**Revista de la**
**Unión Matemática Argentina**
**Volumen 41, 1, 1998.**



# NUMERICAL EXPERIMENTS WITH $u_t = \alpha(u)_{xx}$

Julio E. Bouillet*, Javier I. Etcheverry†

## 1 INTRODUCTION

The equation $u_t = \alpha(u)_{xx}$ , $x \in \mathbf{R}^1$ , is a very general model for evolution of initial concentrations on the line $\mathbf{R}^1$. The function $\alpha(u)$ is assumed to be nondecreasing in $u$, and such that $u(0) = 0$. This covers many of the common diffusion examples, like the heat equation ($\alpha(u) = u$), the porous media equations ($\alpha(u) = u^m$, $m > 1$), the Stefan problem with many phases, etc. Depending on the specific case, the function $\alpha(u)$ may have a high order zero at $u = 0$ (e.g. $u^m$, $m > 1$), or no derivative at all ($\alpha = u^+$), or have a flat portion within the range of the solution $u$ (e.g. $\alpha = (u-1)^+$, for $u \geq 0$). It is well known that when the function $\alpha(u)$ has a flat portion, or zero derivative for some value of the solution, the problem degenerates (for instance in the porous media equation for $u = 0$, or in the Stefan problem). In this case the solution is not longer classical, but there appear interfaces or free boundaries. This makes more involved the numerical treatment of these equations, because it is necessary not only to solve accurately the equation where it is not degenerate, but also to follow accurately the free boundaries in time. For instance, standard finite difference methods tend to smear out the interfaces, due to the introduction of some artificial viscosity, and some special techniques are needed to track the evolution of the interfaces. In what follows, we develop a numerical scheme that can handle in a unified way the general case, where $\alpha(u)$ is a monotone non decreasing function, and we show on several numerical examples the convergence of the numerical solutions to the true solutions. We also demonstrate the ability of the numerical method to follow accurately the free boundaries, allowing for instance to show the existence of waiting times.

## 2 MAIN RESULTS

We discretise the initial data $u_I$ as a piecewise constant function, with constant values $u_i$, and we discretise in the same way the function $\alpha(u)$. Discontinuities of $\alpha(u)$ may not be so frequent, but as soon as a discretisation of the initial data $u_I(x)$,

*This work appeared in 1992, in the internal report Trabajos de Matemática 190, IAM, CONICET, Buenos Aires, Argentina.

†Dept. Mathematics, FCEyN, University of Buenos Aires, Argentina

$x \in \mathbf{R}^1$ , is performed, we may think that a discretised $\alpha(u)$ is also employed, ie. a discontinuous constitutive function $\alpha$ which is monotone nondecreasing in $u$, with jumps at each point of the discretised values $u_i$ of the initial data $u_I$.
Take as basic features of the evolution the conservation of mass $\int u\ dx$, $t > 0$ and of first moment $\int x \cdot u\ dx$ , $t > 0$ (hence also of center of mass). The first one leads to the equation of our interest (written in conservation law form):

$$u_t + [-\alpha_x]_x = 0 \tag{1}$$

where $\alpha = \alpha(x,t)$. We want to derive from (1) a conservation law for $\int x \cdot u\ dx$ , $t > 0$ that we can formally write:

$$(xu)_t + [-x\alpha_x + \alpha]_x = 0 \tag{2}$$

As we look forward to accept discontinuous functions $u$ (i.e., the discretised approximations) we must check the jump condition (Rankine-Hugoniot) for discontinuous solutions to be the same in (1) and (2):

$$\dot{s}(t) = \frac{[-\alpha_x]}{[u]} = \frac{[-x\alpha_x + \alpha]}{[xu]} \tag{3}$$

A sufficient condition for equality is: $\alpha$ is a continuous function of $x$ for fixed $t$, while $\alpha$(x,t) must coincide with $\alpha(u(x,t)) \equiv \alpha_i$ when $u = u(x,t) \neq u_i$ if $\alpha$(u) jumps at $u = u_i$. A moment's reflexion suggests that $u(x,t) \equiv u_i$ on an interval where $\alpha$(x,t) be a linear function of $x$ for each $t$, that joins continuously with the constant values $\alpha(x,t) = \alpha_i$ of the discretisation $\alpha_i = \alpha(u_i)$. Satisfaction of the jump condition yields, for the intervals $(x_i, x_{i-1})$ where $u = u_i =$ constant:

$$\dot{x}_i(t) = \frac{1}{u_{i+1} - u_i}\left[\frac{\alpha_{i+1} - \alpha_i}{x_{i+1} - x_i} - \frac{\alpha_i - \alpha_{i-1}}{x_i - x_{i-1}}\right] \tag{4}$$

# 3 NUMERICAL RESULTS

We have worked out several examples of initial-value problems employing the scheme (4) to give the evolution of the interval $x_i \leq x \leq x_{i+1}$ where $u$ is equal to $u_i$. Most of the examples deal with monotonic, symmetric initial values (for simplicity), and our aim is to provide an insight of how the present method can handle a variety of problems with different classical approach in a unified way. We intend also to show how it can reflect well known properties of the solution of these classes of equations; namely, the evolution of the support of a compactly supported initial data and the existence of waiting times (cf. [1]). In all cases, the evolution of a given initial profile is shown at equally spaced times.
Figure 1 shows a solution of the heat equation $u_t = u_{xx}$. Some of the presented examples deal with the porous media equation $u_t = (u^m)_{xx}$ for various values of m. A source of examples are the well known Barenblatt self-similar solutions, that we write:

$$v(x,t) = \max\left[0, \frac{1}{\lambda(t)}\left[1 - \left(\frac{x}{\lambda(t)}\right)^2\right]^{\frac{1}{m-1}}\right] \tag{5}$$

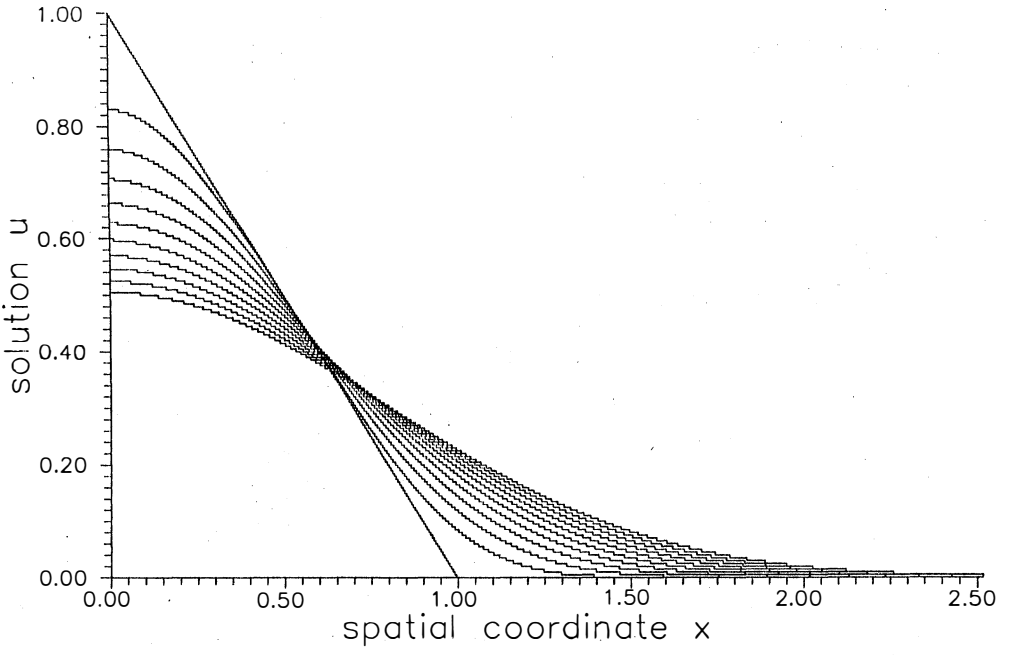


Figure 1: Solution of the heat equation corresponding to a triangular initial profile.

where the support $\lambda(t)$ is given by:

$$\lambda(t) = \left[\frac{2m(m+1)}{m-1}(t+1)\right]^{\frac{1}{m+1}} \tag{6}$$

We have obtained the numerical solutions corresponding to an initial data given by (5) at such a $t^*$ that $\lambda(t^*) = 1$. See Figs. 2, 3 for $m = 2$, and Figs. 4, 5 for $m = 3$. We include the plot of the exact solutions (5) for comparison. It is worth noting that the present method can follow the support of a solution in a natural and accurate way (see Fig. 6), in contrast with others that cannot (cf. [2]), or at least not so naturally (cf. [3]). We have worked out, in particular, an example presented in [3], for the porous media equation with $m = 2$, and initial data:

$$\frac{1}{2}(i\cos^2\theta + j\cos^4\theta), \quad \text{with } i+j=2, i=0,1,2, \text{ and } -\pi \leq \theta \leq \pi$$

Typical obtained profiles are shown in Figs. 7, 8, and the evolution of the support in Fig. 9. Note that there is a one half factor of difference with respect to Fig. 6 of [3]. Interestingly, a close examination of Figs. 7, 8 shows the convergence, as $t$ goes to infinity, of the solution of $u_t = u^2{}_{xx}$ with the given initial data, to a Barenblatt self-similar solution. This is a well known fact, that we show for $m = 3$ in Fig. 10. Looking for a classical Stefan problem we have employed the equation:

$$u_t = (u-1)^+{}_{xx}$$

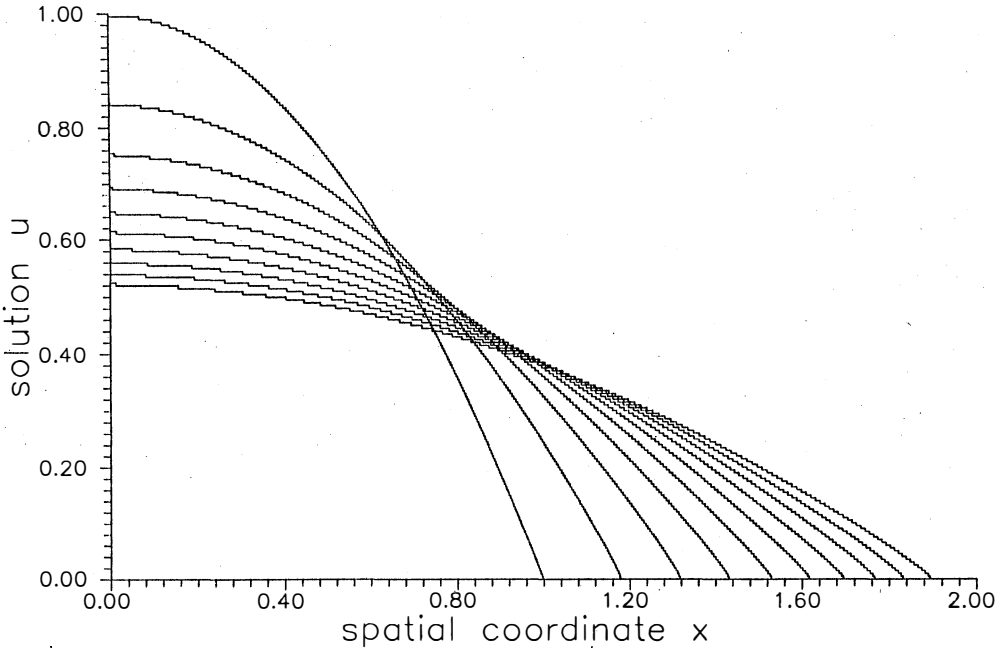


Figure 2: Numerical solution of the porous media equation with $m = 2$, and initial data $1 - x^2$.

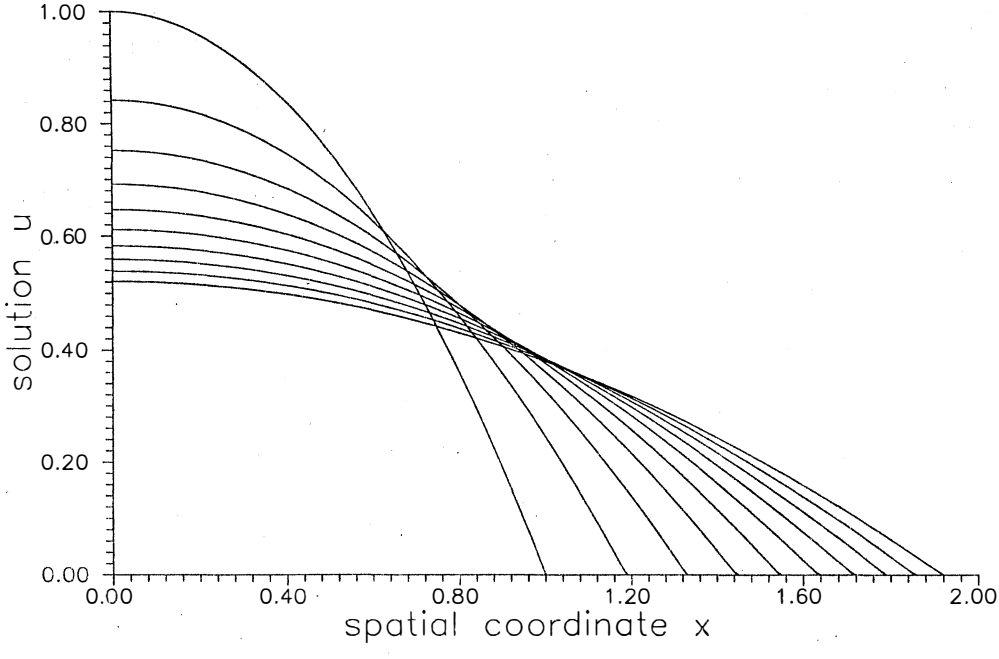


Figure 3: Exact solution of the porous media equation with $m = 2$, and initial data $1 - x^2$.

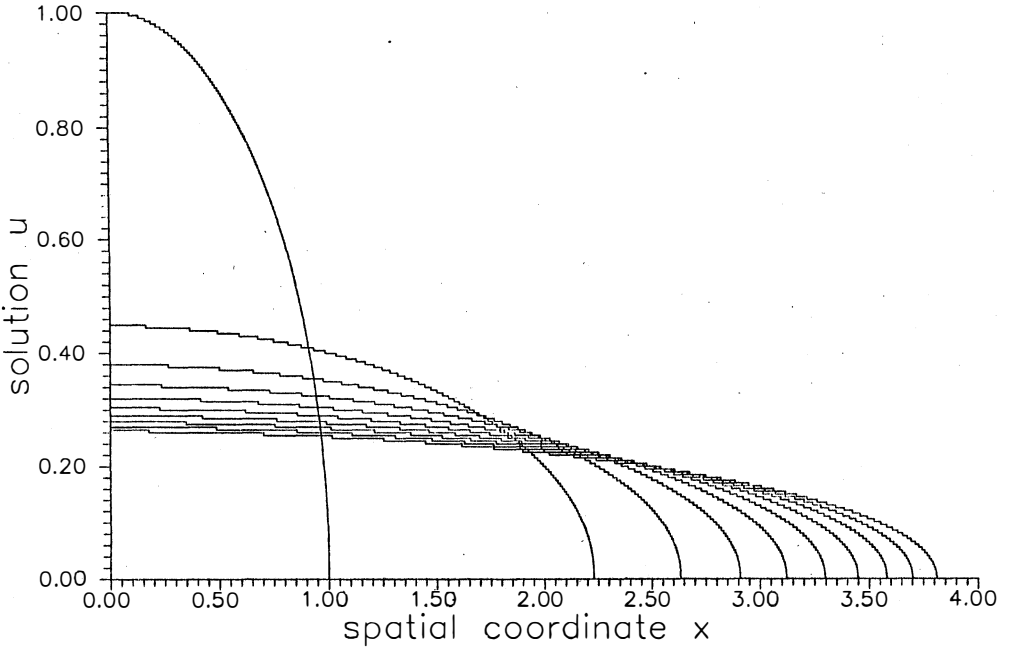


Figure 4: Numerical solution of the porous media equation with $m = 3$, and initial data $(1 - x^2)^{1/2}$.

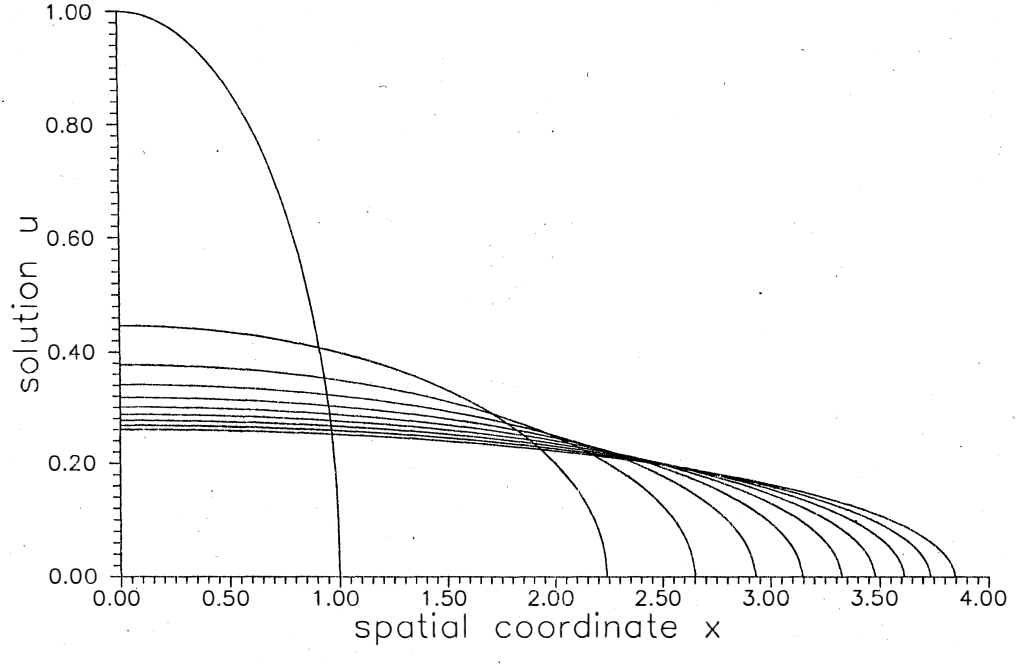


Figure 5: Exact solution of the porous media equation with $m = 3$, and initial data $(1 - x^2)^{1/2}$.

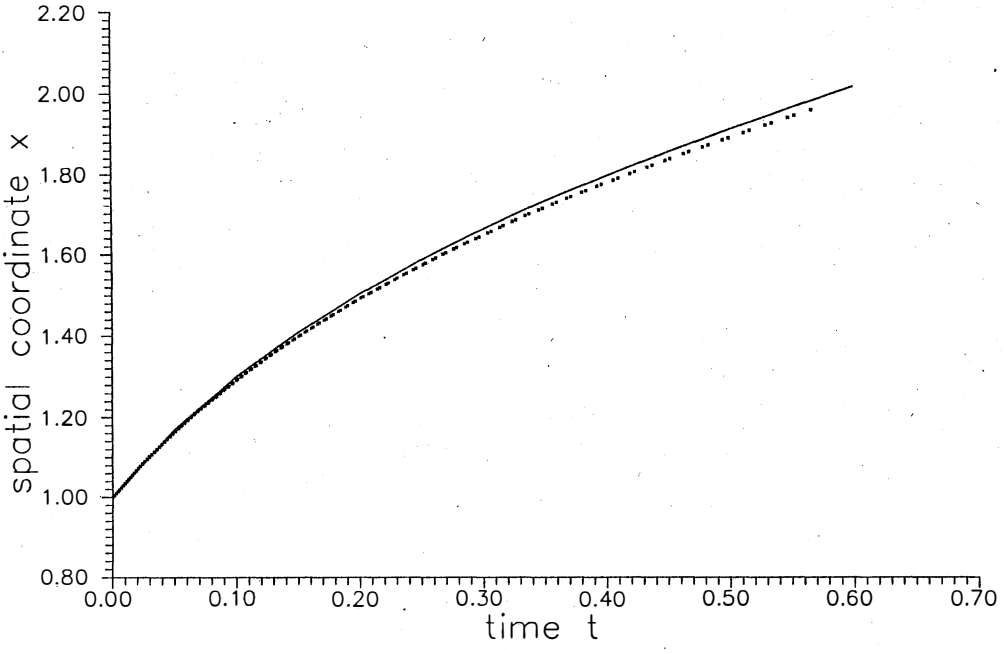


Figure 6: Exact and numerical interface curves for the porous media equation with $m = 2$.

with a triangular initial profile. Its evolution is shown in Fig. 11. As a nonclassical example we include Figs. 12, 13. One may suspect that a positive waiting time will be present, by comparison with the analogous problem for the porous media equation (Fig. 7). Numerical evidence for this waiting time is shown in Fig. 14. Practically in all cases we begin by discretising the range of the initial data, typically in 200 intervals, and taking as its approximation a linear combination of step functions. We have named $x_i$ the abscissas of the discontinuities of the numerical initial profile. The following step is to discretise $\alpha(u)$ according with the assumed discretisation for $u_i$. Then, we integrate in time the system of ordinary differential equations (4), by using a standard variable step, fourth-fifth order Runge-Kutta integrator routine. We have implemented a standard ZBRENT type routine to detect the exact time at that disappears a region of constant $u$ (at a maximum, for instance). The numerical solutions presented were run in a AT 386 based machine, and take about 30-60 minutes each. In some of the examples, mainly in those dealing with the porous media equation for m=3, the system of equations becomes stiff, but not to the extent to force a change of the integrator routine, nor a modification of the system (4) itself. The stiffness of the system is closely related with evolving data that develop vertical edges at u=0. No attempt has been made to modify the system (4) in these situations.

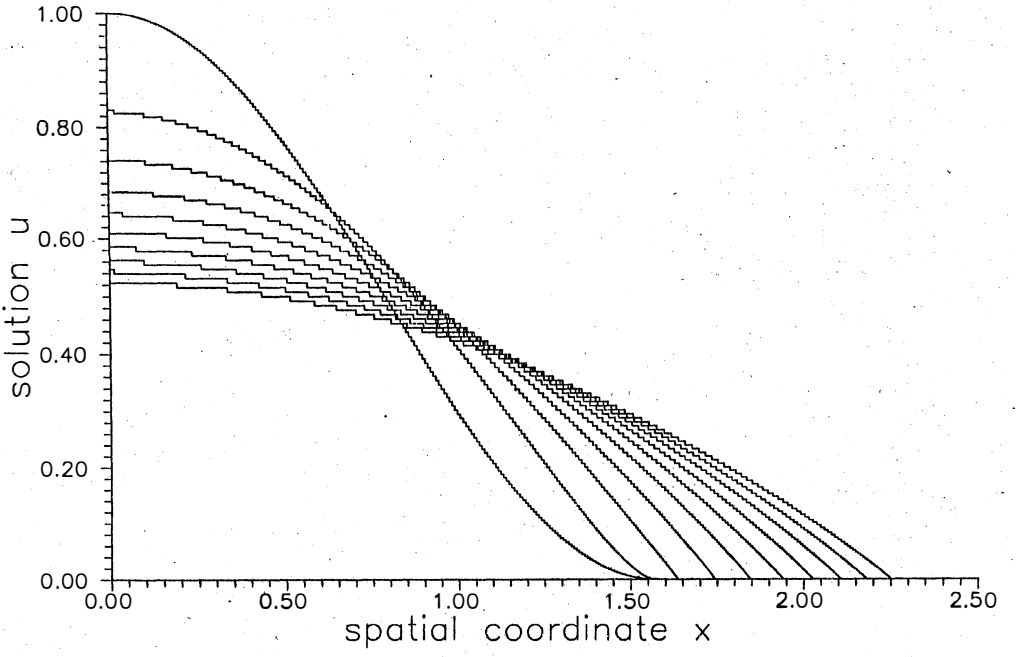


Figure 7: Numerical solution of the porous media equation with $m = 2$, and initial data $\cos^2(x)$.

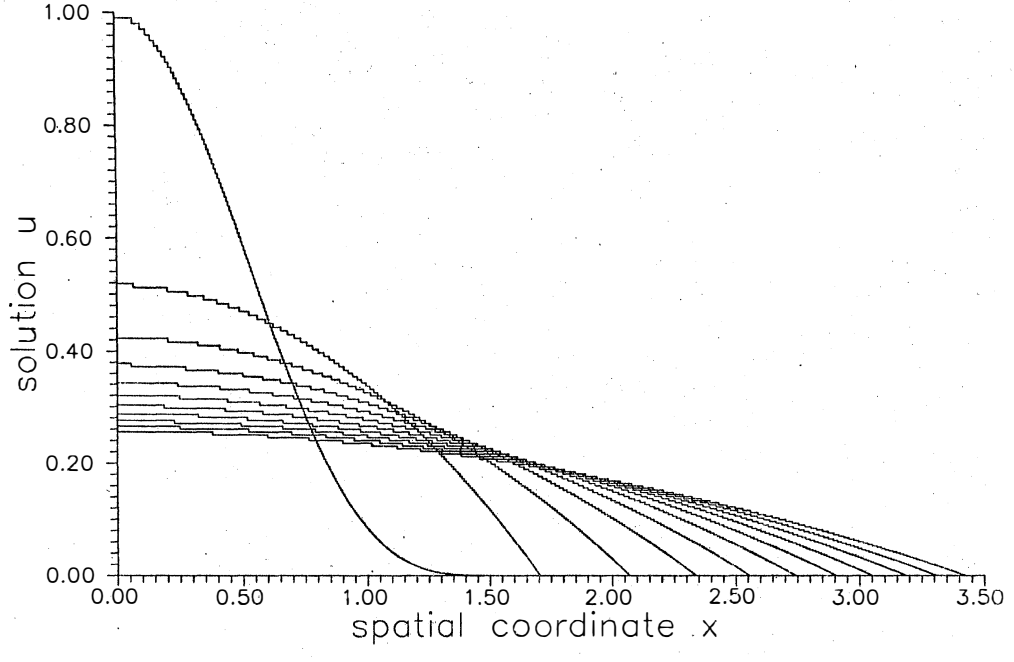


Figure 8: Numerical solution of the porous media equation with $m = 2$, and initial data $\cos^4(x)$.

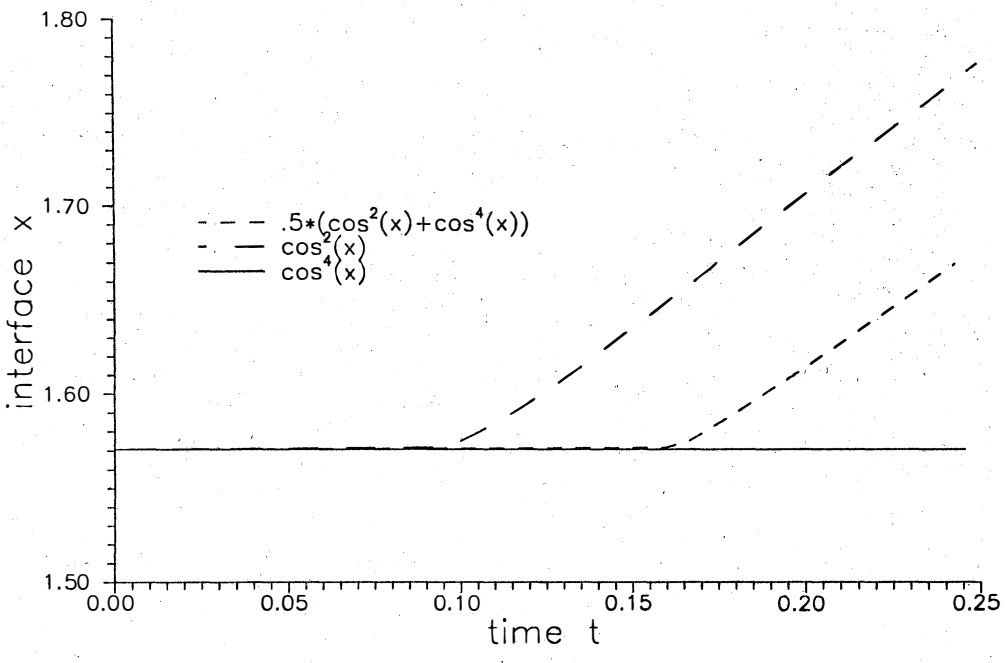


Figure 9: Numerical interface curves for the porous media equation with $m = 2$.

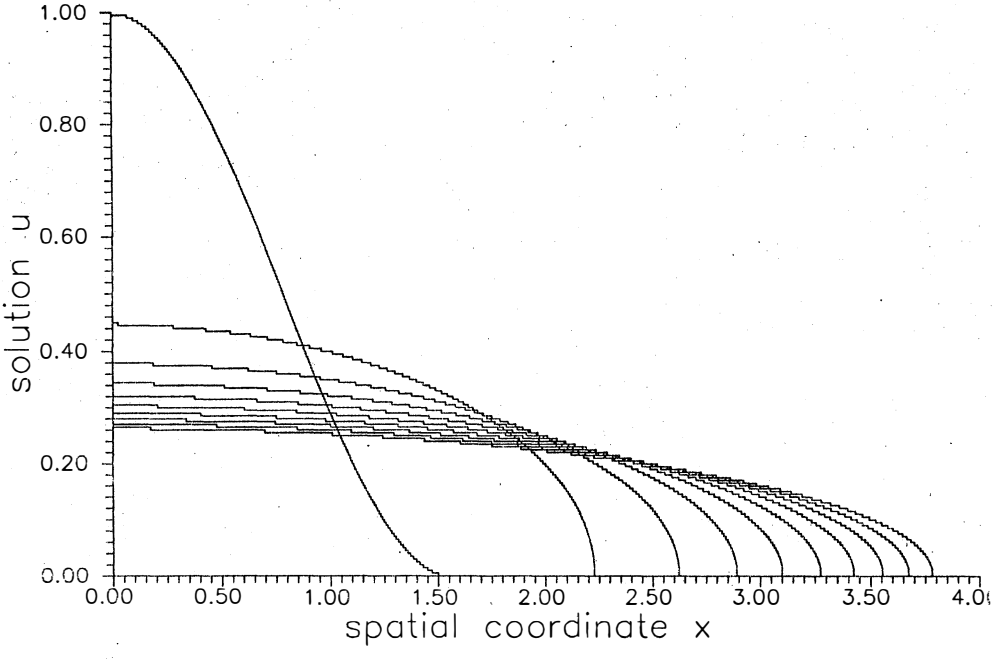


Figure 10: Numerical solution of the porous media equation with $m = 3$, and initia data $\cos^2(x)$.

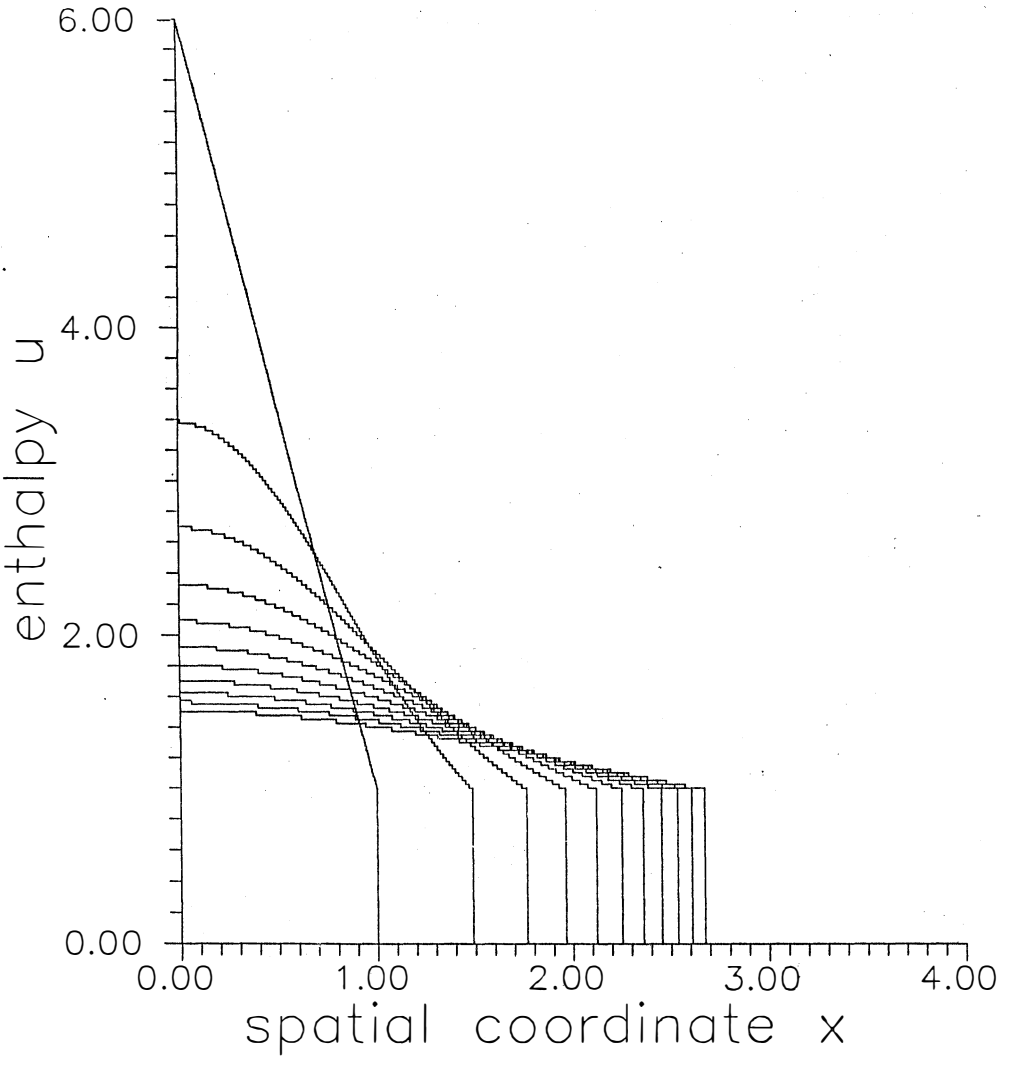


Figure 11: Solution of the Stefan problem $u_t = (u-1)^+_{xx}$ with a sharp peaked initial profile.

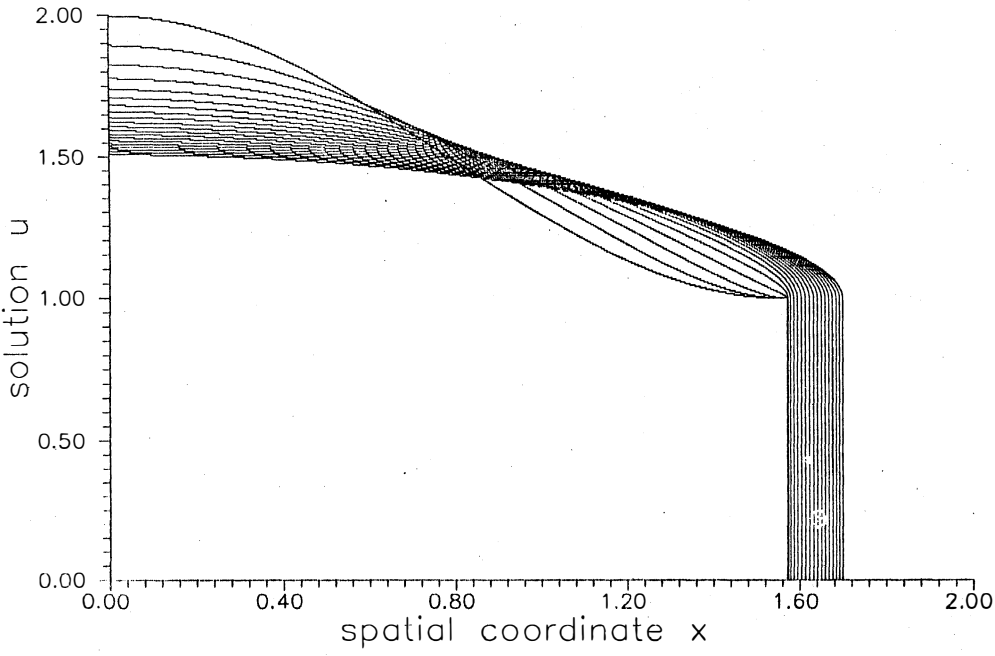


Figure 12: Solution of the problem $u_t = ((u-1)^+)^2_{xx}$ with $1+\cos^2(x)$ as initial data.

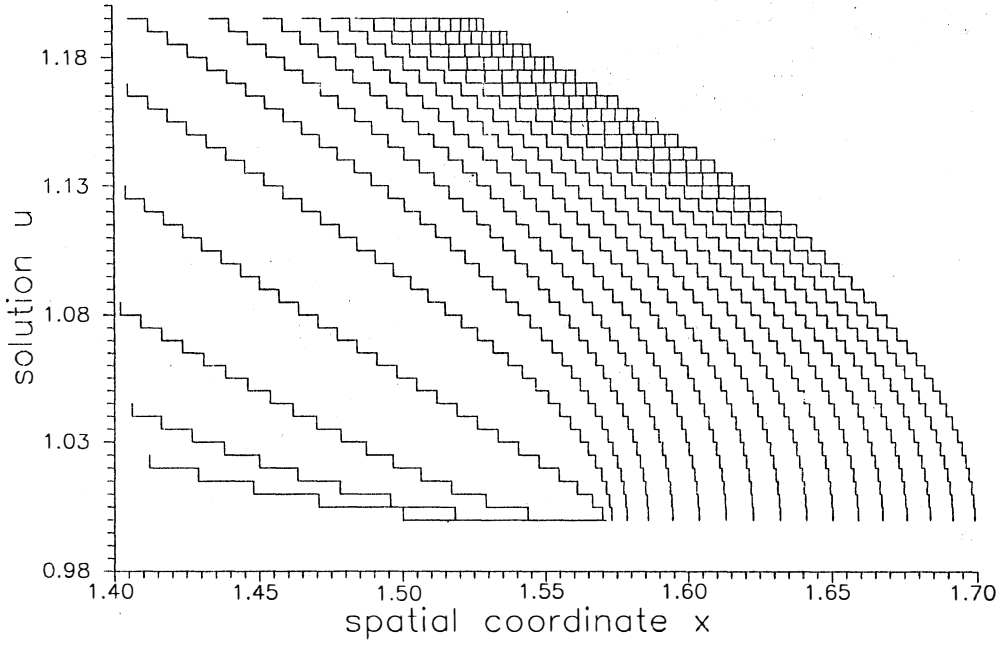


Figure 13: Detail of the support evolution for Fig. 12.

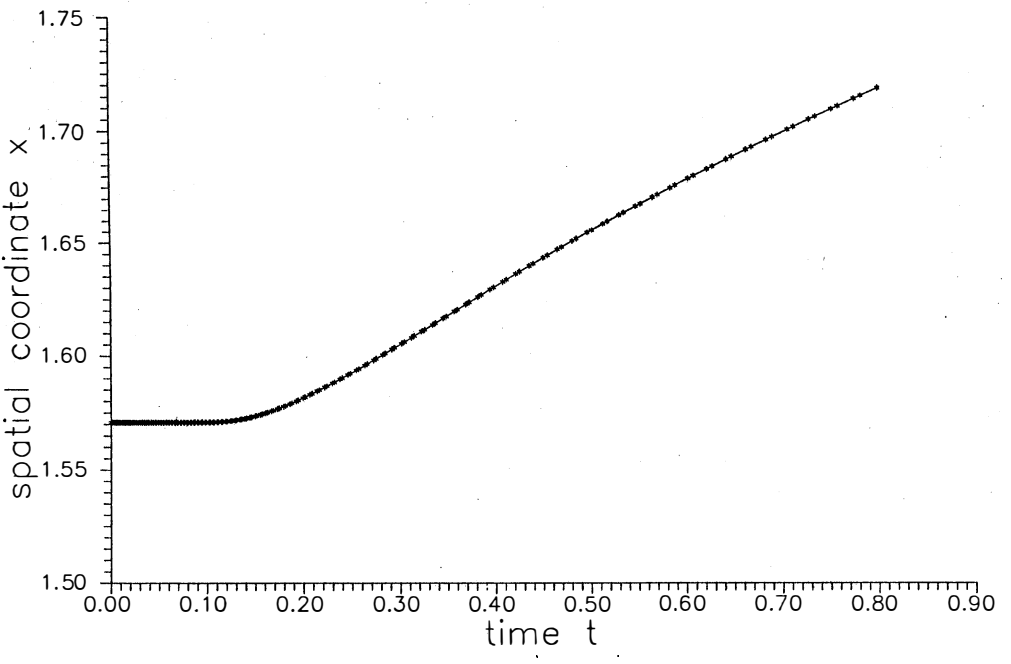


Figure 14: Numerical interface curve for the solution of Figs. 12, 13.

# References

[1] D.G. Aronson. *The porous medium equation*, in Nonlinear Diffusion Problems. A. Fasano, M. Primicerio, Editors. Lecture Notes in Mathematics 1224. Springer-Verlag, 1986.

[2] J.L. Graveleau and P. Jamet. *A finite difference approach to some degenerate nonlinear parabolic equations.* SIAM J. Appl. Math. **20** (1971), 199-223.

[3] K. Tomoeda and M. Mimura. *Numerical approximations to interface curves for a porous media equation.* Hiroshima Math. J. **13** (1983), 273-294.