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SINGULAR LIMITS AND THE "MESA" PROBLEM

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ABSTRACT: For $x \in R$, we discuss the "mesa" type limit of the one-phase Stefan problem in enthalpic variables. This limit is the same as for the porous medium equation, and coincides with the asymptotic limit when time tends to infinity of the soluition of the Stefan problem. We discuss a degenerate diffusion problem where the diffusivity is concentrated on the free boundary, related to the limit when $m \to 0$ in the porous medium equation. The solution to this diffusion reaches in finite time a constant state, which turns out to be the same as in the first three cases: a function $0 \leq u_\infty \leq 1$, which coincides with the initial datum in a set which can be identified by a variational inequality. We show an examle where for $n > 1$ the speed of the free boundary describing the extinction of the zone $u_I(x) > 1$ tends to $-\infty$ as $t \to t_*$.

We wish to honour the dear memory of our late teacher, advisor, and friend Julio E. Bouillet by contributing to this volume with a paper - hitherto in preprint form - written jointly with him. We kept the paper (see [BKM]) in its original form (including the dedication to Prof. Mischa Cotlar). Since this paper has been written, many results have been obtained on the problem of singular limits; we have added some references ([BI1], [BI2], [BBH], [I], [BKM], and [S1]); and the behaviour of weak solutions to equation (1.3) has been extensively discussed ([AK], [K1], [K2]). However, we think that the computations of this paper may still have some interest: as an application of the results of section 4, concerning an elliptic-parabolic problem where the diffusion is concentrated on the free boundary, one can find in [BE] a numerical method for the treatment of solutions to diffusions of the type $u_t = (\alpha(u))_{xx}$, in which ϕ is seen as a limit of a sequence of linear combinations of step functions.

Dedicated to Professor Mischa Cotlar on his 75th. birthday.

1. INTRODUCTION.

The singular limit $m \to \infty$ for solutions $u = u_m$ of

$$\begin{aligned} u_t &= \Delta(|u|^{m-1} u), \quad t > 0 \\ u(x,0) &= u_I(u) \end{aligned} \tag{1.1}$$

was studied by [CF], [FH], [EHKO], [S]. The limit u_∞ is independent of t, equals 1 on a set that, roughly speaking, contains the points where $u_I(x) > 1$, and is equal to $u_I(x)$ (< 1) otherwise. We shall restrict attention to one space dimension operators and initial values $u_I(x)$, sufficiently smooth, such that $\{x : u_I(x) > 1\} = (x_1, x_2)$ and, whenever necessary, $u_I \in L^1(R)$. These

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"parabolic" operators, of a degenerate nature, are

A)

$$u_t = \{m(u-1)^+\}_{xx}\ , \quad x \in R\,,\ t > 0\ ; \tag{1.2}$$

We obtain u_∞ observing that , by re-scaling, it is equivalent to finding $lim_{t\to\infty} u(x,t)$ for a solution $u(x,t)$ of

$$\begin{aligned} u_t &= \{(u-1)^+\}_{xx}\ , \quad t > 0 \\ u(x,0) &= u_I(x)\ . \end{aligned} \tag{1.3}$$

This function u_∞ is the same as in (1.1).

B) The limit for the operator in (1.1) as $m \to 0^+$ is $u_t = \{sgn\ u\}_{xx}$: We change variables to give

$$\begin{aligned} u_t &= \{H(u-1)\}_{xx}\ , \quad t > 0 \\ u(x,0) &= u_I(x) \end{aligned} \tag{1.4}$$

where H is the Heaviside function.
We find the solution $u_0(x,t) = u(x,t)$ to this problem. Under our assumptions on u_I , there is a $t_* > 0$ such that

$$u_0(x,t) \equiv u_0(x,t_*)\ , \quad t \geq t_*\ ,$$

and $u_0(x,t_*)$ is the same singular limit found in (1.1) and (1.3).
In other words, the function $u_\infty(x)$ is obtained instantly, as soon as $t > 0$, as a solution of the evolution operator $u_t = \varphi_\infty(x)_{xx}$ (φ_∞ is the limit of the graphs $|u|^{m-1}u$ as $m \to \infty$, cf. [S]); it is found as $lim_{t\to\infty} u(x,t)$ for $u(x,t)$ solution of (1.3), and is reached in finite time by the solution of (1.4) (in some sense, (1.4) inherits the finite extinction time property of solutions to (1.1) with $m < 1$).

The singular limit u_∞ of (1.1) can be described by a variational inequality [CF]. We discuss the variational inequalities - obtained via the Baiocchi transformation - for both (1.3) and (1.4). While the variational inequalities, depending on t as a parameter, are substantially the same for (1.1) and (1.3), the inequality in (1.4) differs in the definition of the convex set K_t for $t \leq t_*$.

We present also an example of (1.4) in R^n with radial symmetry that shows that for $n > 1$ the speed of the free boundary describing the extinction of the zone $u_I(|x|) > 1$ tends to $-\infty$ as $t \to t_*^-$.

2. PIECEWISE SMOOTH SOLUTIONS.

We note that in both (1.3) and (1.4) the function $\alpha(u)$ is such that $\alpha' \equiv 0$ on intervals of values of u . Therefore if $\alpha(u(x,t)) \equiv constant$ is to hold for (x,t) in an open set, we must have $u_t(x,t) \equiv 0$ there. Moreover, for $u(x,t) > 1$ in (1.3) the eqn. reduces to $u_t = u_{xx}$. Thus we are led to consider generalised solutions to $u_t = \alpha(u)_{xx}$ that are piecewise classical, i.e., are obtained by juxtaposition of classical solutions along smooth curves, in a way similar to the fitting of classical solutions along a shock discontinuity of a first order conservation law (for a particular case, cf. [W]).

Furthermore, when $\alpha = H(u-1)$ in (1.4) we will be forced to redefine $\alpha(u(x,t))$ when $u(x,t) \equiv 1$: we shall take H to be

$$\begin{aligned} H(u-1) &= 1 \quad \text{if } u > 1 \,, \\ &= [0,1] \quad \text{if } u = 1 \,, \\ &= 0 \quad \text{if } u < 1 \,, \end{aligned} \tag{2.1}$$

and put, for all $x, t > 0$,

$$\begin{aligned} U(x,t) &\in H(u(x,t)-1) \,, \\ u_t &= U_{xx} \,, \end{aligned} \tag{2.2}$$

thus if $u(x,t) = 1$ on an open set we must have $u_t(x,t) = 0$ there, and

$$U_{xx} = 0$$

so $U(x,t)$ must be linear in x for fixed t .

Now if a piecewise classical, bounded function $u(x,t)$ is a weak solution of $u_t = \alpha(u)_{xx}$, $u(x,0) = u_I$, in the usual sense,

$$\forall \varphi \in C_0^\infty(R^2) \,, \quad \iint u\,\varphi_t = \iint \alpha(u)\,\varphi_{xx} + \int u_I \varphi(x,0) \tag{2.3}$$

and $u = u_s$ in a ball $B((x_0,t_0),r)$ to the left of a differentiable arc γ, $u = u_d$ in B to the right, taking a test function φ supported in B and integrating by parts give

$$\begin{aligned} &\int_\gamma \varphi\{(u_s - u_d)\,dx - (-\alpha(u_s)_x - (-\alpha(u_d)_x))dt\} = \\ &\int_\gamma (\alpha(u_s) - \alpha(u_d))\varphi_x \, dt \end{aligned} \tag{2.4}$$

whence we obtain

$$(i) \qquad u_s = u_d \quad \text{if } \gamma = \{t = \mathit{constant}\} \text{ at } (x_0, t_0) \,;$$

$$(ii) \begin{cases} \alpha(u_s) &= \alpha(u_d), \ \mathit{and} \\ \alpha(u_s)_x &= \alpha(u_d)_x \end{cases} \text{if } \gamma = \{x = \mathit{constant}\} \text{ at } (x_0, t_0); \tag{2.5}$$

$$(iii) \begin{cases} \alpha(u_s) &= \alpha(u_d) \,, \text{ and, e.g.} \\ \gamma'(t) &= \frac{-\alpha(u_s)_x - (-\alpha(u_d)_x)}{u_s - u_d} = \frac{[-\alpha(u)_x]}{[u]} \end{cases} \quad \text{on } x_0 = \gamma(t_0) \,.$$

(ii) or (iii) imply that two classical solutions $u_s \geq 1$, $u_d \leq 1$ of $u_t = H(u-1)_{xx}$ ((1.4)) cannot join along a curve $\gamma(t)$, and that a region where $u(x,t) \equiv 1$ is needed to let $U(x,t) \in H(u(x,t)-1)$ go from 1 to zero linearly for fixed t (cf. [B]).

3. PROBLEM (1.3), $\alpha(u) = (u-1)^+$.

Solving

$$\begin{aligned} u_t &= \{(u-1)^+\}_{xx} \,, \quad t > 0 \\ u(x,0) &= u_I(x) \,, \quad \{x \,:\, u_I > 1\} = (x_1, x_2) \end{aligned} \tag{1.3}$$

is accomplished by joining the solution $u(x,t) \equiv u_I(x)$ for $x \notin [x_1, x_2]$ with a solution of the heat equation, as follows

(i) $u_t = u_{xx} \quad t > 0\,,\ s(t) < x < d(t)\,,$

(ii) $u(x,0) = u_I(x)\,, \quad x_1 \le x \le x_2\,,$ (i.e. $u_I(x) > 1$)

(iii) $-u_x(s(t)^+, t) = (1 - u_I(s(t)^-)).s'(t)\,, \quad s(0) = x_1\,, \quad (3.2)$

(iv) $-u_x(d(t)^-, t) = (1 - u_I(d(t)^+)).d'(t)\,, \quad d(0) = x_2\,,$

(v) $u(x,t) \equiv u_I(x)$ whenever $x < s(t)$ or $x > d(t)$.

We observe here that (3.2) (i-v) is a Stefan problem with variable latent heat.
Clearly, $s'(t) \le 0\,,\ d'(t) \ge 0$. The behaviour of, say, $s(t)$ as $t \to 0^+$ was studied by [HN], [N], who found that

$$s(t) - x_1 = -\kappa\sqrt{t} + O(\sqrt{t}) \tag{3.3}$$

where κ depends on

$$p := \frac{u_I'(x_1^+)}{u_I'(x_1^-)}$$

through a nonlinear equation.
As for the behaviour as $t \to +\infty$, notice that integration by parts and use of (3.2)(iii-iv) give

$$\int_{s(t)}^{d(t)} u(x,t)\,dx = \int_{s(t)}^{d(t)} u_I(x)\,dx\,; \tag{3.4}$$

By comparison with

$$\frac{K.e^{-(x-x_1)^2/4(t+\tau)}}{\sqrt{2\pi(t+\tau)}} + 1\,,$$

for suitable K , $\tau > 0$, one finds that $\overline{lim}_{t\to\infty} u(x,t) \le 1$, and in particular $lim_{t\to\infty} u(x,t) = 1$ for $s(\infty) < x < d(\infty)$: passing to the limit in (3.4) gives

$$d(\infty) - s(\infty) = \int_{s(\infty)}^{d(\infty)} u_I(x)\,dx \tag{3.5}$$

and therefore $d(\infty) < \infty\,,\ s(\infty) > -\infty$ if $u_I(x) \in L^1(R)$.
From eqs. (3.2) we also obtain

$$\frac{\partial}{\partial t}\int_{s(t)}^{d(t)} x\,u(x,t)\,dx = \frac{\partial}{\partial t}\int_{s(t)}^{d(t)} x\,u_I(x)\,dx$$

Integrating in $(0,T)$ and letting $T \to \infty$ yields

$$\frac{d(\infty)^2 - s(\infty)^2}{2} = \int_{s(\infty)}^{d(\infty)} x\,u_I(x)\,dx$$

$$\frac{d(\infty) + s(\infty)}{2} = \frac{\int_{s(\infty)}^{d(\infty)} x\,u_I(x)dx}{\int_{s(\infty)}^{d(\infty)} u_I(x)dx} \tag{3.6}$$

In fact, (3.5) and (3.6) completely determine $u_\infty(x) := lim_{t\to\infty} u(x,t)$, as will be seen below (cf. §4).

4. PROBLEM (1.4).

We discuss now the initial value problem

$$\begin{aligned} &U(x,t) \in H(u(x,t)-1)\,, \quad t>0\,, \\ &u_t = U_{xx}\,, \quad t>0\,, \\ &u(x,0) = u_I(x)\,, \quad \{x : u_I > 1\} = (x_1, x_2)\,. \end{aligned} \tag{1.4}$$

As $u_t \equiv 0$ on open sets where $u > 1$ or $u < 1$, and $\int u(x,t)\varphi(x) \to \int u_I\varphi(x)$ as $t \to 0^+$, it is clear that $u \equiv u_I(x)$ for $x \neq x_1$, $x \neq x_2$ and <u>small</u> $t > 0$. However, regions where $u(x,t) > 1$ and $u(x,t) < 1$ cannot join along an arc γ, due to the condition $\alpha(u_s) = \alpha(u_d)$ of (2.5)(ii-iii): we need a region where $u \equiv 1$ and $U(x,t) \in [0,1]$ satisfies $U_{xx} = 0$. Consider the right end-point x_2 of $\{x : u_I(x) > 1\}$, to fix ideas.
A region $\{(x,t) : s(t) < x < d_2(t),\ t > 0,\ s_2(0) = d_2(0) = x_2\}$ separates $x < s_2(t)$ where (locally) $u \equiv u_I > 1$, and $U \equiv H(u-1) = 1$, from $\{x > d_2(t)\}$, where $u \equiv u_I < 1$, and $U \equiv H(u-1) = 0$; taking into account the region (where $u \equiv 1$ also) $\{(x,t) : s_1(t) < x < d_1(t),\ t > 0,\ s_1(0) = d_1(0) = x_1\}$, we have a function $U(x,t)$, piecewise linear in x, defined by

$$\begin{aligned} U(x,t) &= 0\,, \quad x < s_1(t) \\ &= \frac{x - s_1(t)}{d_1(t) - s_1(t)}\,, \quad s_1(t) \le x \le d_1(t) \quad (u \equiv 1) \\ &= 1\,, \quad d_1(t) < x < s_2(t) \\ &= \frac{x - s_2(t)}{s_2(t) - d_2(t)}\,, \quad s_2(t) \le x \le d_2(t) \quad (u \equiv 1) \\ &= 0\,, \quad d_2(t) < x \end{aligned} \tag{4.2}$$

This expression for U holds as long as $d_1(t) \le s_2(t)$. The existence of a $t_* > 0$ such that $d_1(t_*) = s_2(t_*)$ implies $U(x,t_*^+) \equiv 0$, and the evolution ends, giving $u(x,t) \equiv u(x,t_*)$, $t \ge t_*$.
We obtain a system of differential equations for $s_2(t)$, $d_2(t)$ (and likewise for s_1, d_1) by satisfying (2.5)(iii):

$$s_2'(t) = \frac{1}{1 - u_I(s_2(t))} \cdot \frac{1}{d_2(t) - s_2(t)} \quad (\le 0)\,, \quad s_2(0) = x_2\,, \tag{4.3}$$

$$d_2'(t) = \frac{1}{1 - u_I(d_2(t))} \cdot \frac{1}{d_2(t) - s_2(t)} \quad (\ge 0)\,, \quad d_2(0) = x_2\,. \tag{4.3}$$

We pause now to consider some examples.

I.

$$\begin{aligned} u_I(x) &= a > 1\,, \quad x \in [-x_1, x_1]\,, \\ &= b < 1\,,\ b > 0\,, \quad x \notin [-x_1, x_1]\,. \end{aligned} \tag{4.4}$$

In this case

$$s' = \frac{1}{1-a}\,\frac{1}{d-s}\,; \quad d' = \frac{1}{1-b}\,\frac{1}{d-s}\,; \quad s(0) = d(0) = x_1$$

$$(d-s)(d-s)' = \frac{1}{1-b} - \frac{1}{1-a} \Rightarrow d-s = \sqrt{2\left(\frac{b-a}{(1-b)(1-a)}\right)t}$$

$$\Rightarrow s' = \frac{1}{(1-a)} \cdot \frac{1}{\sqrt{2\frac{(b-a)}{(1-a)(1-b)}t}} \Rightarrow s(t) = x_1 - \sqrt{\frac{2(1-b)}{(a-b)(a-1)}t}$$

$$\Rightarrow d' = \frac{1}{1-b} \cdot \frac{1}{\sqrt{2\frac{a-b}{(a-1)(1-b)}t}} \Rightarrow d(t) = x_1 + \sqrt{\frac{2(a-1)}{(a-b)(1-b)}t}$$

we see that $s = 0$ when $t = \frac{(a-b)(a-1)}{2(1-b)} x_1^2 = t_*$.

At this time $u(x,t) = 1$ on $\left[-\frac{a-b}{1-b}x_1, \frac{a-b}{1-b}x_1\right]$ and $u(x,t) = b < 1$ outside.

II.

$$u_I(x) = -x + 1 \; ; \tag{4.5}$$

$$s' = \frac{1}{-s}\frac{1}{d-s} , \quad d' = \frac{1}{d}\frac{1}{d-s} , \; s(0) = d(0) = 0 \, .$$

$dd' - ss' = 0 \Rightarrow d = -s$. It follows that

$$s(t) = -\left(\frac{3}{2}t\right)^{1/3} , \quad d(t) = \left(\frac{3}{2}t\right)^{1/3}$$

III.

$$\begin{aligned} u_I(x) &= -ax + 1 \, , \quad x < 0 \, , \quad a > 0 \, , \\ &= -bx + 1 \, , \quad x > 0 \, , \quad b > 0 \, . \end{aligned} \tag{4.6}$$

$$s' = \frac{1}{as}\frac{1}{d-s} , \quad d' = \frac{1}{bd}\frac{1}{d-s} , \; s(0) = d(0) = 0 \, .$$

$bdd' - ass' = 0 \Rightarrow \sqrt{b}d = -\sqrt{a}s$. The system becomes

$$s' = \frac{-1}{a\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{b}}\right)} \cdot \frac{1}{s^2} \; ; \quad d' = \frac{1}{b\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}}\right)}\frac{1}{d^2}$$

$$\Rightarrow s(t) = -\left(\frac{3\sqrt{b}}{a(\sqrt{a}+\sqrt{b})}t\right)^{1/3} , \quad d(t) = \left(\frac{3\sqrt{a}}{b(\sqrt{a}+\sqrt{b})}t\right)^{1/3}$$

We observe that for smooth $u_I(x)$, the "plateau" $u \equiv 1$ expands as $t^{1/3}$, while for discontinuous initial values the behaviour of the free boundaries resembles that of the self-similar solutions: $s, d \sim \sqrt{t}$.
We note in passing the dependence of the speeds of s and d on the side derivatives of the initial values $u_I(x)$ at points where $u_I = 1$, reminiscent of (3.3).
Inspection of the ODE system (4.3) yields a first integral

$$d(t) - s(t) = \int_{s(t)}^{d(t)} u_I(x)\, dx \tag{4.7}$$

which compares with (3.4), (3.5): it actually means conservation of mass, as before. Conservation of the first moment furnishes another first integral, as in (3.6): in fact,

$$\int x\,u(x,t)\,dx = \int_{\{u(x,t)=u_I(x)\}} x\,u_I(x)\,dx + \int_{s(t)}^{d(t)} x\,dx\,, \text{ say}\,.$$

$$\text{But} \quad \int_s^d x\,dx = \frac{d^2(t)-s^2(t)}{2} = \frac{d(t)+s(t)}{2}\int_{s(t)}^{d(t)} u_I dx$$

from above. Now integration by parts in $\int_{s(t)}^{d(t)} x\,u_I(x)dx$ gives the same expression, hence

$$\int x\,u(x,t)\,dx = \int x\,u_I(x)\,dx\,,$$

$$\text{and} \quad \frac{d^2(t)-s^2(t)}{2} = \int_{s(t)}^{d(t)} x\,u_I(x)\,dx\,. \tag{4.8}$$

A "pictorial" way of finding $u(x,t_*)$ employing these integrals is the following, which hints at the possibility of discussion of initial values giving rise to many "mesas" as well: Put $F(x) = \int_0^x u_I(r)dr$. Assume, for simplicity, $u_I \geq 0$, $\{x : u_I > 1\} = (x_1, x_2)$. $F(x)$ is a monotone increasing function that has slope 1 at $x = x_1$ and $x = x_2$. Consider the tangent lines to F at these points, and the portion of the graph of F contained between these parallel lines. The evolution from $u_I(x)$ proceeds by moving the lower line upwards from $(x_1, F(x_1))$, and the upper line downwards from $(x_2, F(x_2))$, keeping their slope. The evolution ends at a certain position where the two lines meet, dividing portions of $F(x)$ that cut equal area above and below the said line of unit slope.

5. THE VARIATIONAL INEQUALITIES CORRESPONDING TO PROBLEM (1.3).

Let $u(x,t)$ be the solution to the following problem:

$$\begin{aligned} u_t &= (\alpha(u))_{xx} && \text{in} \quad R\times(0,\infty) \\ u|_{t=0} &= u_I && \text{in} \quad R \end{aligned} \tag{1.3}$$

where $\alpha(u) = (u-1)^+$.
We want to obtain the set $\{x : lim_{t\to\infty} u(x,t) = 1\}$ from the solution to a variational inequality. Using the Baiocchi transformation we define a function w as follows:

$$w(x,t) = \int_0^t \alpha(u(x,\tau))d\tau \qquad \text{in} \quad R\times(0,\infty)\,;$$

then $w(x,t) = 0$ if $x < s(t)$ or $x > d(t)$; besides, $w(x,t) > 0$ and $w_{xx}(x,t) = u(x,t) - u_I(x)$ if $s(t) < x < d(t)$. In fact, if $d(0) < x < d(t)$

$$w(x,t) = \int_{d^{-1}(x)}^t (\alpha(u))(x,\tau)d\tau$$

then

$$w_x(x,t) = \int_{d^{-1}(x)}^t (\alpha(u))_x(x,\tau)d\tau$$

and

$$\begin{aligned}
w_{xx}(x,t) &= \int_{d^{-1}(x)}^{t} (\alpha(u))_{xx}(x,\tau)d\tau - (\alpha(u))_x(x,d^{-1}(x)).(d^{-1})'(x) \\
&= \int_{d^{-1}(x)}^{t} u_t(x,\tau)d\tau + d'(d^{-1}(x))(1-u_I(x))(d^{-1})'(x) \\
&= u(x,t) - u(x,d^{-1}(x)) + 1 - u_I(x) \\
&= u(x,t) - u_I(x) \ .
\end{aligned}$$

The proof is similar in the case $s(t) < x \le d(0)$.
Let us see that

$$\lim_{x \to d(t)^-} w_x(x,t) = 0 \ .$$

As $u_t \in L^1_{loc}(\{u \ge 1\})$ we have

$$\int_{d^{-1}(x)}^{t} \int_{x}^{d(\tau)} u_t(\xi,\tau)d\xi\, d\tau \to 0\ , \ \text{if } x \to d(t)^- \ ,$$

but

$$\begin{aligned}
\int_{d^{-1}(x)}^{t} \int_{x}^{d(\tau)} u_t(\xi,\tau)d\xi\, d\tau &= \int_{d^{-1}(x)}^{t} \int_{x}^{d(\tau)} (\alpha(u))_{xx}(\xi,\tau)d\xi\, d\tau \\
&= \int_{d^{-1}(x)}^{t} (\alpha(u))_x(d(\tau),\tau)d\tau - \int_{d^{-1}(x)}^{t} (\alpha(u))_x(x,\tau)d\tau
\end{aligned}$$

and

$$\int_{d^{-1}(x)}^{t} (\alpha(u))_x(d(\tau),\tau)d\tau \to 0\ , \quad \text{if} \quad x \to d(t)^-$$

because we are assuming that d is sufficiently smooth. Then

$$\int_{d^{-1}(x)}^{t} (\alpha(u))_x(x,\tau)d\tau \to 0\ , \quad \text{if} \quad x \to d(t)^-$$

that is what we wanted to prove.
In the same way we can see that

$$\lim_{x \to s(t)^+} w_x(x,t) = 0 \ .$$

As $u(x,t) - 1 > 0$ if $s(t) < x < d(t)$ we can write

$$u(x,t) - u_I(x) = (u(x,t)-1)^+ + 1 - u_I(x)$$

(we shall use this decomposition below).
We define a family of functions $\{w^t\}_{t>0}$ as

$$w^t(x) = w(x,t)\ , \qquad x \in R\ , \qquad t > 0\ ,$$

and consider an interval, bounded by (3.5), $(-M, M)$ such that

$$-M < s(\infty) < s(t) < d(t) < d(\infty) < M \ .$$

In $(-M,M)$, w^t is a C^1 function

$$\begin{aligned}
&w^t = 0 \qquad \text{in} \quad [-M, s(t)] \cup [d(t), M]\ , \\
&w^t > 0\ , \quad \text{and} \\
&(w^t)'' = (u^t - 1)^+ + 1 - u_I \qquad \text{in} \quad (s(t), d(t))\ ,
\end{aligned}$$

where $u^t(x) = u(x,t)$, $t > 0$. It is easy to see that w^t is "the" solution to the following problem: Find $w^t \in K$ such that

$$\int_{-M}^{M} (w^t)'(v - w^t)' \geq \int_{-M}^{M} (u_I - 1 - (u^t - 1)^+)(v - w^t) \ \forall v \in K, \ t > 0 \tag{5.1}$$

where $K = \{v \in H_0^1(-M,M) : 0 \leq v\}$.
If $w_* \in K$ satisfies

$$\int_{-M}^{M} w_*'(v - w_*)' \geq \int_{-M}^{M} (u_I - 1)(v - w_*) \qquad \forall v \in K \tag{5.2}$$

then $w^t \leq w_*$; in fact, if we consider $v = -(w^t - w_*)^+ + w^t$ and $v = (w^t - w_*)^+ + w^t$ as test functions in (5.1) we obtain

$$-\int_{-M}^{M} (w^t)'(w^t - w_*)^{+'} \geq -\int_{-M}^{M} (u_I - 1 - (u^t - 1)^+)(w^t - w_*)^+$$

and

$$\int_{-M}^{M} (w^t)'(w^t - w_*)^{+'} \geq \int_{-M}^{M} (u_I - 1 - (u^t - 1)^+)(w^t - w_*)^+$$

so

$$\int_{-M}^{M} (w^t)'(w^t - w_*)^{+'} = \int_{-M}^{M} (u_I - 1 - (u^t - 1)^+)(w^t - w_*)^+ . \tag{5.3}$$

Besides, if we consider $v = (w^t - w_*)^+ + w_*$ as a test function in (5.2) we obtain

$$\int_{-M}^{M} w_*'(w^t - w_*)^{+'} \geq \int_{-M}^{M} (u_I - 1)(w^t - w_*)^+ , \tag{5.4}$$

then

$$\int_{-M}^{M} [(w^t - w_*)^{+'}]^2 \leq \int_{-M}^{M} -(u^t - 1)^+(w^t - w_*)^+ \leq 0 ,$$

so

$$w^t \leq w_* , \qquad \forall t > 0 ,$$

and we obtain

$$\{w_* > 0\} \supseteq (s(\infty), d(\infty)) .$$

To prove the other inclusion recall that

$$\|w^t - w_*\|_{H_0^1} \leq C \, \|u_I - 1 - (u^t - 1)^+ - (u_I - 1)\|_{(H_0^1)'} , \tag{5.5}$$

(see [KS]) and as

$$\left|\int_{-M}^{M} (u^t - 1)^+ v\right| \leq \|(u^t - 1)^+\|_{L^2} \|v\|_{L^2}$$
$$\leq C \, . \|(u^t - 1)^+\|_{L^2} \|v\|_{H_0^1} , \quad \forall v \in H_0^1$$

we have

$$\|(u^t - 1)^+\|_{(H_0^1)'} \leq C \, \|(u^t - 1)^+\|_{L^2} ;$$

from §3 $(u^t - 1)^+ \to 0$ a.e. as $t \to \infty$, so we have

$$\|(u^t - 1)^+\|_{L^2} \to 0 \qquad \text{as} \quad t \to \infty$$

and by (5.5)

$$w^t \to w_* \qquad \text{in} \quad H_0^1 \qquad \text{and a.e.}$$

Then we can deduce that

$$\{w_* > 0\} \subseteq (s(\infty), d(\infty))$$

and obtain

$$\{w_* > 0\} = (s(\infty), d(\infty))\,.$$

6. THE VARIATIONAL INEQUALITIES CORRESPONDING TO PROBLEM (1.4).

Let $u(x,t)$ be the solution to the following problem

$$\begin{aligned} u_t &= (\alpha(u))_{xx} && \text{in} \quad R \times (0,\infty) \\ u|_{t=0} &= u_I && \text{in} \quad R \end{aligned} \tag{1.4}$$

where $\alpha(u) = H(u-1)$ and

$$H(r) = \begin{cases} 0\,, & \text{if } r < 0 \\ [0,1]\,, & \text{if } r = 0 \\ 1\,, & \text{if } r > 0\,. \end{cases}$$

We know that $u(x,t) = u(x,t_*)$, for $t \geq t_*$. We want to obtain the set $\{x : u(x,t_*) = 1\}$ from the solution to the variational inequality which has been studied in §5 and also find t_* from it.

Using the Baiocchi transformation we can define a function w as

$$w(x,t) = \int_0^t U(x,\tau)\,d\tau\,,\; U(x,t) \in H(u(x,t)-1)\,,$$
$$\text{in } R \times (0,\infty), \quad \text{cf. (4.2)},$$

then $w(x,t) = 0$ if $x < s_1(t)$ or $x > d_2(t)$ and $w(x,t) = t$ if $d_1(t) < x < s_2(t)$; besides $0 < w(x,t) < t$ and $w_{xx}(x,t) = 1 - u_I(x)$ if $s_1(t) < x < d_1(t)$ or $s_2(t) < x < d_2(t)$.

The proof is similar to that of §5 and we can also prove that

$$\lim_{x \to d_i(t)^-} w_x(x,t) = 0 \qquad\qquad i = 1,2$$

and

$$\lim_{x \to s_i(t)^+} w_x(x,t) = 0 \qquad\qquad i = 1,2\,.$$

Let us define a family of functions $\{w^t\}_{t>0}$ as

$$w^t(x) = w(x,t) \qquad\qquad \text{for} \quad x \in R\,,\; t > 0$$

and consider an interval $(-M, M)$, bounded by (3.5), such that

$$-M < s_1(t_*) < d_2(t_*) < M\,.$$

In $(-M,M)$, w^t is a C^1 function

$$\begin{aligned} w^t &= 0 && \text{in} \quad [-M, s_1(t)] \cup [d_2(t), M], \\ w^t &= t && \text{in} \quad [d_1(t), s_2(t)], \end{aligned}$$

$0 < w^t < t$ and $(w^t)'' = 1 - u_I$ in $(s_1(t), d_1(t)) \cup (s_2(t), d_2(t))$. For each t , w^t is "the" solution to the problem:

Find $w^t \in K_t$ such that

$$\int_{-M}^{M} (w^t)'(v - w^t)' \geq \int_{-M}^{M} (u_I - 1)(v - w^t) \quad \forall v \in K_t \tag{6.1}$$

where $K_t = \{v \in H_0^1(-M, M) \,:\, 0 \leq v \leq t\}$.

If $t = t_*$, as $d_1(t_*) = s_2(t_*)$ we have

$$\{x \,:\, w^{t_*}(x) = t_*\} = d_1(t_*) = s_2(t_*)$$

and it is easy to see that

$$(w^{t_*})''(d_1(t_*)) = 1 - u_I(d_1(t_*))$$

and

$$\int_{-M}^{M} (w^{t_*})'(v - w^{t_*})' \geq \int_{-M}^{M} (u_I - 1)(v - w^{t_*}) \quad \forall v \in K$$

to obtain

$$w^{t_*} \equiv w_*$$

and then

$$t_* = \max\{w_*(x) \,:\, -M \leq x \leq M\}$$
$$(s_1(t_*), d_2(t_*)) = \{w_* > 0\}$$

If $t > t_*$, $w_* \in K_t$ and

$$\int_{-M}^{M} w_*'(v - w_*)' \geq \int_{-M}^{M} (u_I - 1)(v - w_*) \,, \qquad \forall v \in K_t$$

so

$$w^t \equiv w_* \qquad \forall t \geq t_* \,.$$

7. AN EXAMPLE.

Let us next discuss an example of difussion under the law (1.4) in R^n . To this aim, take $u_I(x) = u_I(|x|) = u_I(r)$, continuous, radially decreasing, $u_I(r) > 1$ for $r \in [0, r_1]$, and $u_I(r) < 1$ for $r_1 < r < r_2$, $supp\, u_I(r) \subset B(0, r_2)$.
As done in (2.2), one should solve

$$\begin{aligned} &U(r,t) \in H(u(r,t) - 1) \\ &\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}u\right) = 0 \\ &U_d = 0 \,, \quad U_s = 1 \end{aligned} \tag{7.1}$$

where $u \equiv 1$, that is, for (r, τ) in $s(\tau) < r < d(\tau)$, $t > r$. The shape of $U(r,t)$ is obtained in a straightforward manner:

$$U(r,t) = \frac{s(t)^{n-2}\, d(t)^{n-2}}{d(t)^{n-2} - s(t)^{n-2}}\, r^{2-n} - \frac{s(t)^{n-2}}{d(t)^{n-2} - s(t)^{n-2}} \,,$$

$$U_r(r,t) = \frac{(2-n)s(t)^{n-2}d(t)^{n-2}}{d(t)^{n-2}-s(t)^{n-2}}\, r^{1-n}\,, \quad \text{for } n>2\,, \text{ and}$$

$$U(r,t) = \frac{\log\, r/d(t)}{\log\, s(t)/d(t)}\,,$$

$$U_r(r,t) = \frac{1}{r\, \log\, s(t)/d(t)}\,, \quad \text{for } n=2\,.$$

The free boundaries are obtained from condition (2.5)(iii), which gives

$$s'(t) = \frac{(2-n)d(t)^{n-2}}{(s(t)d(t)^{n-2}-s(t)^{n-2})}\,\frac{1}{(u_I(s(t))-1)}$$

$$d'(t) = \frac{(2-n)s(t)^{n-2}}{d(t)(d(t)^{n-2}-s(t)^{n-2})}\,\frac{1}{(u_I(d(t))-1)}\,, \quad \text{for } n>2\,, \text{ and}$$

$$s'(t) = \frac{1}{s(t)(u_I(s(t))-1)\,\log\, s(t)/d(t)}$$

$$d'(t) = \frac{1}{d(t)(u_I(s(t))-1)\,\log\, s(t)/d(t)}\,, \quad \text{for } n=2\,.$$

As in the one-dimensional case, it can be seen that in either case $s'(t)$ is bounded above by a negative constant, for $t>\delta>0$. Therefore, the mesa is reached in finite time. Also,

$$\lim_{t\uparrow t_*} s'(t) \sim \lim_{r\to 0+} -\frac{1}{r} = -\infty \quad (n>2)\,, \quad \text{and}$$

$$\lim_{t\uparrow t_*} s'(t) \sim \lim_{r\to 0+} \frac{1}{r\,\log\, r} = -\infty\,,$$

while in the one-dimensional case (cf. (4.3)), $s'(t_*-)$ is finite.

For a similar behaviour in the standard porous medium equation see [A, lecture 5].

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