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SOME REMARKS ON THE ANTIMAXIMUM PRINCIPLE

J.-P.GOSSEZ[1]

Département de Mathématique, C.P.214

Université Libre de Bruxelles

1050 Bruxelles

1 INTRODUCTION

A function u on an interval $[a,b]$ which satisfies $-u'' \geq 0$ on $[a,b]$ achieves its minimum at a or b (because u is concave). If in addition $u(a) = u(b) = 0$, then $u \geq 0$ on $[a,b]$, and in fact either $u \equiv 0$ sur $[a,b]$, or $u > 0$ on $]a,b[$ with $u'(a) > 0$ and $u'(b) < 0$. These observations from calculus represent the most elementary form of the *maximum principle*, which is a fundamental result in the theory of elliptic partial differential equations.
Let us consider the Dirichlet problem

$$(1) \qquad \begin{cases} -\Delta u = f(x) \text{ in } \Omega, \\ u = 0 \text{ sur } \partial\Omega, \end{cases}$$

où Ω is a smooth bounded domain in $\mathbb{R}^N$ and Δ is the Laplacian operator. The *maximum principle* for (1.1) states that is $f \in L^\infty(\Omega)$ is ≥ 0 a.e. in Ω,then the solution u of (1.1) (which belongs to $C^1(\bar{\Omega})$) is ≥ 0 in Ω; moreover either $u \equiv 0$ in Ω, or u is > 0 in Ω with $\partial u/\partial n < 0$ on $\partial\Omega$ ($\partial/\partial n$ represents the exterior normal derivative). More generally the same conclusion holds for the problem

$$(2) \qquad \begin{cases} -\Delta u = \lambda u + f(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

provided the real parameter λ satisfies $\lambda < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the problem

$$(3) \qquad \begin{cases} -\Delta u = \lambda u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

In other words, when going from f to the solution u, the sign is preserved and even reinforced : $f \geq 0, \not\equiv 0$ implies $u > 0$. Standard references for the various versions of the maximum principle include [Pr-We], [Gi-Tr].

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In 1979, P.Clément and L.Peletier studied problem (1.2) in the case where the parameter λ satisfies $\lambda > \lambda_1$.They derive for certain values of λ a conclusion which is totally opposed to the preceding one : $f \geq 0, \not\equiv 0$ implies $u < 0$, which suggested the name of *antimaximum principle.* In a precise manner : given $f \in L^\infty(\Omega)$ with $f \geq 0$ a.e. in Ω, $f \not\equiv 0$, there exists $\delta > 0$ such that if u is a solution of (1.2) with $\lambda_1 < \lambda < \lambda_1 + \delta$, then $u < 0$ in Ω and $\partial u/\partial n > 0$ sur $\partial\Omega$.
The purpose of this Note is to present some recent results dealing with this antimaximum principle. In §2 a relation is exhibited between this principle and the Fučik spectrum of the differential operator. In §3 this principle is extended to the case of the p-laplacian operator :

$$\Delta_p u = div(|gradu|^{p-2}gradu) \tag{4}$$

where $1 < p < \infty$.Let us also mention a recent work by P.Takáč [Ta] dealing with an abstract version of the antimaximum principle in the setting of ordered Banach spaces.

2 FUČIK SPECTRUM

Let us denote by $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$ the positive and negative parts of u. The Fučik spectrum is defined as the set Σ of those $(\alpha,\beta) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} -\Delta u = \alpha u^+ - \beta u^- \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1}$$

has a nontrivial solution. The usual spectrum corresponds to the case $\alpha = \beta$.
This generalized spectrum was introduced in the 70's by S.Fučik [Fu] and N.Dancer [Da$_1$] in connexion with the study of semilinear problems of the form

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2}$$

The description of this set Σ remains a largely open question. For instance, while it is classical that the usual spectrum (i.e. the intersection of Σ with the diagonal in $\mathbb{R}^2$) is made of a sequence going to infinity, it is not known whether $\Sigma \subset \mathbb{R}^2$ has a nonempty interior (cf. [Da$_2$] for a partial result in that direction).
The beginning of the Fučik spectrum was studied recently in [Cu-Go], [DF-Go]. It is easily seen that Σ is contained in $\{(\alpha,\beta) \in \mathbb{R}^2; \alpha \text{ and } \beta \geq \lambda_1\}$ and contains the straight lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$. The existence of a first non trivial curve in Σ going through the second eigenvalue (λ_2, λ_2) and extending to infinity was established in [Cu-Go], [DF-Go]. A variational characterization of this curve in the line of the classical Courant-Hilbert formulas was also given. (Another variation characterization, using the mountain pass theorem, can also be given, cf. the work of M.Cuesta, D.De Figueiredo, B.Ruf and J.-P. Gossez described in [Cu]). This first nontrivial curve can be written

$$\mathcal{C}_2 = \{(\alpha(t),\beta(t)) \in \mathbb{R}^2; t \in \mathbb{R}\},$$

it is contained in $\{(\alpha, \beta) \in \mathbb{R}^2; \alpha \text{ and } \beta > \lambda_1\}$, is symmetric with respect to the diagonal, and the functions $\alpha(t)$, $-\beta(t)$ are continuous, strictly increasing and such that $\alpha(t) \to +\infty$ as $t \to +\infty$ and $\beta(t) \to +\infty$ as $t \to -\infty$.
The asymptotic behaviour of this curve $\mathcal{C}_2$ is connected with the antimaximum principle. In the statement of this principle as given in the introduction, the number $\delta > 0$ depends on f: $\delta = \delta(f)$. Let us say that the **uniform** antimaximum principle holds if there exists $\delta > 0$ such that if u is a solution of (1.2) with $\lambda_1 < \lambda < \lambda_1 + \delta$ and $f \in L^\infty(\Omega)$, $f \geq 0$ a.e., $f \not\equiv 0$, then $u < 0$ in Ω and $\partial u/\partial n > 0$ on $\partial\Omega$. So here δ does not depend on f.

PROPOSITION 2.1. *(cf. [DF-Go]) The uniform antimaximum principle holds if and only if*

(3) $$\lim_{t \to +\infty} \beta(t) > \lambda_1.$$

Moreover, in this case, the largest number δ admissible in the statement of this uniform principle is equal to

(4) $$\lim_{t \to +\infty} \beta(t) - \lambda_1.$$

The situation described in this proposition is in fact rather exceptional. Indeed the strict inequality holds in (2.3) when the dimension $N = 1$ and the Dirichlet boundary condition $u = 0$ is replaced in (1.2), (2.1) by the Neumann boundary condition $\partial u/\partial n = 0$. On the contrary equality holds in (2.3) (and so the uniform version of the antimaximum principle does not hold) in the case of the Dirichlet boundary condition (with N arbitrary), the Neumann boundary condition with $N \geq 2$ (cf. [DF-Go]) and the mixed boundary condition (with N arbitrary) (cf. [Go-Ma]).

3 P-LAPLACIEN

The p-laplacian operator (1.4) is a quasilinear generalization of the classical laplacian (which corresponds to $p = 2$ in (1.4)). It appears e.g. in some questions from fluid mechanics. Its interest comes also from the fact that it is the derivative of the Sobolev norm $\int_\Omega |grad\ u|^p$.
The problem analogous to (1.2) is now written

(1) $$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + f(x) \text{ dans } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

and the corresponding eigenvalue problem is

(2) $$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

The study of the spectrum of the p-laplacian, i.e. the values of $\lambda \in \mathbb{R}$ for which (3.2) has a nontrivial solution, is also a question which remains largely open. Several

works in the 80's (in particular those of A.Anane [An]) contributed to show that the first eigenvalue λ_1 of (3.2) enjoys properties which are rather similar to those in the linear case $p = 2$ (simplicity, strictly positive eigenfunction,...). Let us mention that recently A.Anane and N.Tsouli studied the nodal domains of the eigenfunctions and derived a variational characterization of the second eigenvalue λ_2 (cf. [An-Ts]). Let us also observe that, as in the case of the Fučik spectrum, it is not known whether the spectrum of the p-laplacian has nonempty interior.
A maximum principle was derived for (3.1) when $\lambda < \lambda_1$: if $f \in L^\infty(\Omega)$ is ≥ 0 a.e. and $\not\equiv 0$, then the solution u of (3.1) (which exists, is unique and of class $C^1(\bar{\Omega})$) is > 0 in Ω and satisfies $\partial u/\partial n < 0$ on $\partial\Omega$ (cf. [Va], [Fl-He-dT]). And recently the antimaximum principle was also extended to the p-laplacian : given $f \in L^\infty(\Omega)$, ≥ 0 a.e., $\not\equiv 0$, there exists $\delta > 0$ such that if u is a solution of (3.1) with $\lambda_1 < \lambda < \lambda_1 + \delta$, then $u < 0$ in Ω and $\partial u/\partial n > 0$ on $\partial\Omega$ (cf. [Fl-Go-Ta-dT$_1$]).
The proof of this antimaximum principle for (3.1) is very different from that of [Cl-Pe] relative to the linear case $p = 2$. It is based on the following nonexistence result of independant interest.

PROPOSITION 3.1. *(cf. [Fl-Go-Ta-dT$_1$]) Let $f \in L^\infty(\Omega)$, ≥ 0 a.e. and $\not\equiv 0$. Then (3.1) with $\lambda = \lambda_1$ has no solution.*

Let us recall here the classical result of Fredholm alternative when $p = 2$. Let $f \in L^\infty(\Omega)$; then problem (1.2) with $\lambda = \lambda_1$ has a solution if and only if $\int_\Omega f(x)\varphi_1(x) = 0$, where φ_1 is the eigenfunction (> 0 in Ω) associated to λ_1. In particular (1.2) with $\lambda = \lambda_1$ has no solution if $f \not\equiv 0$ does not change sign. Proposition 3.1 appears in this respect as a first step towards an extension of this Fredholm alternative to the quasilinear case $p \neq 2$. The characterization of those functions f for which (3.1) with $\lambda = \lambda_1$ is solvable remains an open question.
Let us observe finally that this antimaximum principle as well as the above nonexistence result have been extended recently to some strictly cooperative elliptic systems involving the p-laplacian (cf. [Fl-Go-Ta-dT$_2$]).

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