E}) \qquad -\Delta v = \underset{\sim}{f} \quad \text{on} \;\; \underset{\sim}{\Omega} \;\;,\quad v = 0 \quad \text{on} \;\; \partial\underset{\sim}{\Omega}\,.$$

This has been used by J.L.Vazquez in [V], together with non linear semi group theory in $L^1(\Omega)$, to obtain symetrization results for the problem

$$(P_1) \qquad u_t \in \Delta\varphi(u) \quad \text{on} \;\; Q =]0,\infty[\times\mathbb{R}^N \;\;,\;\; u(0,.) = u_0 \quad \text{on} \;\; \mathbb{R}^N$$

where φ is a maximal monotone graph in $\mathbb{R}$ with $0 \in \varphi(0)$.

The proofs of the above results study the rearrangement $\underset{\sim}{u}$ of a function $u \in H_0^1(\Omega)$ using the Polya inequality

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(1) If E is a mesurable set in $\mathbb{R}^N$, $|E|$ is the N-Lebesgue measure of E.

$$\| \underset{\sim}{u} \|_{H_0^1(\underset{\sim}{\Omega})} \leq \|u\|_{H_0^1\Omega)}$$

and the isoperimetric inequality (with perimeter in the sense of De Giorgi)

$$\text{perimeter of } \{ \underset{\sim}{u} > k \} \leq \text{perimeter of } \{ \mid u \mid > k \}$$

In this paper we propose a completly different approach based on studying the distribution function

$$u^*(k) = \mid \{ \mid u \mid > k \} \mid ,$$

or more precisely it's integral

$$\int_k^{\infty} u^*(s)ds = \int_{\Omega} (\mid u(x) \mid -k)^+ dx ,$$

and using the Sobolev inequality

$$(SI) \qquad \lambda_N \|u\|_{L^{\frac{N}{N-1}}} \leq \|gradu\|_{L^1} \quad \forall u \in \mathcal{D}(\Omega)$$

where

$$(2) \qquad \lambda_N = N\omega_N^{\frac{1}{N}} \quad \text{with } \omega_N = \mid B(0,1) \mid$$

Use of the distribution function u^* instead of the spherical rearrangement $\underset{\sim}{u}$ is explained by the following elementary result on rearrangement (see for instance [B-S] , [B-C]).

Lemma 0. Let $(\Omega_1, \mathcal{B}_1, \mu_1), (\Omega_2, \mathcal{B}_2, \mu_2)$ be measure spaces with $\mu_1(\Omega_1) = \mu_2(\Omega_2)$, $u_1 \in L^1(\Omega_1)$, $u_2 \in L^2(\Omega_2)$ and $\underset{\sim}{u_1}, \underset{\sim}{u_2}$ be their rearrangement on $\underset{\sim}{\Omega} = B(0,R)$ in $I\!R^N$ with $\mid \underset{\sim}{\Omega} \mid = \mu_i(\Omega_i)$. Then the following properties are equivalent:

$$(i) \qquad \int_{\Omega_1} (\mid u_1 \mid -k)^+ d\mu_1 \leq \int_{\Omega_2} (\mid u_2 \mid -k)^+ d\mu_2 \quad \forall k \geq 0$$

$$(ii) \qquad \int_{B(0,r)} \underset{\sim}{u_1}(x)dx \leq \int_{B(0,r)} \underset{\sim}{u_2}(x)dx \quad \forall r \in]0, R[$$

Then, since for the solution v of (P_0) , clearly $v(t) = \underset{\sim}{v}(t)$, conclusion (1) of theorem 0 may be as well stated:

$$\int_{\Omega} (\mid u(t,x) \mid -k)^+ dx \leq \int_{\underset{\sim}{\Omega}} (v(t,x) - k)^+ dx \quad \forall (t,x) \in]0,T[\times I\!R^+$$

Applying lemma 0 with $N=1$, the properties (i) and (ii) are still equivalent to

$$(iii) \qquad \int_0^r u_{1,*}(s)ds \leq \int_0^r u_{2,*}(s)ds \quad \forall r \in]0,R_0[$$

where $R_0 = \mu_i(\Omega_i)$ and u_* is the rearrangement on $]0,R_0[$ of u , which is actually only the inverse function of u^* ; hence $\int_r^{R_0} u_*(s)ds$ and $\int_\Omega (|u|-k)^+$ are conjugate convex functions. In this sense our method of symetrization appears to be " dual " of the classical one.

Notice also that the Sobolev inequality (SI) with the best constant λ_N follows by the isoperimetric inequality such that , in some sense, our method implicitely use this inequality (in a classical case) . However, as we will see, without knowing the best constant, we will obtain some symetrization result; such result indeed, with our method, is only related to existence of some Sobolev inequality. Also the method allows to handle as well quite general quasi linear problem.

As example of the results we can obtain with our method, we give two statements:

Theorem 1. Let Ω be any open set in $I\!R^N$, $Q=]0,T[\times\Omega$, $a : Q \times I\!R^{N+1} \to I\!R^N$ be Caratheodory and satisfy

$$a(t,x,k,\xi).\xi \geq \alpha(|k|)|\xi|^2 - F(t,x,k).\xi \quad \forall (t,x,k,\xi) \in Q \times I\!R^{N+1}$$

where $\alpha : I\!R \to]0,\infty[$ is Hölder continuous, $F : Q \times I\!R \to I\!R^N$ is Caratheodory and satisfies

$$(4) \quad \sup_{|k|\leq R} |F(.,k) \in L^2(Q)\, \forall R>0 \quad \text{and} \quad div(kF(t,.,k)) \leq 0 \text{ in } \mathcal{D}'(\Omega)\, \forall (t,k) \in]0,T[\times I\!R$$

Let $u \in \mathcal{C}([0,T[;L^1(\Omega)) \cap L^2_{loc}(]0,T[;W^{1,2}_0(\Omega))$ with $a(.,u,gradu) \in L^2_{loc}(]0,T[;L^2(\Omega))$ satisfy

$$(5) \qquad u_t = div\ a(.,u,gradu) \quad \text{in } \mathcal{D}'(Q)\ .$$

On the other hand let $\widetilde{\Omega}$ be a ball $B(0,R) \subset I\!R^N$ with $|\Omega| \leq |\widetilde{\Omega}|$, $v_0 \in L^1(\widetilde{\Omega}) \cap L^\infty(\widetilde{\Omega})$ with $\underset{\sim}{v_0} = v_0$ and v be the solution of

$$(6) \qquad \begin{cases} v_t = \Delta\varphi(v) & \text{on }]0,T[\times\widetilde{\Omega}\ , \\ v=0 \quad \text{on }]0,T[\times\partial\widetilde{\Omega}\ , & v(0,.) = v_0 \quad \text{on } \widetilde{\Omega} \end{cases}$$

where $\varphi(v) = \int_0^v \alpha(k)dk$.

If $\int_\Omega (|u_0|-k)^+dx \leq \int_{\widetilde{\Omega}} (v_0-k)^+dx$ for any $k \geq 0$, then

$$(7) \qquad \int_\Omega (|u(t)| - k)^+ dx \le \int_{\underset{\sim}{\Omega}} (v(t) - k)^+ dx \quad \forall (t,k) \in]0,T[\times I\!R^+$$

While the proof we will give is new and quite simple, the result of theorem 1 is not surprising: however notice that we can handle parabolic equation (5) with first order derivative in x like the equation

$$(8) \qquad u_t + divF(.,u) = \Delta\varphi(u)$$

with $F : Q \times I\!R \to I\!R^N$ Caratheodory satisfying (4).

In Theorem 1, we have assumed that $\varphi \in C^1(I\!R)$ with $\varphi' > 0$ on $I\!R$; actually this can be relaxed and we may assume that $\varphi : I\!R \to I\!R$ is only continuous non decreasing, according that the solution of (8) we consider is the limit of sufficiently smooth solutions to approximate problems.

More important is that the method allows to relax in some sense the Dirichlet boundary condition $u = 0$ on $]0,T[\times\Gamma$. As an example we state the following simple case:

Theorem 2. Let Ω be a bounded open set in $I\!R^N$ with smooth boundary Γ and Γ_+ be a smooth closed set in Γ with positive superficial measure. There exists a constant $c = c(\Omega, \Gamma_+)$ such that, if $a_{ij} = a_{ji} \in L^\infty(\Omega)$ satisfy the ellipticity condition

$$\sum a_{ij}\xi_i\xi_j \ge c^2|\xi|^2 \quad \forall \xi \in I\!R^N \ , \ a.e. \text{on } \Omega \ ,$$

if $\varphi : I\!R \to I\!R$ is continuous non decreasing and odd,
if $u \in C([0,T[; L^1(\Omega))$ satisfies

$$(9) \qquad \begin{cases} \varphi(u) \in L^2(0,T; W^{1,2}(\Omega)), \ \varphi(u) = 0 \text{ on }]0,T[\times\Gamma_+ \\ \displaystyle\int\int_Q u\zeta_t = \int\int_Q \sum a_{ij}\varphi(u)_{x_i}\zeta_{x_j} \quad \forall \zeta \in \mathcal{D}(]0,T[\times(I\!R^N\backslash\Gamma_+)) \ , \end{cases}$$

if $\widetilde{\Omega} = \underset{\sim}{\Omega}$ and v is the solution of (6) with $v_0 = \underset{\sim}{u(0)}$,
then (7) holds.

We will see that

$$(10) \qquad c = \lambda_N \sup\{\|\zeta\|_{L^{\frac{N}{N-1}}(\Omega)} \ ; \ \zeta \in \mathcal{D}(I\!R^N\backslash\Gamma_+) \ , \ \int_\Omega |grad\ \zeta| = 1 \ \}$$

In particular, if $\Gamma_+ = \Gamma$ then, using isoperimetric inequality, $c = 1$.

At the end of this introduction, we want to explain how these results are related to Julio Bouillet to whom this paper is dedicated. Actually we started with his very nice paper [Bo] on comparison of heat flux for different diffusion laws. Julio Bouillet was considering a Stefan problem in one dimension

$$(11)\qquad \begin{cases} u_t = (\alpha(u)u_x)_x \quad \text{on } G = \{(t,x); 0 < x < s(t)\} \\ u(t,s(t)) = u_0 \ , \ (k(u)u_x)(t,s(t)) + \lambda \dot{s}(t) = f(t) \quad \text{on }]0,T[\\ u(t,0) = \varphi(t) \quad \text{non decreasing on } [0,T[\\ u(0,x) = \psi(x) \quad \text{non increasing on } [0,\infty[\ ; \end{cases}$$

he proved that for u_0, λ, f, given , and two different data $(\alpha_1,\varphi_1,\psi_1),(\alpha_2,\varphi_2,\psi_2)$ with $\alpha_1 \geq \alpha_2, \varphi_1 \geq \varphi_2$, and $\psi_1 \geq \psi_2$, the free boundaries $s_1 \geq s_2$; the proof was based on studying the function $\int_0^{s(t)} (u(t,x)-k)^+dx$. Using the same idea, the first author extended in [A1] the result to more general non linear heat equation still in one dimension. At this stage we realize that one can get results assuming only that $\varphi_2, -\psi_2$ are non decreasing, φ_1, ψ_1 being any function. The extension to the N-dimensional case brought us to the method of symetrization developped here and first presented in [A2].

This first contact with Julio Bouillet was later developped by a strong collaboration between him and the second author who wants to express, by dedicating this article, how much he has appreciated Julio Bouillet for mathematics and also how much he liked him for friendship.

2. A general result.

Let Ω be an open set in $\mathbb{R}^N$, $Q =]0,T[\times\Omega$, $\widetilde{\Omega}$ be a ball $B(0,R)$ in $\mathbb{R}^N$ with $0 < R \leq +\infty$ and $|\Omega| \leq |\widetilde{\Omega}|$, $v_0 \in L^1(\widetilde{\Omega}) \cap L^\infty(\widetilde{\Omega})$ with $v_0 = \underset{\sim}{v_0}$, and $\varphi \in C^{1+\epsilon}(\mathbb{R}^+)$ with $\varphi' > 0$ in $\mathbb{R}^+$. We consider the problem

$$(12)\qquad \begin{cases} v_t = \Delta\varphi(v) \quad \text{on } \widetilde{Q} =]0,T[\times\widetilde{\Omega} \\ v = 0 \quad \text{on }]0,T[\times\partial\widetilde{\Omega} \\ v(0,.) = v_0 \quad \text{on } \widetilde{\Omega} \end{cases}$$

It has a unique solution $v \in C([0,T[;L^1(\widetilde{\Omega})) \cap C^{1,2}(]0,T[\times\widetilde{\Omega})$; more precisely, $v(t,x) = g(t,|x|)$ on $]0,T[\times\widetilde{\Omega}$ with $g :]0,T[\times[0,R[\to \mathbb{R}^+$ satisfying :

$$(13)\qquad \begin{cases} g > 0 \ , \ \dfrac{\partial g}{\partial r} < 0 \quad \text{on }]0,T[\times]0,R[\\ \dfrac{\partial g}{\partial r}(t,0) = 0 \quad \text{for } \ t \in]0,T[\\ \text{if } R < \infty \ , \ \lim\limits_{r\to R} g(t,r) = 0 \quad \text{for } t \in]0,T[\\ \text{if } R = \infty \ , \ \displaystyle\int_0^\infty r^{N-1} g(t,r)dr = \frac{1}{N\omega_N}\int_{\mathbb{R}^N} v_0 \quad \text{for } t \in]0,T[\end{cases}$$

Let $\mathcal{V}$ be a closed subspace of $W^{1,2}(\Omega)$ containing $\mathcal{D}(\Omega)$; assume there exists a constant $c > 0$ such that

$$(14) \qquad \lambda_N \|\zeta\|_{L^{\frac{N}{N-1}}(\Omega)} \leq c\|grad\ \zeta\|_{L^1(\Omega)} \quad \forall \zeta \in \mathcal{V} \cap L^1(\Omega) \cap L^\infty(\Omega)$$

and that

$$(15) \qquad \zeta \in \mathcal{V} \ \Rightarrow p(\zeta) \in \mathcal{V} \quad \forall p \in \mathcal{P}_0$$

where $\mathcal{P}_0 = \{p \in C^1(I\!R) \ ; \ p(0) = 0 \ , p' \ \geq 0 \ , supp\ p' \ \text{ is compact and } 0 \notin supp\ p' \ \}$.

We have the following general result:

Theorem 3. Let $u \in \mathcal{C}([0,T[; L^1(\Omega)) \cap L^2_{loc}(]0,T[; \mathcal{V})$. Assume that

$$(16) \qquad \begin{cases} \dfrac{d}{dt} \displaystyle\int_\Omega j(|u(t)|) + c^2 \int_\Omega \varphi'(|u(t)|)\ p'(|u(t)|)\ |\nabla u(t)|^2 \leq 0 \quad \text{in } \mathcal{D}'(]0,T[) \\ \text{for any } p \in \mathcal{P}_0 \text{ with } j = \int_0 p(k)dk, \end{cases}$$

where c is the constant in (14). If $\displaystyle\int_\Omega (|u(0)| - k)^+ dx \leq \int_{\widetilde{\Omega}} (v_0 - k)^+ dx$ for any $k \geq 0$, then (7) holds.

We will show in section 3 the relation between the equation (5) and the assumption (16). Let us here only give the proof of theorem 3 which relies on comparing the functions

$$(17) \qquad U(t,k) = \int_\Omega (|u(t)| - k)^+ dx \ , \ V(t,k) = \int_{\widetilde{\Omega}} (v(t) - k)^+ dx.$$

This will be done in three steps.

Proof of theorem 3, step 1.

Introduce the function $r(t,k)$ defined on

$$(18) \qquad G = \{(t,k) \ ; \ t \in]0,T[\ , 0 < k < g(t,0) = \|v(t)\|_{L^\infty}\}$$

by

$$(19) \qquad r = r(t,k) \Longleftrightarrow g(t,r) = k$$

This is well defined according to (13). We have

$$(20) \qquad \begin{cases} V(t,k) = 0 \quad \text{for } t \in]0,T[, \ k \geq g(t,0) \\ V(t,k) = N\omega_N \displaystyle\int_0^{r(t,k)} r^{N-1} g(t,r)dr \quad \text{for } (t,k) \in G \end{cases}$$

Also by derivation in (17), we have

$$\frac{\partial V}{\partial k}(t,k) = -|B(0,r(t,k))| = -\omega_N r(t,k)^N \quad \text{on } G\ ; \tag{21}$$

then

$$\begin{cases} \dfrac{\partial V}{\partial k} \in \mathcal{C}([0,T]\times\,]0,\infty[)\,,\ \dfrac{\partial V}{\partial k}(t,k) = 0 \text{ for } k \geq g(t,0) \\ \text{if } R < \infty\ ,\ \lim\limits_{k\to 0} \dfrac{\partial V}{\partial k}(t,k) = -|\widetilde{\Omega}| \end{cases} \tag{22}$$

and

$$\frac{\partial^2 V}{\partial k^2} \in \mathcal{C}(G)\ ,\ \frac{\partial^2 V}{\partial k^2} = -N\omega_N r(t,k)^{N-1}\ \frac{\partial r}{\partial k}(t,k) > 0 \text{ on } G\ . \tag{23}$$

On the other hand

$$\frac{\partial V}{\partial t}(t,k) = \int_{B(0,r(t,k))} \frac{\partial v}{\partial t}(t,x)dx = \int_{B(0,r(t,k))} \Delta\varphi(v(t)) =$$

$$= N\omega_N [r^{N-1}\ \tfrac{\partial}{\partial r}\varphi(g(t,r))]_{r=0}^{r=r(t,k)} = N\omega_N\ \varphi'(k)\ \frac{\partial g}{\partial r}(t,r(t,k))r(t,k)^{N-1}$$

Using (21), (23) and

$$\frac{\partial g}{\partial r}(t,r(t,k))\frac{\partial r}{\partial k}(t,k) = 1 \quad \text{on } G\ ,$$

we obtain the p.d.e.

$$V_t = -\frac{\varphi'(k)\lambda_N^2(-V_k)^{\frac{2}{N'}}}{V_{kk}} \quad \text{on } G \tag{24}$$

with $N' = \frac{N}{N-1}$.

This equation (24) is interesting by itself and has been used in [B-B] to obtain sharp estimates on the solution v of (12). Actually we will use it under a more classical form of parabolic equation: set

$$\mu = \frac{\lambda_N(-V_k)^{\frac{1}{N'}}}{V_{kk}} \in \mathcal{C}(G)\ ; \tag{25}$$

then V satisfies

$$V_t = \mu^2\ \varphi'\ V_{kk} - 2\lambda_N\ \mu\ \varphi'(-V_k)^{\frac{1}{N'}} \quad \text{on } G \tag{26}$$

Proof of theorem 3, step 2.

We now study the function $U(t,k)$. Let $p \in \mathcal{C}^1(I\!R)$ with $p' \geq 0$, $p(r) = 0$ for $r \leq 0$, $p(r) = 1$ for $r \geq 1$, and set $j_n(r) = \int_0^r p(ns)ds$. We study the approximation of U:

$$U_n(t,k) = \int_\Omega j_n(|u(t)| - k)dx \ .$$

It is clear that

$$\frac{\partial U_n}{\partial k} = -\int_\Omega p(n(|u(t)| - k))dx$$

$$\frac{\partial^2 U_n}{\partial k^2} = n\int_\Omega p'(n(|u(t)| - k))dx$$

are continuous functions on $[0,T[\times[0,\infty[$. Now, by (16)

$$\frac{\partial U_n}{\partial t} \leq -nc^2\int_\Omega \varphi'(|u(t)|)\ p'(n(|u(t)| - k))\ |\nabla u(t)|^2 dx \quad \text{in } \mathcal{D}'(]0,T[\times]0,\infty[)$$

For $a.e.t \in]0,T[$ and any $k \in]0,\infty[$, according to (14) and (15), one has

$$\lambda_N \|p(n(|u(t)| - k))\|_{L^{N'}(\Omega)} \leq cn\int_\Omega p'(n(|u(t)| - k))\ |\nabla u(t)|dx \ ,$$

and then for $\mu(t,k) \geq 0$

$$2\lambda_N\ \mu(t,k)(\min_{[k,k+\frac{1}{n}]}\varphi')\ \|p(n(|(u(t)| - k))\|_{L^{N'}} \leq$$

$$2n\int_\Omega \varphi'(|u(t)|)\ p'(n(|u(t)| - k)\ c\ |\nabla u(t)|\mu(t,k)dx \leq$$

$$nc^2\int_\Omega \varphi'(|u(t)|)\ p'(n(|u(t)|-k)\ |\nabla u(t)|^2 dx + \mu(t,k)^2(\max_{[k,k+\frac{1}{n}]}\varphi')n\int_\Omega p'(n(|u(t)|-k))dx.$$

In other words, for any function $\mu \in L^2_{loc}(]0,T[\times]0,\infty[)$, $\mu \geq 0$, one has

$$(27)\qquad \begin{cases} \dfrac{\partial U_n}{\partial t} \leq \mu^2(\max\limits_{[k,k+\frac{1}{n}]}\varphi')\ \dfrac{\partial^2 U_n}{\partial k^2} - \\ \qquad 2\lambda_N\mu(\min\limits_{[k,k+\frac{1}{n}]}\varphi')\|p(n(|u(t)| - k))\|_{L^{N'}} \ \text{ in } \mathcal{D}'(]0,T[\times]0,\infty[). \end{cases}$$

As $n \to \infty$,

$$U_n(t,k) \to U(t,k) \ \text{ in } \mathcal{C}([0,T[\times[0,\infty[)$$

$$\|p(n(|u(t)| - k))\|_{L^{N'}}^{N'} \to |\{|u(t)| > k\}| = -\frac{\partial U}{\partial k}(t,k) \ \text{ in } L^1_{loc}([0,T[\times[0,\infty[).$$

Then, at the limit in (27), we get

$$(28)\qquad \frac{\partial U}{\partial t} \leq \mu^2\varphi'\frac{\partial^2 U}{\partial k^2} - 2\lambda_N\mu\varphi'(-\frac{\partial U}{\partial k})^{\frac{1}{N'}} \ \text{ in } \mathcal{D}'(]0,T[\times]0,\infty[);$$

this is well defined and justified for any function $\mu \in \mathcal{C}(]0,T[\times]0,\infty[)$ since $U(t,k)$ being convex in k, $\dfrac{\partial^2 U}{\partial k^2}$ is a non negative Radon measure on $]0,T[\times]0,\infty[$.

Proof of theorem 3, step 3.

According to steps 1 and 2, the functions U and V are respectively subsolution and solution of the same parabolic equation; since by assumption $U(0,k) \leq V(0,k)$ for any $k \geq 0$, one think to use parabolic maximum principle. However, due the lack of regularity, we have not see how to use a classical result. For this reason, we will prove directly the comparison $U \leq V$ in our situation, using strongly the convexity with respect to k.
Notice first that as a consequence of (16), $U(t,k)$ is non increasing with respect to $t \in [0,T[$.

As classical, let

$$W(t,k) = e^{-t}(U(t,k) - V(t,k)) \in \mathcal{C}([0,T[\times[0,\infty[)$$

and assume $\sup W > 0$. Using $W(0,k) \leq 0$ and $W(t,k) \leq e^{-t}U(0,k) \to 0$ as $k \to \infty$, there exists $(\underline{t},\underline{k}) \in]0,T[\times[0,\infty[$ such that

$$W(\underline{t},\underline{k}) = \max_{[0,\underline{t}]\times[0,\infty[} W > 0 \quad . \tag{29}$$

The derivatives $\dfrac{\partial W}{\partial k}(t,k+)$ and $\dfrac{\partial W}{\partial k}(t,k-)$ exists for any $(t,k) \in]0,T[\times]0,\infty[$, and since $\dfrac{\partial V}{\partial k} \in \mathcal{C}(]0,T[\times]0,\infty[)$ we have $\dfrac{\partial W}{\partial k}(t,k+) \geq \dfrac{\partial W}{\partial k}(t,k-)$; then

$$\frac{\partial W}{\partial k}(\underline{t},\underline{k}+) = \frac{\partial W}{\partial k}(\underline{t},\underline{k}-) = 0 \quad \text{if } \underline{k} > 0 \, . \tag{30}$$

It follows, using notations of step 1, that

$$0 \leq \underline{k} < g(\underline{t},0) \; ;$$

otherwise, we would have $\dfrac{\partial U}{\partial k}(\underline{t},\underline{k}+) = e^{\underline{t}} \dfrac{\partial W}{\partial k}(\underline{t},\underline{k}+) = 0$ and $W(\underline{t},\underline{k}) = e^{-\underline{t}}U(\underline{t},\underline{k}) = 0$.
We show that $\underline{k} > 0$. In the case $R = \infty$, we have $V(\underline{t},0) = V(0,0) \geq U(0,0) \geq U(\underline{t},0)$ and then $W(\underline{t},0) \leq 0$. Consider now the case $R < \infty$; we may assume $R > (\dfrac{|\Omega|}{\omega_N})^{\frac{1}{N}}$, since if the theorem is true for any such R, it will be true at the limit for $R = (\dfrac{|\Omega|}{\omega_N})^{\frac{1}{N}}$ (for $\epsilon > 0$, consider a problem (12) with $\widetilde{\Omega} = B(0, R+\epsilon)$ which solution converges to v as $\epsilon \to 0$) . Then we have

$$\frac{\partial W}{\partial k}(\underline{t},0_+) = e^{-\underline{t}}(|\widetilde{\Omega}| - |\{|u(t)| > 0\}|)) > 0$$

such that $\underline{k} > 0$.

So

(31) $$(\underline{t}, \underline{k}) \in G$$

Instead of using (28) involving measure, we will use (27) for the approximation U_n of U. Set $W_n = e^{-t}(U_n - V)$. Clearly $W_n \to W$ uniformly on $[0, t_0] \times [0, \infty[$ for any $t_0 \in]0, T[$, and we may choose $(\underline{t}, \underline{k})$ such there exists $(t_n, k_n) \in [0, T[\times[0, \infty[$ such that

$$W_n(t_n, k_n) = \max_{[0,t_n]\times[0,\infty[} W_n \quad , \ (t_n, k_n) \to (\underline{t}, \underline{k}) \ .$$

For n large we will have $(t_n, k_n) \in G$. On G , W_n is bounded variation with respect to t and twice continuously differentiable with respect to k. Using (27) and (26) with μ given by (25), we have

(32)
$$\begin{cases} 0 \le \liminf\limits_{t \to t_n^-} \dfrac{W_n(t_n, k_n) - W_n(t, k_n)}{t - t_n} \le \\ -W_n(t_n, k_n) + e^{-t_n}\ \mu(t_n, k_n)^2 (\min\limits_{[k_n, k_n + \frac{1}{n}]} \varphi')(\dfrac{\partial^2 U_n}{\partial k^2}(t_n, k_n) \\ -\dfrac{\partial^2 V}{\partial k^2}(t_n, k_n) - 2\lambda_N \mu(t_n, k_n)(\max\limits_{[k_n, k_n + \frac{1}{n}]} \varphi')(\|p(n(|u(t_n)| - k_n))\|_{L^{N'}} - (-V_k)(t_n, k_n)^{\frac{1}{N'}}) \end{cases}$$

Using (30), $|\ \{\ |u(\underline{t}\ )| = \underline{k}\ \}\ | = 0$ and then

$$\|\ p(n(|u(t_n)| - k_n))\ \|_{L^{N'}}^{N'} \ \to\ |\ \{\ |u(\underline{t})| > \underline{k}\ \}\ | = -V_k(\underline{t}, \underline{k}) \quad \text{as } n \to \infty \ .$$

Since $\dfrac{\partial^2 W_n}{\partial k^2}(t_n, k_n) = e^{-t_n}\ (\dfrac{\partial^2 U_n}{\partial k^2}(t_n, k_n) - \dfrac{\partial^2 V}{\partial k^2}(t_n, k_n)\ ) \le 0$, we obtain at the limit in (32), $0 < W(\underline{t}, \underline{k}) \le 0$, a contradiction ◇

Remark . With the same proof, but some more technical arguments, one could
a) replace (12) by a more general equation

$$v_t = div\ (\alpha(v, |\nabla v|)\nabla v) \quad ;$$

b) have the space $\mathcal{V}$ defining the boundary conditions be dependant on $t \in]0, T[$; in this case (14) will involve a constant $c(t)$;

c) have (16) replaced by a more general form

$$\frac{d}{dt}\int_\Omega j(|u(t)|) + \int_\Omega p'(|u(t)|) J(t, u(t), |\nabla u(t)|) \le 0$$

where $J(t, k, s)$ is convex in s and satisfies

$$J(t, k, s)\ \ge\ \alpha(|k|, c(t)s)\ c(t)^2 s^2$$

3. Proofs of Theorems 1 and 2.

We now give the proofs of theorems 1 and 2 as examples of application of Theorem 3.

Proof of theorem 1 :

We apply theorem 3 with $\mathcal{V} = W_0^{1,2}(\Omega)$ which satisfies (15) and (14) with $c = 1$. We have only to show that, under assumptions of theorem 1, the function u satisfies (16).

Since $u \in L^2_{loc}(]0,T[;\mathcal{V})$ and

$$u_t = div\ a(.,u,gradu) \in L^2_{loc}(]0,T[;\mathcal{V}') ,$$

we have for any $p \in \mathcal{P}_0$, $\int_\Omega j(|u|) \in W^{1,1}_{loc}(]0,T[)$ and

$$\frac{d}{dt}\int_\Omega j(|u(t)|) =< p(|u(t)|)signu(t), u'(t) >_{\mathcal{V},\mathcal{V}'} =$$

$$-\int_\Omega a(t,.,u(t),gradu(t))p'(|u(t)|)gradu(t) \ \leq$$

$$-\int_\Omega \varphi'(|u(t)|)p'(|u(t)|)|gradu(t)|^2 + \int_\Omega p'(|u(t)|)F(t,.,u(t))gradu(t) \text{ for a.e. } t \in (0,T) \ .$$

Using assumption (4)

$$\int_\Omega p'(|\xi|)F(t,.,\xi).grad\xi \ \leq \ 0 \quad \forall \xi \in \mathcal{D}(\Omega)$$

and then approximating $u(t) \in W_0^{1,2}(\Omega)$, one has

$$\int_\Omega p'(|u(t)|)F(t,.,u(t)).gradu(t) \leq 0 \quad \text{for a.e. } t \in (0,T)$$

and u satisfies (16) $\diamond$

Proof of theorem 2 .

Let $\mathcal{V}$ be the closure in $W^{1,2}(\Omega)$ of $\{\xi_{|\Omega} \ ; \ \xi \in \mathcal{D}(\mathbb{R}^N\backslash\Gamma_+)\}$. Due to the smoothness assumptions on Γ and Γ_+ , $\mathcal{V} = \{u \in W^{1,2}(\Omega) \ ; \ u = 0 \text{ on } \Gamma_+\}$. It is clear that $\mathcal{V}$ satisfies (15) and (14) with c given by (10).

Let assume first that $\varphi \in \mathcal{C}^{1+\epsilon}(\mathbb{R})$ with $\varphi' > 0$ and $u \in L^\infty(Q)$. Then $v_0 = \underset{\sim}{u(0)} \in L^\infty(\widetilde{\Omega})$, and the conclusion holds if we verify that u satisfies (16). This is clear, since by (9) and the regularity on φ, $u \in L^2(0,T;\mathcal{V}), u_t \in L^2(0,T;\mathcal{V}')$ and

$$\frac{d}{dt}\int_\Omega j(|u(t)|) =< p(|u(t)|)signu(t), u'(t) >_{\mathcal{V},\mathcal{V}'} =$$

$$-\int_\Omega p'(|u(t)|)\varphi'(u(t))\sum a_{ij}\frac{\partial u(t)}{\partial x_i}\frac{\partial u(t)}{\partial x_j} \ \leq \ -\int_\Omega c^2p'(|u(t)|)\varphi'(u(t))|gradu(t)|^2.$$

Let now φ, u be general as in the theorem. Let $\varphi_n \in \mathcal{C}^{1+\epsilon}(\mathbb{R})$ with $\varphi_n' > 0$ such that $\varphi_n \to \varphi \in \mathcal{C}(\mathbb{R})$ and $u_{0,n} \in L^\infty(\Omega)$ such that $u_{0,n} \to u(0) \in L^1(\Omega)$. There exists a unique solution $u_n \in \mathcal{C}([0,T[;L^1(\Omega))$ of (9) corresponding to φ_n with $u_n(0) = u_{0,n}$; it is clear that $u_n \in L^\infty(Q)$ and $u_n \to u$ in $\mathcal{C}([0,T[;L^1(\Omega))$. In the same way if v_n is the solution of (6) corresponding to φ_n and $v_{0,n} = \underset{\sim}{u}_{0,n}$, since $\underset{\sim}{u}_{0,n} \to \underset{\sim}{u}(0) \in L^1(\underset{\sim}{\Omega})$, we have $v_n \to v$ in $\mathcal{C}([0,T[;L^1(\Omega))$. It is clear then, at the limit in (7) for (u_n, v_n), that (7) holds for (u,v) $\diamond$

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