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STABILITY CRITERIA THROUGH CHARACTERISTIC EQUATIONS OF LINEAR OPERATORS

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Dedicated to the memory of J. Bouillet

ABSTRACT. *In [Na-4] we presented an abstract framework in which the spectrum of a linear operator can be computed through a characteristic equation. In the present note this is combined with the Perron - Frobenius spectral theory for positive operators in order to obtain simple stability criteria for the solutions of the associated Cauchy problem.*

1. INTRODUCTION

"Stability" is the most searched for property of solutions of linear abstract Cauchy problems

$$u'(t) = Au(t)\ , \quad u(0) = x_0\ . \tag{ACP}$$

Here, $A : \mathcal{D}(A) \subset X \to X$ is a linear operator on the Banach space X and, supposing (ACP) to be well posed, the (mild) solution is given by

$$u(t) = T(t)x_0$$

with $(T(t))_{t\geq 0}$ the semigroup generated by A. It is well known that stability properties of the semigroup $(T(t))_{t\geq 0}$ can be characterized by spectral properties

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of the generator A. We refer to [Ne] for a recent and systematic discussion of these phenomena (see also [Ar-2]) and concentrate here on *(uniform) exponential stability* which means that there exist constants $M \geq 1, \epsilon > 0$ such that

$$\|T(t)\| \leq Me^{-\epsilon t}$$

for all $t \geq 0$. In many (but not all) cases uniform exponential stability is implied by the negativity of the *spectral bound* of A, i.e.,

$$s(A) := \sup\{\text{Re } \lambda : \lambda \in \sigma(A)\} < 0$$

(see [Ne], Chapter 3). It therefore remains a primordial task to determine (or to estimate) the spectrum $\sigma(A)$ or the spectral bound $s(A)$ of the operator A. In [Na-4] we proposed an abstract framework how this can be done through so called *characteristic equations*. However, these equations can be quite complex making it difficult to estimate $s(A)$. In this note we show how positivity assumptions on the operators involved facilitate considerably this task and yield some simple stability criteria. For the necessary background on positive semigroups and their spectral theory we refer to [Na-1].

2. ABSTRACT FRAMEWORK

Our setup will be similar to the one in [Na-4], but we now add hypotheses involving ordered Banach spaces and positive operators.

2.1 Assumptions. Throughout this section we always assume the following.

(A_1) The Banach space $X := X_1 \times X_2$ is the product of two Banach lattices X_1 and X_2.

(A_2) The unperturbed operator $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset X \to X$ is the generator of a strongly continuous semigroup of positive operators. Hence, it has a *positive resolvent* (see [Ar-1]) which is given as a 2×2 operator matrix, i.e., there exists $\omega \in \mathbb{R}$ such that

$$0 \leq R(\lambda, \mathcal{A}_0) = (R_{ij}(\lambda))_{2\times 2}$$

for all $\lambda > \omega$.

(A_3) The perturbation $\mathcal{B} : \mathcal{D}(\mathcal{A}_0) \subset X \to X$ is a positive operator such that $0 \leq \mathcal{B}R(\lambda, \mathcal{A}_0) \in \mathcal{L}(X)$ for all $\lambda > \omega$.

(A_4) The operator we are interested in is

$$\mathcal{A} := \mathcal{A}_0 + \mathcal{B}$$

with $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_0)$.

Under these assumptions we can apply Theorem 2.4 from [Na-4] in order to determine (parts of) the spectrum $\sigma(\mathcal{A})$ through a characteristic equation in X_1. However, being mainly interested in $s(\mathcal{A})$ only, we now use the Perron - Frobenius spectral theory from [Na-1]. Its fundamental result is stated in the following lemma (see [Na-1], C-III, Theorem 1.1).

2.2 Lemma. *Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator with positive resolvent on the Banach lattice X. Then its spectral bound $s(A)$ is characterized as*

$$s(A) = \inf\{\lambda \in \rho(A) \cap \mathbb{R} : R(\lambda, A) \geq 0\}$$

and satisfies

$$s(A) \in \sigma(A).$$

Note also that for bounded positive operators $T \in \mathcal{L}(X)$ the spectral bound $s(T)$ coincides with the spectral radius $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$ and therefore the same statements hold for $r(T)$. Combining the perturbation techniques from [Na-4] with the above lemma enables us to obtain our main result.

2.3 Theorem. *Let the operators $\mathcal{A}, \mathcal{A}_0$ and $\mathcal{B}$ satisfy the assumptions $(A_1)-(A_4)$ and assume that $\mathcal{B}$ is of the form $\mathcal{B} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. Then for every $\mu > \omega$ the following assertions are equivalent.*

(a) $s(\mathcal{A}) < \mu$.

(b) $r(BR_{21}(\mu)) < 1$.

(b)* $(Id - BR_{21}(\mu))$ *is invertible with positive inverse.*

*(b**)* $\sum_{n=o}^{\infty} \|(BR_{21}(\mu))^n\| < \infty$.

Before proving this result we briefly comment on the implication $(b) \Rightarrow (a)$ stating that in order to have $\operatorname{Re}\lambda < \mu$ for all complex numbers $\lambda \in \sigma(\mathcal{A})$ it suffices to look at the value of the real valued function $\xi : \lambda \to \xi(\lambda) := r(BR_{21}(\lambda))$ at the point $\lambda = \mu$. Here, $BR_{21}(\lambda)$ is a positive operator on the Banach lattice X_1 satisfying $BR_{21}(\lambda_1) \leq BR_{21}(\lambda_2)$ for $\omega < \lambda_2 \leq \lambda_1$. Therefore $\xi(\cdot)$ is a decreasing function and in order to have $s(\mathcal{A}) < \mu$ it suffices to test the value of $\xi(\cdot)$ in $\lambda = \mu$ only.

Proof of Theorem 2.3. We first observe that for $\lambda > \omega$ one has $\lambda \in \rho(\mathcal{A}_0)$ and therefore the operator

$$(\lambda - \mathcal{A}) = [Id - \mathcal{B}R(\lambda, \mathcal{A}_0)](\lambda - \mathcal{A}_0) \tag{$*$}$$

is invertible if and only if

$$\begin{aligned}[Id - \mathcal{B}R(\lambda, \mathcal{A}_0)] &= \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix} - \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ R_{21}(\lambda) & R_{22}(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} Id - BR_{21}(\lambda) & -BR_{22}(\lambda) \\ 0 & Id \end{pmatrix}\end{aligned} \tag{$**$}$$

is invertible. This is the case if and only if $(Id - BR_{21}(\lambda))$ is invertible on X_1. The matrix rules for operator matrices (see [Na-2] or [En]) then yield

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ R_{21}(\lambda) & R_{22}(\lambda) \end{pmatrix} \begin{pmatrix} (Id - BR_{21}(\lambda))^{-1} & (Id - BR_{21}(\lambda))^{-1} BR_{22}(\lambda) \\ 0 & Id \end{pmatrix}.$$

We now show $(a) \Rightarrow (b)$. From Lemma 2.2 we know that $s(\mathcal{A}) < \mu$ if and only if $\mu \in \rho(\mathcal{A})$ and $R(\mu, \mathcal{A})$ is positive. Then the above identities $(*)$ and $(**)$ imply $1 \in \rho(BR_{21}(\mu))$. Since

$$r(BR_{21}(\mu)) = \inf\{\lambda \in \rho(BR_{21}(\mu)) \cap \mathbb{R} : R(\lambda, BR_{21}(\mu)) \geq 0\} \tag{$***$}$$

(see the observation following Lemma 2.2) it suffices to show that

$$R := R(1, BR_{21}(\mu)) = (Id - BR_{21}(\mu))^{-1}$$

is a positive operator. Assume to the contrary that there exists $0 < x_1 \in X_1$ such that $Rx_1 \not\geq 0$. Since $R(\mu, \mathcal{A}_0)^* X_+^*$ is weak*-dense in X_+^*, we find $0 \leq \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in X_1^* \times X_2^* = X^*$ satisfying

$$\left\langle \begin{pmatrix} Rx_1 \\ 0 \end{pmatrix}, R(\mu, \mathcal{A}_0)^* \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle < 0.$$

This implies that

$$\left\langle R(\mu, \mathcal{A}) \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} R & RBR_{22}(\mu) \\ 0 & Id \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, R(\mu, \mathcal{A}_0)^* \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle$$

is negative, thus contradicting the positivity of $R(\mu, \mathcal{A})$. $(b) \Rightarrow (a)$. By the above considerations we immediately obtain $\mu \in \rho(\mathcal{A})$. Moreover, since $R(\mu, \mathcal{A}_0)$ is positive, we conclude from the above matrix representation that $R(\mu, \mathcal{A})$ is positive whenever $(Id - BR_{21}(\mu))^{-1}$ is positive. This follows since $BR_{21}(\mu)$ is a positive operator satisfying $r(BR_{21}(\mu)) < 1$. The equivalences $(b) \Leftrightarrow (b^*) \Leftrightarrow (b^{**})$ are consequences of standard spectral theory and the characterization of the spectral radius of a bounded, positive operator as in $(***)$. ◇

In order to obtain a stability criterion it suffices to assume $\omega < 0$ and take $\mu = 0$ in the above theorem. This becomes particularly useful if the factor space X_1 is finite dimensional. In this case we identify the operator $BR_{21}(0) \in \mathcal{L}(X_1)$ with a $n \times n$ - matrix and write $(BR_{21}(0))_{k\times k}$ for the upper left $k \times k$ - submatrix for each $1 \le k \le n$.

2.4 Corollary. *In addition to the assumptions of Theorem 2.3 let $\omega < 0$ and suppose $\dim X_1 = n$. Then the following assertions are equivalent.*

(a) $s(\mathcal{A}) < 0$

(b) $r(BR_{21}(0)) < 1$

(c) $(-1)^{k+1} \det((BR_{21}(0) - Id)_{k\times k}) < 0$ *for* $1 \le k \le n$.

Proof. It suffices to observe that $BR_{21}(0) - Id$ is a matrix with positive off-diagonal elements and that (b) implies

$$s(BR_{21}(0) - Id) < 0.$$

Apply now one of the many characterizations from [Be-Pl] for so called M-matrices in order to obtain (c) or other equivalent statements. ♢

3. EXAMPLE

All the examples discussed in Section 3 of [Na-4] fit into the above framework once (in Example 3.1 and 3.2) some natural positivity assumptions are added. We therefore restrict ourselves to one more example and consider hyperbolic systems with dynamic boundary conditions as studied in [N-Sr-L], [Na-3], Section 4 or [En], Chapter II, Example 2.18. To the equations

$$\begin{aligned} \dot{u}(t,x) &= au_x(t,x) \\ \dot{v}(t,x) &= -dv_x(t,x) \end{aligned}$$

for $0 \le x \le 1, 0 \le t$ we associate the dynamic boundary conditions

$$\begin{aligned} \frac{d}{dt}(-\alpha u(t,0) + v(t,0)) &= \quad u(t,0) + v(t,0)) \\ \frac{d}{dt}(u(t,1) - \beta v(t,1)) &= \quad u(t,1) + v(t,1) \end{aligned}$$

and the initial conditions

$$u(0,\cdot) = u_0, \qquad v(0,\cdot) = v_0.$$

This situation can be described by the following operator theoretic model. On $X := (L^1[0,1] \times L^1[0,1]) \times (\mathbb{C} \times \mathbb{C})$ take the operator matrix

$$\mathcal{A}_0 := \begin{pmatrix} \begin{pmatrix} a\frac{d}{dx} & 0 \\ 0 & -d\frac{d}{dx} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathcal{A}_0) := \left\{ \begin{pmatrix} f \\ g \\ x \\ y \end{pmatrix} : f,g \in W^{1,1}; x,y \in \mathbb{C}; \begin{array}{rcl} -\alpha f(0) + g(0) & = & x \\ f(1) - \beta g(1) & = & y \end{array} \right\}.$$

Given

$$\mathcal{B} := \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{pmatrix} \delta_0 & \delta_0 \\ \delta_1 & \delta_1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$$

as perturbing operator we want to estimate the spectral bound $s(\mathcal{A})$ of

$$\mathcal{A} := \mathcal{A}_0 + \mathcal{B}.$$

In order to satisfy the positivity assumptions in Theorem 2.3 we assume

$$0 < a, d, \alpha, \beta.$$

As a first task we have to compute $R(\lambda, \mathcal{A}_0)$. This can be taken from [Na-3], Section 4, Step 4. One only has to observe that we changed here the sign in the first boundary condition in order to obtain positivity. Using essentially the notation from that paper and noting that the operator $\mathcal{A}_0$ has positive resolvent (see [Na-3], Section 3.3) we obtain the following explicit representation for $R(\lambda, \mathcal{A}_0)$, thus satisfying Assumption (A_2) for $\omega := \sup(0, \frac{ad}{a+b}\ln(\alpha\beta))$. Here, we write $\epsilon_\mu(s) := e^{\mu s}$ for $0 \leq s \leq 1$ and $\mu \in \mathbb{C}$.

3.1 Lemma. *Take* $0 \neq \lambda \in \mathbb{C}$ *and* $c := -(a+d)/ad$ *such that*

$$\xi(\lambda) := 1 - \alpha\beta e^{c\lambda} \neq 0$$

and define

$$K_\lambda := \frac{1}{\xi(\lambda)} \begin{pmatrix} \beta e^{\lambda c}\epsilon_{\lambda/a} & e^{-\lambda/a}\epsilon_{\lambda/a} \\ \epsilon_{-\lambda/d} & \alpha e^{-\lambda/a}\epsilon_{-\lambda/d} \end{pmatrix}.$$

Then $\lambda \in \rho(\mathcal{A}_0)$ *and the resolvent is*

$$R(\lambda, \mathcal{A}_0) = \begin{pmatrix} R(\lambda, A_0) & \frac{1}{\lambda}K_\lambda \\ 0 & \frac{1}{\lambda} \end{pmatrix}.$$

Since the perturbing operator $\mathcal{B}$ is positive we can apply Theorem 2.3 and Corollary 2.4 in order to obtain the following stability criterion.

3.2 Proposition. *Under the above assumptions and for $\mu > \omega$ the following assertions are equivalent.*

(a) $s(\mathcal{A}) < \mu$.

(b) $r(\frac{1}{\mu}BK_\mu) < 1$.

(b*) *The spectral radius of the 2×2-matrix*

$$BK_\mu = \frac{1}{\xi(\mu)} \begin{pmatrix} 1+\beta e^{\mu c} & (1+\alpha)e^{-\mu/a} \\ (1+\beta)e^{-\mu/d} & 1+\alpha e^{\mu c} \end{pmatrix}$$

is smaller than μ.

(c) *The inequalities*

$$\frac{1+\beta e^{\mu c}}{\mu\xi(\mu)} < 1$$

and

$$(1+\alpha)(1+\beta)e^{\mu c} < (1+\alpha e^{\mu c} - \mu\xi(\mu))(1+\beta e^{\mu c} - \mu\xi(\mu))$$

hold.

REFERENCES

[Ar-1] W. Arendt: Positive resolvent operators. Proc. London Math. Soc. **54** (1987), 321-349.

[Ar-2] W. Arendt: Spectrum and growth of positive semigroups. In: G. Ferreyra, G. Ruiz Goldstein, F. Neubrander (eds.): Evolution Equations, Lect. Notes Pure Appl. Math. **168**, 21-28. Marcel Dekker 1995.

[Be-Pl] A. Berman, R. Plemmons: *Nonnegative Matrices in the Mathematical Sciences.* Academic Press 1979.

[En] K.-J. Engel: *Operator Matrices and Systems of Evolution Equations.* Book manuscript 1996.

[Na-1] R. Nagel (ed.): *One-parameter Semigroups of Positive Operators.* Lect. Notes Math. **1184**. Springer-Verlag 1986.

[Na-2] R. Nagel: Towards a "matrix theory" for unbounded operator matrices. Math. Z. **201** (1989), 57-68.

[Na-3] R. Nagel: The spectrum of unbounded operator matrices with non-diagonal domain. J. Funct. Analysis **89** (1990), 291-302.

[Na-4] R. Nagel: Characteristic equations for the spectrum of generators. Ann. Scuola Norm. Sup. Pisa (to appear).

[Ne] J. v. Neerven: *The Asymptotic Behaviour of a Semigroup of Linear Operators.* Operator Theory Adv. Appl. **88**. Birkhäuser Verlag 1996.

[N-SR-L] A. F. Neves, H. Souza Ribeiro, O. Lopez: On the spectrum of evolution operators generated by hyperbolic systems. J. Funct. Analysis **67** (1986), 320-344.