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THE RELATION BETWEEN PETROV-GALERKIN AND COLLOCATION METHODS USING SPLINE MULTIRESOLUTION ANALYSES

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Abstract

In this work we consider numerical schemes for partial differential equations based on two frameworks. On one hand are the Petrov-Galerkin type schemes based on spline biorthogonal multiresolution analyses. On the other hand are the collocation schemes based on interpolating functions obtained from these biorthogonal basic functions. Our aim is to show that both methods are equivalent when applied to periodic initial value problems for constant coefficient differential equations. In order to reach this purpose we use the characterization of biorthogonal spline wavelets by means of derivatives, primitives, and interpolatory properties. Particularly, we establish the relation between the interpolatory multiresolution analyses coming from convolution of biorthogonal spline families with other ones already existing in the literature and known under different names. We also conclude that both collocation and Petrov- Galerkin methods are equivalent to a Galerkin procedure using Daubechies' orthogonal scaling functions.
The reason for such property comes from the fact that the refinement masks of all the involved basic functions correspond to different factorizations of a same trigonometric polynomial.

1 INTRODUCTION

In some cases, collocation methods can be interpreted as Galerkin methods. For instance, in [30] Swartz and Wendroff consider a periodic initial value problem for a constant coefficient differential equation. They prove that for this kind of problem, the Galerkin scheme arising from splines of order μ has precisely the same solution as the collocation scheme using a basis of cardinal splines of order 2μ. A thorough explotation of this point of view, by combining the stability of Galerkin methods with error bounds for spline interpolation, can be used to obtain the convergence of both methods.

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Our aim in this paper is to extend the results in [30]. We establish a correspondence between the Petrov-Galerkin methods based on biorthogonal spline scaling functions and collocation methods using an associated interpolatory framework. In this sense, our results are also related with those obtained by Bertoluzza and Naldi. For an elliptic boundary value model problem, these authors stablish in [3] the equivalence of the linear system arising in a wavelet-Galerkin formulation using Daubechies' wavelets of compact support and those coming from a collocation method based on approximating trial spaces generated by autocorrelation of these wavelets.

In order to reach our purpose, two aspects of biorthogonal spline multiresolution analyses must be studied, namely, their interpolatory properties and their compatibility with derivatives and primitives.

First we shall analyse the interpolatory multiresolution analyses coming from convolution of biorthogonal spline scaling functions. We establish the relation between such interpolatory framework with other ones already existing in the literature.

Another important aspect about biorthogonal multiresolution analyses is that they are compatible with derivatives and primitives. This means that given a pair of biorthogonal wavelets $\{\Psi, \Psi^*\}$, then differentiating Ψ and integrating Ψ^* we obtain another pair of biorthogonal wavelets $\{\tilde{\Psi}, \tilde{\Psi}^*\}$ [20]. In the case of biorthogonal spline wavelets, the corresponding primitives and derivatives are also within the family of biorthogonal spline wavelets.

This paper is organized as follows. First we recall the main properties of biorthogonal multiresolution analyses in Section 2. Section 3 is dedicated to the analysis of these properties for the particular case of spline biorthogonal wavelets. Finally, in Section 4, we establish the correspondence between the Petrov-Galerkin methods based on biorthogonal spline functions and collocation methods using their associated interpolatory framework.

2 THE BIORTHOGONAL FRAMEWORK

Through this paper $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of integer and real numbers, $L^2(\mathbf{R})$ is the vector space of measurable, square-integrable one dimensional real functions $f(x)$ for which $\langle f, g\rangle$, $||f||$ and $\hat{f}$ stand for the usual inner product, norm and Fourier transform. In $L^2(\mathbf{R})$ we shall consider multiresolution analyses $\{V_j\}$ and associated scaling functions Φ, in the sense defined, for example, in [25] or [24] (see also [10]). As it is well known, a scaling function must satisfy a two scale relation

$$\Phi(x) = 2\sum_{n\in\mathbf{Z}} h(n)\Phi(2x-n). \tag{1}$$

This relation is very important, since it characterizes all the properties of Φ, and consequently of the multiresolution analysis. In the Fourier domain it can be written as

$$\hat{\Phi}(\xi) = H(\xi/2)\hat{\Phi}(\xi/2),$$

where

$$H(\xi) = \sum_{n \in \mathbf{Z}} h(n) e^{-in\xi}.$$

$H(\xi)$ is called the scaling filter for $\{V_j, \Phi\}$ and $h(n)$ the filter coefficients. It is usual to normalize Φ such that $\int_{\mathbf{R}} \Phi(x)dx = 1$, or equivalenttly, $H(0) = \sum_n h(n) = 1$. Another required property is $H(\pi) = 0$, and an important parameter is $p+1$, the order of the zero of $H(\xi)$ at $\xi = \pi$, i.e.,

$$\left.\frac{d^k H(\xi)}{d\xi^k}\right|_{\xi=\pi} = 0, \quad 0 \le k \le p.$$

As a consequence,

$$\hat{\Phi}(0) = 1; \quad \hat{\Phi}(2k\pi) = 0, k \neq 0.$$

Furthermore, at $\xi = 2k\pi$, $\hat{\Phi}$ has zeros of order $p+1$, the same order of the zero of $H(\xi)$ at $\xi = \pi$. This property is known as Strang-Fix condition (of order p). It is a necessary and sufficient condition for the polynomials $1, x, \cdots, x^p$ to be reproduced exactly by the translates of Φ. In turn, it determines the best order $O(2^{-j(p+1)})$ of approximation of smooth functions from V_j in the L^2 norm [28].

In a multiresolution analysis of $L^2(\mathbf{R})$ the wavelet functions appear when complement spaces W_j are considered, such that $V_{j+1} = V_j + W_j$. A systematic procedure for the construction of such complement spaces uses the concept of biorthogonal multiresolution analysis [7].

A biorthogonal multiresolution analysis consists of a pair $\{V_j, \Phi\}$ and $\{V_j^*, \Phi^*\}$ of multiresolution analyses of $L^2(\mathbf{R})$ related by

$$\left\langle \Phi_{j,k}, \Phi^*_{j,l} \right\rangle = \delta_{kl}, \tag{2}$$

for each fixed j, where, as usual

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \; j, k \in \mathbf{Z}$$

whichever may be the function ϕ. In this case, the corresponding scaling filters $H(\xi)$ and $H^*(\xi)$ satisfy

$$H(\xi)\overline{H^*(\xi)} + H(\xi+\pi)\overline{H^*(\xi+\pi)} = 1, \tag{3}$$

for all ξ. Associated to a biorthogonal multiresolution analysis there are spaces $W_j = V_j \bigcap V_j^{*\perp}$ and $W_j^* = V_j^* \bigcap V_j^{\perp}$, which are complements, not necessarily orthogonal, of V_j in V_{j+1} and V_j^* in V_{j+1}^*, respectively. Riesz bases for W_j (resp. W_j^*) are formed by the families $\Psi_{j,k}(x)$ (respectively $\Psi^*_{j,k}(x)$) associated to the wavelets

$$\Psi(x) = 2 \sum_{n \in \mathbf{Z}} g(n) \Phi(2x - n), \tag{4}$$

$$\Psi^*(x) = 2 \sum_{n \in \mathbf{Z}} g^*(n) \Phi^*(2x - n), \tag{5}$$

where $g(n) = (-1)^n h^*(-n+1)$ and $g^*(n) = (-1)^n h(-n+1)$. The following biorthogonality relations hold

$$\left\langle \Phi_{j,k}, \Psi^*_{j,l} \right\rangle = \left\langle \Phi^*_{j,k}, \Psi_{j,l} \right\rangle = 0,$$

and

$$\left\langle \Psi_{j,k}, \Psi^*_{m,l} \right\rangle = \delta_{jm} \delta_{kl}.$$

A biorthogonal multiresolution analysis provides a useful tool for studying functions in $L^2(\mathbf{R})$. For instance, the projection of a function $f \in L^2(\mathbf{R})$ onto V_j, parallel to V_j^*,

$$\Pi_j f(x) = \sum_{k \in \mathbf{Z}} \left\langle f, \Phi^*_{j,k} \right\rangle \Phi_{j,k}(x) \tag{6}$$

gives an approximation of f at a resolution 2^j. Since $V_{j-1} \subset V_j$, and $\Pi_j f \to f$ as $j \to \infty$, we have a convergent approximation process. Furthermore, the *detail* of f at a higher resolution 2^j, is given by the projection of f onto W_{j-1} parallel to $(W^*_{j-1})^\perp$, i.e.

$$Q_{j-1} f(x) = (\Pi_j - \Pi_{j-1}) f(x) = \sum_{k \in \mathbf{Z}} \left\langle f, \Psi^*_{j-1,k} \right\rangle \Psi_{j-1,k}(x). \tag{7}$$

Similarly, the projection of a function $f \in L^2(\mathbf{R})$ onto V_j^*, parallel to V_j is given by

$$\Pi^*_j f(x) = \sum_{k \in \mathbf{Z}} \langle f, \Phi_{j,k} \rangle \Phi^*_{j,k}(x). \tag{8}$$

2.1 INTERPOLATORY MULTIRESOLUTION ANALYSES

To each pair of conjugate scaling functions $\Phi(x)$ and $\Phi^*(x)$, we can associate a function $\theta(x)$ defined by

$$\theta(x) = \int_{\mathbf{R}} \Phi(y) \Phi^*(y - x) dy.$$

The biorthogonality relation (2) implies that

$$\theta(k) = \delta_{0k}, \tag{9}$$

for all integers k (see [2]). This interpolatory function $\theta(x)$ satisfies the scaling relation

$$\theta(x) = 2\sum_n p(n)\theta(2x-n), \tag{10}$$

where

$$p(n) = \sum_l h(l)h^*(l-n).$$

Thus, the corresponding scaling filter is

$$P(\xi) = H(\xi)\overline{H^*(\xi)},$$

which satisfies, as a consequence of (3),

$$P(\xi) + P(\xi+\pi) = 1, \tag{11}$$

for all real ξ. Using (9) in the scaling relation (10), we obtain

$$\theta(k/2) = 2\sum_n p(n)\theta(k-n) = 2p(k), \tag{12}$$

for all integers k. This relation implies that the interpolatory scaling filter coefficients vanish for all non zero even indices.

Note that the interpolatory property (9) can be interpreted in the distributional sense as

$$\langle\theta(x-k), \delta(x-l)\rangle = \delta_{kl}, \tag{13}$$

where $\delta(x)$ is the Dirac distribution. This expression can be interpreted as a biorthogonal duality relation. Since $\delta(x)$ also satisfies the scaling relation

$$\delta(x) = 2\delta(2x),$$

then $\delta(x)$ and $\theta(x)$ can be viewed as dual scaling functions. The associated dual wavelets are

$$\eta(x) = \sum_n(-1)^n p(-n+1)\delta(x-\frac{n}{2})$$

and

$$\eta^*(x) = -2\theta(2x-1),$$

respectively. Since $\langle f, \delta_{j,k}\rangle = 2^{-j/2}f(2^{-j}k)$, then the corresponding projection to $\Pi_j^* f$ is

$$I_j f(x) = \sum_k f(2^{-j}k)\theta(2^j x-k), \tag{14}$$

which is an interpolation operator, i.e., $I_j f(2^{-j}k) = f(2^{-j}k)$. Similarly to (7), we define the difference operator

$$Q_{j-1}f(x) = I_jf(x) - I_{j-1}f(x) = \sum_k \langle f, \eta_{j-1,k}\rangle \, \eta^*_{j-1,k}(x). \tag{15}$$

Let $f_j(k) = \langle f, \delta_{j,k}\rangle = 2^{-j/2}f(2^{-j}k)$. The values $f_{j-1}(k)$, at the next coarser grid, are formed from f_j by setting

$$f_{j-1}(k) = \sqrt{2}f_j(2k). \tag{16}$$

Denoting

$$d_{j-1}(k) = \langle f, \eta_{j-1,k}\rangle\,,$$

the values $d_{j-1}(k)$ can also be obtained from f_j by

$$\begin{aligned} d_{j-1}(k) &= 2^{\frac{-j+1}{2}}\sum_n(-1)^n p(-n+1)f(2^{-j}(2k+n)) \\ &= \sqrt{2}\sum_n(-1)^{n+1}p(n)f_j(2k-n+1). \end{aligned} \tag{17}$$

Equations (16) and (17) are the decomposition formulas corresponding to the transition from the one-level basis $\{\theta_{j,k}(x)\}$ to the two-level basis $\{\theta_{j-1,k}(x)\} \cup \{\eta^*_{j-1,k}(x)\}$.

2.2 COMPATIBLY WITH DERIVATIVES AND PRIMITIVES

An important fact about biorthogonal multiresolution analyses is that they are compatible with derivatives. For instance, under certain regularity conditions (e.g. $\Phi \in H^1(\mathbf{R})$), it was pointed out in [20] that differentiating $\Phi(x)$ and integrating $\Phi^*(x)$ the following formulae hold

$$\Phi'(x) = \tilde{\Phi}(x) - \tilde{\Phi}(x-1) \tag{18}$$

and

$$\tilde{\Phi}^*(x) = \int_x^{x+1} \Phi^*(y)dy, \tag{19}$$

where $\{\tilde{\Phi}, \tilde{\Phi}^*\}$ are also biorthogonal scaling functions. The corresponding scaling filters $\{H(\xi), H^*(\xi)\}$ and $\{\widetilde{H}(\xi), \widetilde{H}^*(\xi)\}$ are related by

$$\widetilde{H}(\xi) = \frac{2}{1+e^{-i\xi}}H(\xi) \tag{20}$$

$$\widetilde{H}^*(\xi) = \frac{1+e^{-i\xi}}{2e^{-i\xi}}H^*(\xi). \tag{21}$$

As a consequence the dual wavelets $\{\Psi, \Psi^*\}$ and $\{\tilde{\Psi}, \tilde{\Psi}^*\}$ satisfy

$$\Psi'(x) = 4\tilde{\Psi}(x), \quad \tilde{\Psi}^{*\prime}(x) = -4\Psi^*(x). \tag{22}$$

The relations (18) and (19) imply the commutation formula

$$\frac{d}{dx} \circ \Pi_j = \tilde{\Pi}_j \circ \frac{d}{dx}, \tag{23}$$

where

$$\tilde{\Pi}_j f(x) = \sum_{k \in \mathbf{Z}} \left\langle f, \tilde{\Phi}^*_{j,k} \right\rangle \tilde{\Phi}_{j,k}(x)$$

is the projection operator on the biorthogonal multiresolution analysis defined by $\{\tilde{\Phi}, \tilde{\Phi}^*\}$. We refer to [32] for a generalization of these results to several dimensions.

3 BIORTHOGONAL SPLINE SCALING FUNCTIONS

A family of biorthogonal scaling functions $\{\Phi, \Phi^*\}$, based on the B-splines, was constructed in [7]. For even $N = 2l$, $\Phi = \Phi_N$ is the symmetric B-spline, centered on 0, and

$$H_N(\xi) = \left(\cos\frac{\xi}{2}\right)^N. \tag{24}$$

For each $N^* = 2l^*$ there exist a conjugate scaling function $\Phi^* = \Phi_{N,N^*}$. The corresponding scaling filter $H^*_{N,N^*}(\xi)$ is given by

$$H^*_{N,N^*}(\xi) = \left(\cos\frac{\xi}{2}\right)^{N^*} \sum_{m=0}^{l+l^*-1} \binom{l+l^*-1+m}{m} \left(\sin^2\frac{\xi}{2}\right)^m. \tag{25}$$

Similarly, for odd $N = 2l+1$, $\Phi = \Phi_N$ is the symmetric B-spline centered on $\frac{1}{2}$ such that

$$H_N(\xi) = e^{-i\xi/2}\left(\cos\frac{\xi}{2}\right)^N. \tag{26}$$

For each odd index $N^* = 2l^*+1$ there exists a conjugate scaling function $\Phi^* = \Phi_{N,N^*}$ such that

$$H^*_{N,N^*}(\xi) = e^{-i\xi/2}\left(\cos\frac{\xi}{2}\right)^{N^*} \sum_{m=0}^{l+l^*} \binom{l+l^*+m}{m} \left(\sin^2\frac{\xi}{2}\right)^m. \tag{27}$$

The basic functions Φ, Ψ, Φ^* and Ψ^* have compact support in all the cases, and both Φ and Ψ are C^{N-2} piecewise polynomials of degree $N-1$. Their duals Φ^* and Ψ^* have increasing regularity with increasing N^* (see [7] and also [10]).

3.1 CHARACTERIZATION BY MEANS OF INTERPOLATORY PROPERTIES

Now we shall consider the particular case of interpolatory scaling functions obtained from the spline biorthogonal family. For each pair of conjugate scaling functions $\Phi = \Phi_N$ and $\Phi^* = \Phi_{N,N^*}$, let $\theta(x) = \theta_{N,N^*}(x)$ be the associated interpolatory function. If $N = 2l$ and $N^* = 2l^*$ are even integers, then the corresponding scaling filter is given by

$$P_{N,N^*}(\xi) = \left(\cos\frac{\xi}{2}\right)^{N+N^*} \sum_{m=0}^{l+l^*-1} \binom{l+l^*-1+m}{m} \left(\sin^2\frac{\xi}{2}\right)^m .$$

Similarly, if $N = 2l+1$ and $N^* = 2l^*+1$ are odd integers, then

$$P_{N,N^*}(\xi) = \left(\cos\frac{\xi}{2}\right)^{N+N^*} \sum_{m=0}^{l+l^*} \binom{l+l^*+m}{m} \left(\sin^2\frac{\xi}{2}\right)^m .$$

Note that, in both cases, $M = N + N^*$ is an even integer, and the interpolatory scaling filter depends only on M, e.g., $P_{N,N^*} = P_M$. Indeed, $M = 2(l+l^*)$ in the first case and $M = 2(l+l^*+1)$ in the second one. The next theorem summarizes the main properties of such interpolatory multiresolution analyses.

Theorem 3.1 *To every even integer $M = 2K, K \geq 1$ it is associated an interpolatory scaling filter*

$$P_M(\xi) = \left(\cos\frac{\xi}{2}\right)^M \sum_{m=0}^{K-1} \binom{K-1+m}{m} \left(\sin\frac{\xi}{2}\right)^{2m} \tag{28}$$

and an interpolatory scaling function $\Theta_M(x)$ which satisfy the following properties:

1. *$P_M(\xi)$ is symmetric around $\xi = 0$, i.e. $P_M(\xi) = P_M(-\xi)$.*

2. *$P_M(\xi) \geq 0$, for all $\xi \in \mathbf{R}$, and $P_M(\xi) = 0$ if and only if $\xi = k\pi, k \in \mathbf{Z}, k \neq 0$. Furthermore, for $\overline{\xi} = 0$ and $\overline{\xi} = \pi$,*

$$\frac{d^k P_M(\xi)}{d\xi^k}(\overline{\xi}) = 0, \quad 1 \leq k \leq M-1.$$

3. *The filter coefficients $p_M(n)$ are symmetric around $n = 0$, and $p_M(n) = 0$ for all $n \leq -M$ and $n \geq M$. $p_M(n)$ also vanish for all even integer $n \neq 0$.*

4. *$\Theta_M(x)$ is supported in $[-M+1, M-1]$ and it is symmetric around $x = 0$.*

5. *$\hat{\Theta}_M(0) = 1$ and $\hat{\Theta}_M(2k\pi) = 0$ for all $k \in \mathbf{Z}, k \neq 0$; Furthermore,*

$$\frac{d^l}{d\xi^l}\hat{\Theta}_M(2k\pi) = 0, \quad 1 \leq l \leq M-1,$$

for all $k \in \mathbf{Z}$ (including k=0).

6. *$\Theta_M(x)$ coincides with all interpolatory scaling functions $\theta_{N,N^*}(x)$ coming from biorthogonal spline scaling functions Φ_N and Φ_{N,N^*}, for any positive integers N and N^* such that $N + N^* = M$.*

Proof. Assertions 1 and 2 can be easily obtained from formula (28). 5 is a consequence of 2, and 6 follows from the comments just before the statement of the theorem. We can thus think in $\Theta_M(x)$ as being $\theta_{1,M-1}(x)$. Therefore, 4 is implied by the facts that $\Phi_1(x)$ and $\Phi_{1,M-1}(x)$ are symmetric around the same point $x = 1/2$, $\Phi_1(x) = 0$ for $x < 0$ and $x \geq 1$, and that support $\Phi_{1,M-1}(x) = [-M+2, M-1]$ (see [7]). Finally 3 comes from 4 together with expressions (9) and (12). □

The remainder of this section is dedicated to some consequences of Theorem 3.1. We shall emphasize that interpolatory multiresolution analyses already known in the literature under different three names coincide to those coming from biorthogonal spline wavelets.

Remark 3.1 In filter bank theory the filters satisfying properties 1, 2 and 3 are refered as *linear-phase halfband maxflat filters.* One way to obtain two-channel filter banks is just to find spectral factorizations $P_M(\xi) = F_0(\xi)F_1(\xi)$ of such filters [29]. The factorizations mentioned above, of P_M in terms of spline biorthogonal scaling filters, are some examples. There is also another important factorization $P_M(\xi) = |F_0(\xi)|^2$ (i.e. $F_1(\xi) = \overline{F_0(\xi)}$) which is the so called paraunitary factorization. It gives rise to the famous orthogonal Daubechies' filters of lenght $M = 2K$. If we form the interpolatory scaling functions using $\Phi^* = \Phi$, where $\Phi {=}_K\Phi$ are the associated Daubechies scaling functions, then they also have $P_M(\xi)$ as interpolatory scaling filters. Under this point of view, these interpolatory scaling functions are just the autocorrelation functions of ${}_K\Phi$ (see [27] and [26]). Therefore, the interpolatory multiresolution analyses determined by the autocorrelation function of the Daubechies's orthogonal scaling function supported in $[0, 2K-1]$, and by the biorthogonal spline scaling functions Φ_N and Φ_{N,N^*} such that $N + N^* = 2K$, are the same.

Remark 3.2 According to the terminology used in [27], the filters $P_M(\xi)$ are *Lagrange à trous filters*, e.g., the filter coefficients $p_M(n)$ are real and symmetric, with support described by $n \in [-M+1, M-1]$; the "à trous" comes for the "holes" $p_M(2l) = \delta_{0l}$, and the "Lagrange" is due to the interpolation formula

$$L(k/2) = 2\sum_n p_M(k-2n)L(n)$$

which is exact for polynomials of degree $\leq M-1$. This is a consequence of assertion 2 in Theorem 3.1 above. In particular, the following holds for the odd filter coefficients, $p_M(2n-1) = p_M(-2n+1) = L_n^{M-1}(\frac{1}{2})/2, n = 1, \cdots, M/2$, where $L_l^{M-1}(x)$ are the Lagrange polynomials based on the points $l = -M/2+1, \cdots, M/2$. These properties can be interpreted in the following form. Consider the interpolation operator

$$I_j^M f(x) = \sum_n f(2^{-j}n)\Theta_M(2^j x - n), \tag{29}$$

and let

$$L(x) = L(x; M, k),$$

be the $(M-1)$-th degree polynomial that interpolates $f(x)$ at the points $x_l^{j-1} = 2^{-j+1}l, \quad l = k - M/2, \cdots, k + M/2 - 1$. Then

$$I_{j-1}^M f(2^{-j}(2k-1)) = L(2^{-j}(2k-1)).$$

This means that $\Theta_M(x)$ also correspond to the fundamental functions of *Lagrange iterative interpolations*, as described in [13] and [11].

Remark 3.3 One can observe that replacing $x = 2^{-j}(2k+1)$ in equation (15) gives

$$Q_{j-1}f(2^{-j}(2k+1)) = -2^{\frac{j+1}{2}}\sum_l d_{j-1}(l)\theta(2k-2l) = -2^{\frac{j+1}{2}}d_{j-1}(k).$$

Using (15), the "wavelet coefficients" $d_{j-1}(k)$ can also be expressed by

$$\begin{aligned} d_{j-1}(k) &= -2^{\frac{-j-1}{2}}\left[I_j f(2^{-j}(2k+1)) - I_{j-1}f(2^{-j}(2k+1))\right] \\ &= 2^{\frac{-j-1}{2}}\left[I_{j-1}f(2^{-j}(2k+1)) - f(2^{-j}(2k+1))\right]. \end{aligned} \tag{30}$$

Therefore, $d_{j-1}(k)$ is in fact the relative error at $x = 2^{-j}(2k+1)$ of the interpolation formula (29), obtained from the $(j-1)$th grid points. Based on formula (30), A. Harten suggested in [17] a systematic procedure to obtain (generalized) interpolatory multiresolution analyses by just choosing any interpolation technique in the definition of the reconstruction procedure $I_j f(x)$. For instance, in the applications of the subsequent papers ([18] and [5]) the authors used central interpolation where $I_{j-1}f(2^{-j}(2k-1))$ is computed from the $(M-1)$-th degree polynomial that interpolates $f(x)$ at the points $x_l^{j-1} = 2^{-j+1}l, \quad l = k-s, \cdots, k+s-1, \ M = 2s$. From the previous remark, we conclude that there is a one-to-one correspondence between the interpolatory multiresolution analyses determined by the halfband maxfalt filters $P_M(\xi)$ and the interpolatory multiresolution analyses by means of central Lagrange interpolation of degree $M-1$, as introducced by A. Harten in [17].

Remark 3.4 The assertion 5 of Theorem 3.1 means that $\Theta_M(x)$ satisfies the Strang-Fix condition of order $M-1$. In addition it verifies the following moment relation

$$\hat{\theta}_M(\xi) = 1 + O(\xi^M), \quad \text{as } \xi \to 0.$$

Therefore, as described in [28], for smooth functions f in the Sobolev class $H^M(\mathbf{R})$, the following rate of convergence holds

$$||I_j^M f - f||_{H^s} \leq C2^{-j(M-s)}||f||_{H^M}, \tag{31}$$

Next we shall apply these results to show that the multiresolution analyses of *cell averages* constructed independently by A. Harten in [17] and D. Donoho in [12] and the biorthogonal spline family corresponding to $N = 1$ are the same.

Remark 3.5 Consider the cell averages $\overline{f}(k)$ of a function f given by

$$\overline{f}(k) = \int_k^{k+1} f(y)dy = \int_{\mathbf{R}} f(y)\Phi_1(y-k)dy.$$

Using the expression

$$\overline{f}(k) = F(k+1) - F(k),$$

of the cell averages in terms of the point values of the primitive function

$$F(x) = \int_0^x f(y)dy,$$

the " reconstruction via primitives" $R^M(x, \overline{f})$ is then defined by central Lagrange interpolation of degree $M-1$ of the primitive point values followed by derivation (see [17] and [12]). Therefore, from Remark 3.3 we conclude that $R^M(x, \overline{f})$ is precisely $d/dx I_0 F(x)$, where

$$I_0 F(x) = \sum_k F(k)\Theta_M(x-k).$$

If in formulae (32) and (33) we set $N = 1$, and $N^* = M - 1$, and if we apply formula (19), then we get

$$\Phi_{0,M}(x) = \Theta_M(x) = \int_x^{x+1} \Phi_{1,N^*}(x)(y)dy.$$

Therefore,

$$\begin{aligned}
\frac{d}{dx} I_0 F(x) &= \sum_k F(k)\frac{d}{dx}\Theta_M(x-k) \\
&= \sum_k F(k)\left[\Phi_{1,N^*}(x+1-k) - \Phi_{1,N^*}(x-k)\right] \\
&= \sum_k \left[F(k+1) - F(k)\right]\Phi_{1,N^*}(x-k) \\
&= \sum_k \overline{f}(k)\Phi_{1,N^*}(x-k).
\end{aligned}$$

This implies that $R^M(x, \overline{f})$ and the projection operator $\Pi_0 f(x)$ associated to the pair of biorthogonal scaling functions $\{\Phi_1, \Phi_{1,M-1}\}$ are the same.

In the next section we shall give another application of the results in (32) and (33). They will be used in the study of the equivalence between Petrov-Galerkin methods based on biorthogonal spline multiresolution analyses and collocation schemes based on the interpolatory scaling functions $\Theta_M(x)$.

for all $0 \leq s \leq r$, provided that $\Theta_M \in H^r(\mathbf{R})$. We refer to [11] for a method to determine the order of regularity of these interpolatory functions.

3.2 CHARACTERIZATION BY MEANS OF DERIVATIVES AND PRIMITIVES

If in formulae (20) and (21) we replace $H(\xi)$ and $H^*(\xi)$ by the pair of scaling filters defined in expressions (24) and (25), or in (26) and (27)), then we conclude that the spline biorthogonal scaling functions are related by derivatives and primitives, as follows (cf. [16]). If $\Phi = \Phi_N$ and $\Phi^* = \Phi_{N,N^*}$, then the associated functions $\{\tilde{\Phi}, \tilde{\Phi}^*\}$ satisfying (18) and (19) are given by

$$\tilde{\Phi}(x) = \begin{cases} \Phi_{N-1}(x) & \text{if } N \text{ is odd} \\ \Phi_{N-1}(x+1) & \text{if } N \text{ is even} \end{cases} \tag{32}$$

and

$$\tilde{\Phi}^*(x) = \begin{cases} \Phi_{N-1,N^*+1}(x) & \text{if } N \text{ is odd} \\ \Phi_{N-1,N^*+1}(x+1) & \text{if } N \text{ is even} \end{cases} \tag{33}$$

Let $\Pi_j^{(N,N^*)} f$ and $\Pi_j^{*(N,N^*)} f$ be the projectors in (6) and (8) corresponding to $\Phi = \Phi_N$ and $\Phi^* = \Phi_{N,N^*}$. The commutation formula (23), together with expressions (32) and (33), imply that

$$\frac{d}{dx} \Pi_j^{(N,N^*)} f(x) = \Pi_j^{(N-1,N^*+1)} f'(x),$$

and

$$\frac{d}{dx} \Pi_j^{*(N,N^*)} f(x) = \Pi_j^{*(N+1,N^*-1)} f'(x).$$

Note that formulae (32) and (33) also hold for $N = 0$ and $N^* = 2K = M \geq 2$, by just identifying $\Phi_0(x) = \delta(x)$ and $\Phi_{0,M}(x) = \Theta_M(x)$. Denoting $\Psi_{0,M}^*(x) = -2\Theta_M(2x-1)$, and using formula (22) we get the following expression for the derivatives of $\Theta_M(x)$.

Theorem 3.2 *Whenever $\Psi_{s,M-s}^*(x)$ is a well defined function, then*

$$\frac{d^s}{dx^s} \Theta_M(x) = (-2)^{s-1} \Psi_{s,M-s}^* \left(\frac{x+1}{2} - [s/2] \right),$$

where $[s/2]$ stands for the integer part of $s/2$.

As an example, consider $M = 6$. Thus

$$\Theta_6'(x) = \Psi_{1,5^*}(\frac{x+1}{2}), \quad \theta_6''(x) = -2\Psi_{2,4}^*(\frac{x-1}{2}), \quad \text{and} \quad \Theta_6'''(x) = 4\Psi_{3,3^*}(\frac{x-1}{2}).$$

4 PETROV-GALERKIN AND COLLOCATION METHODS

As in [30], we consider here the periodic initial value problem for the constant coefficient differential equation

$$\begin{aligned} &u_t = \textstyle\sum_{p=0}^{\mu} a_p \left(\frac{\partial}{\partial x}\right)^p u \equiv \mathcal{D}u. \\ &u(x,0) = u_0(x), \\ &u(x,t) \ \text{ 1-periodic in space.} \end{aligned} \tag{34}$$

The Galerkin method described in this section is based on biorthogonal frameworks $\{V_j, V_j^*\}$ corresponding to spline dual functions $\Phi = \Phi_N$ and $\Phi^* = \Phi_{N,N^*}$. One of the families of basic functions, say $\Phi^*_{j,k}(x)$, are used as *trial* functions, and its dual family $\Phi_{j,k}(x)$ are the test functions. This means that we shall approximate the exact solution of (34) by

$$u_j(x,t) = \sum_k U_k(t)\Phi^*_{j,k}(x), \tag{35}$$

where $U_k(t) = U_{k+2^j}(t)$ are periodic coefficients that must be determined by requiring $u_{j_t} - \mathcal{D}u_j$ to be orthogonal to V_j. That is,

$$\int_{\mathbf{R}} \left(u_{j_t} - \mathcal{D}u_j\right) \Phi_{j,l}(y)dy = 0, \tag{36}$$

for all $l \in \mathbf{Z}$. It is assumed that the basic functions have the regularity properties required in order that (36) makes sense (using integration by parts when necessary). The initial conditions $U_k(0), k \in \mathbf{Z}$, are the coefficients of some approximation $u_j(x,0)$ of the initial data $u_0(x)$.

The equations in (36) can be expressed as the following system of ordinary differential equations

$$\frac{d}{dt}U = \mathbf{B}U \tag{37}$$

where $U(t)$ is a vector with components $U_k(t)$ and $\mathbf{B}$ is a matrix with entries b_{lk} given by

$$b_{lk} = \sum_{p=0}^{\mu} 2^{pj}\Gamma_p(k-l),$$

and

$$\Gamma_p(m) = \Gamma_{p,N,N^*}(m) = (-1)^s \int_{\mathbf{R}} D^{p-s}\Phi^*(y)D^s\Phi(y+m)dy.$$

Here $D = \frac{d}{dx}$ and the index $0 \le s \le p$ is introducced in order to handle possible lack of regularity of the basic functions. For instance, let $N = 0$ and M be an even integer. For this case, $\Phi_0(x) = \delta(x)$ and $\Phi^*_{0,M}(x) = \Theta_M(x)$, and s is always set equal to 0. Thus, $\Gamma_p(m) = D^p\theta_M(-m)$, which can be calculated using Theorem 3.2. Note that, in this case the above formulation is just the collocation method, and equation (36) reduces to

$$\left[u_{j_t} - \mathcal{D}u_j\right](x_l^j) = 0, \tag{38}$$

for all $l \in \mathbf{Z}$, where $x_l^j = l2^{-j}$.

The following result is a consequence of the relations (32) and (33).

Theorem 4.1 *Let $M = N + N^*$ be an even integer. Then the system of ordinary differential equations (37) depends only on M.*

Proof. Let us consider the case of odd integers N and N^*. Using integration by parts, and the relation of derivatives and primitives (32) and (33), we have

$$\begin{aligned}
\Gamma_{p,N,N^*}(m) &= (-1)^{s+1}\int_{\mathbf{R}} D^{p-s-1}\Phi_{N,N^*}(y)D^{s+1}\Phi_N(y+m)dy \\
&= (-1)^{s+1}\int_{\mathbf{R}} D^{p-s-1}\Phi_{N,N^*}(y)D^{s}\left[\Phi_{N-1}(y+m) - \Phi_{N-1}(y+m-1)\right]dy \\
&= (-1)^{s}\int_{\mathbf{R}} D^{p-s-1}\left[\Phi_{N,N^*}(y+1) - \Phi_{N,N^*}(y)\right]D^{s}\Phi_{N-1}(y+m)dy \\
&= (-1)^{s}\int_{\mathbf{R}} D^{p-s}\Phi_{N-1,N^*+1}(y)D^{s}\Phi_{N-1}(y+m)dy \\
&= \Gamma_{p,N-1,N^*+1}(m).
\end{aligned}$$

The case of even N and N^* can be considered in an analogous way. □

In [3] the authors establish the equivalence of the linear system arising in a wavelet-Galerkin formulation using Daubechies' scaling functions of compact support and those coming from a collocation method based on approximating trial spaces generated by their autocorrelation functions. Therefore, by virtue of the above theorem and of Remark 3.1, the following results hold.

Corollary 4.1 *Let $M = 2K$. Apart from the choice of inicial data $U(0)$, the following numerical methods for problem (34) are equivalent:*

- *The Petrov-Galerkin method using biorthogonal spline scaling functions $\Phi = \Phi_N$ and $\Phi^* = \Phi_{N,N^*}$ with $N + N^* = M$.*
- *The Galerkin method using $\Phi = \Phi^* =_K \Phi$, the orthogonal Daubechies' scaling function supported in $[0, 2K-1]$.*
- *The collocation method based on the interpolatory scaling functions Θ_M.*

We can also prove that the methods mentioned above may provide very accurate approximations of exact regular solutions at the nodes with a superconvergence rate $O(2^{-j(M-\mu)})$, provided that the approximate initial data is chosen properly. The proof uses quite standard arguments that have already been successfully applied for usual spline based methods [31] as well as in the wavelet context ([14], [15]).

5 FINAL REMARKS

In this paper we discuss the use of multiresolution analyses and their associated scaling functions as bases in Galerkin, Petrov-Galerkin and collocation methods. We analyse the specific case of scaling functions whose refinement masks come from different factorizations of the same linear-phase halfband maxflat filter. We show the equivalence of these methods when they are applied to linear constant coefficients differential equations with periodic boundary conditions. However, there are situations where these schemes may be really different. This is the subject of a next paper which is in progress. For instance, in the presence of non-constant coefficients, nonlinear terms or nonperiodic boundary conditions. Furthermore, the main advantages of methods based on multiresolution analyses are better appreciated if the problems are discretized with wavelet basis instead of the scaling function basis [19]. When applied to irregular situations showing localized singular behaviour, the multiresolution structure of wavelet bases provides a simple way to adapt computational refinements to the local regularity of the solution (cf. [22], [23],[1], [18], [5] and [8]). The wavelet framework also allows sparse and well-conditioned representation of operators ([4], [9], [6]). In all these applications, a spline biorthogonal setting may show several good properties, as discussed in [21].

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