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THE H-SURFACE SYSTEM WITH CONSTANT BOUNDARY VALUES

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dedicated to the memory of our friend Julio Bouillet

Abstract: We consider the Dirichlet problem for the mean curvature equation (Dir) here below. If the boundary data g is constant and $H = H_0$ is also a constant, it is known that $X = g$ is the only solution (cf [W]). On one hand we prove that if $g = 0$ and H is even and real analytic then $X = 0$ is the only solution. On the other hand, we obtain results which imply that for any constant g there are $H's$ of class C^1 such that (Dir) has at least two solutions in the Sobolev space H^2 which are continuous up to the boundary.

INTRODUCTION. We consider the Dirichlet problem in the unit disc $B = \{(u,v) \in R^2;\ u^2 + v^2 < 1\}$ for a vector function $X : \overline{B} \to R^3$ which satisfies the equation of prescribed mean curvature

$$\text{(Dir)} \begin{cases} (1)\Delta X &= 2H(X)X_u \wedge X_v \quad \text{in } B \\ (2)X &= g \quad \text{on } \partial B \end{cases}$$

where $X = \dfrac{\partial X}{\partial u}$, $X_v = \dfrac{\partial X}{\partial v}$, "$\wedge$" denotes the exterior product in R^3 and $H : R^3 \to R$ is a given continuous function. When H is bounded and g is in the Sobolev space $H^1(B,R^3)$ we call $X \in H^1(B,R^3)$ a weak solution of (Dir) if for every $\varphi \in C_0^1(B,R^3)$

$$\text{(Sol)} \begin{cases} \int_B (\nabla X \cdot \nabla\varphi + 2H(X)X_u \wedge X_v \cdot \varphi) = 0 \\ X \in T \equiv g + H_0^1(B,R^3) \end{cases}$$

where $H_0^1(B,R^3) = \text{adh}_{H^1} C_0^1(B,R^3)$. It is known that for certain functions H and boundary values g, we can obtain weak solutions of (Dir) as critical points of the functional

$$D_H(X) = D(X) + 2V(X)$$

[H] and [S], with $D(X) = \frac{1}{2}\int_B |\nabla X|^2$ the Dirichlet integral, and $V(X) = \frac{1}{3}\int_B Q(X)\cdot X_u$
$\wedge X_v$ the Hildebrandt volume, where for $\xi = (\xi_1,\xi_2,\xi_3) \in R^3$ the associated function Q to H is

$$Q(\xi) = \left(\int_0^{\xi_1} H(s,\xi_2,\xi_3)ds, \int_0^{\xi_2} H(\xi_1,s,\xi_3)ds, \int_0^{\xi_3} H(\xi_1,\xi_2,s)ds\right)$$

which satisfies $\operatorname{div} Q = 3H$.

For $Q = H_0\xi$ and $g \equiv g_0 =$ const. in R^3, there is only one weak solution $X \equiv g_0$ [W].

We denote $W^{1,p}(B,R^3)$ the usual Sobolev spaces [A] and $H^1(B,R^3) = W^{1,2}(B,R^3)$. For $X \in H^1(B,R^3)$, $\|X\|_{L^2(\partial B,R^3)} = \left(\int_{\partial B} |TrX|^2\right)^{\frac{1}{2}}$ and for $Y \in L^\infty(U,R^n)$ we denote $\|Y\|_\infty = \sup_{w\in U} |Y(w)|$.

Concerning D_H(resp. V) we denote

$$dD_H(\varphi) = \lim_{t\to 0}\left[\frac{D_H(X+t\varphi) - D_H(X)}{t}\right]$$

whenever this limit exists (resp. $DV(X)(\varphi)$).

We prove the uniqueness for $g = 0$ and H analytic and even in 1. In 2. we give conditions on H to have multiple solutions for $g = c$, an arbitrary constant with explicit examples of such $H's$. Finally a few technical lemmas are stated in 3.

1. THE ANALYTIC CASE.

In a similar way to [W] we have

THEOREM 1: *Suppose $H : R^3 \to R$ real analytic and even. If $X \in C^1(\overline{B},R^3)$ is a weak solution of (Dir), with $g = 0$ on ∂B. Then $X \equiv 0$ in B.*

Proof: First we remark that X is real analytic because $2H(X)X_u \wedge X_v \in C(\overline{B},R^3)$, hence $X \in W^{2,p}(B,R^3)$ for $p \geq 1$, so $X \in C^{1,\alpha}(\overline{B},R^3)$ and $\Delta X \in C^{0,\alpha}(\overline{B},R^3)$ gives $X \in C^{2,\alpha}(\overline{B},R^3)$ and by hypoellipticity [M] of this system, X is real analytic.

Second we extend X to R^2 by reflection on ∂B

$$Y(u,v) = \begin{cases} X(u,v) & \text{in} \quad \overline{B} \\ -X(u/r^2, v/r^2) & \text{elsewhere} \end{cases}$$

where $r^2 = u^2 + v^2$. Even if H is non constant, we claim that this odd extension satisfies weakly (1) in (Dir). Procceding as in [W] we take $\varphi \in C_0^\infty(R^2 - \{(0,0)\})$ and descompose

$$\varphi = \frac{\varphi + \hat{\varphi}}{2} + \frac{\varphi - \hat{\varphi}}{2}$$

with $\hat{\varphi}(u,v) = \varphi(\frac{u}{r^2}, \frac{v}{r^2})$. The even part $\varphi_e = \dfrac{\varphi + \hat{\varphi}}{2}$ with respect to ∂B satisfies

$$D'_H(Y)(\varphi_e) = \int_{R^2} Y \cdot \varphi_e + 2\int_{R^2} H(Y) Y_u \wedge Y_v \cdot \varphi_e = 0$$

because Y is odd, so $Y_u \wedge Y_v$ and H is even, so is $H(Y)$, everything with respect to ∂B.

The odd part $\varphi_o = \dfrac{\varphi - \hat{\varphi}}{2}$ is 0 in ∂B, so $\varphi_0|_{\partial B} \in H_0^1(B, R^3)$ and

$$D'_H(Y)(\varphi_o) = D'_H(X)(\varphi_o) + \int_{R^2 - B} Y \cdot \varphi_o + 2\int_{R^2 - B} H(Y) Y_u \wedge Y_v \cdot \varphi_o = 2D'_H(X)(\varphi_o)$$

by changing variables $(u,v) \to (u/r^2, v/r^2)$ in the last two terms and since X is solution of (Dir) in B, $2D'_H(X)(\varphi_o) = 0$. Hence $D'_H(Y)(\varphi) = 0$. The Courant conformal meassure function

$$F(u,v) = |Y_u|^2 - |Y_v|^2 - 2iY_u.Y_v$$

is then holomorphic in all R^2 and

$$\int_{R^2} |F(u,v)| \leq 2\int_{R^2} |Y_u|^2 + |Y_v|^2 \leq 4\int_B |X_u|^2 + |X_v|^2 < +\infty$$

so $F \equiv 0$, i.e. (u,v) are isothermal coordinates for Y. From [G] or [H-W] we know that Y has only isolated branching points, since H is analytic. But Y has ∂B as branching points, a contradiction except if $Y = 0$, so $X = 0$. □

2. NOUNIQUENESS IN THE DIRICHLET PROBLEM WITH CONSTANT BOUNDARY VALUES.

We will prove that for each c in $R^3 - \{(0,0,0)\}$, there exists a class of H's verifying that (Dir) has at least two weak solutions.

For this purpose we will use the following theorem and technical lemmas from section 3.

For $c \in R^3$, $H \in C^1(R^3)$ with $0 < H_0 = \|H\|_\infty < +\infty$ and $k > 0$ in R, we define

$$M_k = \{X \in c + H_0^1(B, R^3); \|X - c\|_\infty \leq 1/H_0, \|\nabla(X - c)\|_\infty \leq k\}$$

and denote ρ the slope of D_H in M_k [LD-M], [S], i.e.

$$\rho(X) = \sup_{Y \in M_k} dD_H(X)(X - Y).$$

Finally, we define for $\beta \in R$,

$$K_\beta = \{X \in M_k; D_H(X) = \beta, \rho(X) = 0\} \text{ and}$$

$$M_k^\beta = \{X \in M_k; D_H(X) < \beta\}.$$

THEOREM 2: *Let $H \in L^\infty(R^3)$, and $Q \in L^\infty(R^3, R^3)$. Then $\inf\limits_{M_k} D_H(X) = \beta_0 \in R$ and K_{β_0} is nonempty.*

Proof: From

$$D_H(X) \geq D(X)(1 - 2/3\|Q\|_\infty) \geq -1/2|B|k^2|1 - 2/3\|Q\|_\infty| \text{ for } X \in M_k,$$

we conclude that $\inf\limits_{M_k} D_H(X) > -\infty$.

For $\bar{\varepsilon} = 1$ and for any neighborhood N of the set K_{β_0} in M_k, there exists a number $\varepsilon \in (0, \bar{\varepsilon})$ and a deformation Φ with the properties stated in Lemma 1. If $K_{\beta_0} = \phi$ we choose $N = \phi$, and then, $\Phi(1, M_k^{\beta_0 + \varepsilon}) \subset M_k^{\beta_0 - \varepsilon}$. But $M_k^{\beta_0 - \varepsilon} = \phi$ by definition of β_0. A contradiction. □

Let $(c_1, c_2, c_3) \in R^3 - \{(0,0,0)\}$. For $\varepsilon > 0$ we choose $H \in C^1(R^3)$ such that

$$H(\xi) = \begin{cases} H_0 & \text{if} \quad \xi_1^2 + \xi_2^2 \leq R^2 \quad \text{and} \quad c_3 \leq \xi_3 \leq c_3 + \delta \\ 0 & \text{if} \quad \xi_1^2 + \xi_2^2 > (R + \varepsilon)^2 \quad \text{or} \quad \xi_3 \notin (c_3 - \varepsilon, c_3 + \delta + \varepsilon) \end{cases}$$

with $H_0 = \|H\|_\infty$. Where R, δ and H_0 positive will be fixed later.

Consider $X_1(u,v) = (c_1 + \alpha f(u,v), c_2, c_3 + \alpha g(u,v))$ with $f, g \in C^\infty(\overline{B})$, $Trf = Trg = 0$, $g \geq 0$ and $\alpha > 0$ in R.

If $c_2 \neq 0$, we fix $R^2 = (|c_1| + \alpha\|f\|_\infty^2)^2 + c_2^2$, $\delta > \alpha\|g\|_\infty$, and $H_0 = \dfrac{1}{\alpha\sqrt{\|f\|_\infty^2 + \|g\|_\infty^2}}$. Then

$$D_H(X_1) = \frac{\alpha^2}{2} \int_B (f_u^2 + f_v^2 + g_u^2 + g_v^2) + \frac{2}{3} \frac{c_2 \alpha^2}{\alpha\sqrt{\|f\|_\infty^2 + \|g\|_\infty^2}} \int_B (g_u f_v - g_v f_u).$$

We now take f, g such that $c_2 \int_B (g_u f_v - g_v f_u) < 0$ and choose α sufficiently close to zero to have $D_H(X_1) < D_H(c) = 0$.

Finally, taking $k = \|\nabla X_1\|_\infty$, we obtain that c and X_1 are in M_k. Hence, from Theorem 2, there exists $X_2 \in M_k$ verifying that $D_H(X_2) = \inf_{M_k} D_H(X)$, $X_2 \neq c$ and from Lemma 2, X_2 is a weak solution of (Dir).

REMARK 1. The solutions are continuous up to the boundary: let X be a solution of (Dir). As H is bounded there exists a positive constant γ verifying that $\gamma + H(X) > 0$. Taking U the solution of

$$(1) \begin{cases} \Delta U &= 2\gamma X_u \wedge X_v \quad \text{in } B \\ U &= 0 \quad \text{on } \partial B \end{cases}$$

we obtain that $X + U - c$ is a solution of

$$(2) \begin{cases} \dfrac{\Delta(X + U - c)}{2(\gamma + H(X))} &= X_u \wedge X_v \quad \text{in } B \\ X + U - c &= 0 \quad \text{on } \partial B. \end{cases}$$

But the solutions of equations (1) and (2) are continuous in $\overline{B}$ [C-L], and we conclude that X is continuous up to the boundary.

Remark 2. If X_2 is smooth on $\overline{B}$, its image $X_2(\overline{B})$ seems not to be a surface with boundary because $X_2(\partial B) = \{c\}$.

3. TECHNICAL LEMMAS.

Exactly as [S] Lemma 1.9 p.36 we obtain the following result:

LEMMA 1: *Let $\beta \in R, \bar{\varepsilon} > 0$, and suppose that N is a neighborhood of K_β in M_k.*

Then there exists a number $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous one parameter family $\Phi : [0,1] \times M_k \to M_k$ of homeomorphisms $\phi(t,.)$ of M_k having the properties

i) $\Phi(t, X) = X$ if $t = 0$, or if $|D_H(X) - \beta| \geq \bar{\varepsilon}$ or if $\rho(X) = 0$.

ii) $D_H(\Phi(t, X))$ is non increasing in t.

iii) $\Phi(1, M_k^{\beta+\varepsilon} \backslash N) \subset M_k^{\beta-\varepsilon}$ and $\Phi(1, M_k^{\beta+\varepsilon}) \subset M_k^{\beta-\varepsilon} \cup N$.

LEMMA 2: *Any $X \in M_k$ with slope $\rho(X) = 0$ is a weak solution of* (Dir).

<u>Proof:</u> If $\rho(X) = 0$, it is known that $dD_H(X)(X - c) < 0$ or X is a weak solution of (Dir) (Lemma 1, [LD-M]).

But

$$dD_H(X)(X-c) \geq \int_B [|\nabla X|^2 - 2H_0|X-c||X_u \wedge X_v|] \geq 0.$$

$\square$

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