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SOME NONLINEAR HEAT CONDUCTION PROBLEMS FOR A SEMI-INFINITE STRIP WITH A NON-UNIFORM HEAT SOURCE

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ABSTRACT. A non-classical initial boundary value problem for the non-homogeneous one-dimensional heat equation for a semi-infinite material $x > 0$, with temperature or heat flux boundary conditions on the face $x = 0$ is studied. It is not an standard heat conduction problem because a heat source $\Phi(x)\ \mathcal{F}_{(t)}[.,.]$ is considered, where Φ is a real function and $\mathcal{F}_{(t)}$ is a functional on the heat flux $u_x(0,.)$, for all $t > 0$. Existence and uniqueness of solution is proved under suitable assumptions on data. A priori estimates, continuous and monotone dependence upon the data and the asymptotic behavior of the solution are also analized for the particular case $\mathcal{F}_{(t)}(V(.),.) = F(V(t),t)$.

I. INTRODUCTION

In this paper, the following nonlinear one-dimensional initial boundary-value problem for the

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KEY WORDS and PHRASES: Nonlinear heat conduction problems, Non-uniform heat source, Integral equations, Qualitative properties of the solution.

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heat conduction equation for a semi-infinite material is considered

$$
(\mathrm{P}) \quad \left| \begin{array}{ll} (1) & u_t(x,t) - u_{xx}(x,t) = \Phi(x) \Big(\mathcal{F}(u_x(0,.),.)\Big)(t) \ , \ x > 0 \ , t > 0 \\ (2) & u(0,t) = g(t) \ , \ \text{for} \ t > 0 \\ (3) & u(x,0) = h(x) \ , \ \text{for} \ x > 0 \ . \end{array} \right.
$$

In problem (P), Φ, h and g are real functions defined on $\mathbb{R}^+$. $\mathcal{F}$ is a functional, that is for any $t > 0$ and $u_x(0,.)$, $\mathcal{F}_{(t)}$ is a given function of t given by $\mathcal{F}_{(t)}(u_x(0,.),.) = \Big(\mathcal{F}(u_x(0,.),.)\Big)(t)$. Some particular and interesting cases are the following

$$
(4) \qquad \mathcal{F}_{(t)}\Big(V(.),.\Big) = F(V(t),t) , \qquad t > 0 ,
$$

$$
(5) \qquad \mathcal{F}_{(t)}\Big(V(.),.\Big) = F\Big(\int_0^t V(\tau)\, d\tau , t\Big), \qquad t > 0 ,
$$

where F is a given function of two real variables.
Such problems can be thought as motivated by the modelling of a system of temperature regulation in isotropic mediums, with the non-uniform source term $\Phi(x) \ \mathcal{F}_{(t)}(u_x(0,.),.)$ which provides a cooling or heating effect depending uppon the properties of $\mathcal{F}$ (or F) related to the course of the heat flux $u_x(0,t)$. For example, in cases such as (4) is supposed, when

$$
(6) \qquad \Phi(x) > 0 \quad \text{and} \quad u_x(0,t)\ F\Big(u_x(0,t),t\Big) > 0 \quad \text{if} \quad u_x(0,t) \neq 0 .
$$

the source term is a cooler if $u_x(0,t) < 0$ and a heather if $u_x(0,t) > 0$.
For the case of a bounded domain, a class of problems when the heat source is uniform and belongs to a given multivalued function from $\mathbb{R}$ into itself, was studied in [Ke Pr] regarding existence, uniquenees and asymptotic behavior. Other references on the subject are [GlSp1, GlSp2, Ke]. Some results concerning the particular situation in which $\Phi(x) = \text{const} > 0$, $g(t) = 0$, $\mathcal{F}_{(t)}$ given by (4), were obtained in [Vi, TaVi].

In Section II, existence and uniqueness of solution for (P) is proved. The solution u of problem (P) has an integral representation given by (II-10) where $V(t) = u_x(0,t)$ must satisfy a functional integral equation of Volterra type given by (II-12). We give sufficient conditions on data in order to obtain the existence and uniqueness for the corresponding integral equation; this result is obtained by a generalization of the theory on integral equation developed in chapter 8 and 20 of [Ca].
A priori estimates, continuous and monotone dependence upon the data and an asymptotic behavior of the solution are explicited in Section III for the case (4).
In Section IV, we also consider the following initial–boundary value problem for the one–dimensional heat equation

$$(P') \quad \left| \begin{array}{ll} (7) & v_t - v_{xx} = \phi(x)\, \mathcal{F}_{(t)}\Big(v(0,.),.\Big) \ , \ x > 0 \ , t > 0 \ , \\ (8) & v(x,0) = h_o(x) \ , \ x > 0 \ , \\ (9) & v_x(0,t) = g_o(t) \ , \ t > 0 \ . \end{array} \right.$$

We recall that problem (P′) can be reduced to the problem (P) by using the transformation (IV-1) or (IV-3).

II. EXISTENCE AND UNIQUENESS

Let K be the fundamental solution of the one-dimensional heat equation, G and N the Green and Neumann functions for $x > 0$, given by

$$(1) \qquad K(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\Big(\frac{-(x-\xi)^2}{4(t-\tau)}\Big), \qquad x\,,\, \xi > 0, \qquad t > \tau,$$

$$(2) \qquad G(x,t;\xi,\tau) = K(x,t;\xi,\tau) - K(-x,t;\xi,\tau), \qquad x\,,\, \xi > 0, \qquad t > \tau,$$

$$(3) \qquad N(x,t;\xi,\tau) = K(x,t;\xi,\tau) + K(-x,t;\xi,\tau), \qquad x\,,\, \xi > 0, \qquad t > \tau,$$

For data $h=h(x)$, $g=g(t)$ and $\mathcal{F}=\mathcal{F}_{(t)}$ in problem (P) we shall consider the following assumptions:

(H1) h and g are continuously differentiable functions on $\mathbb{R}^+$ with

$$(4) \qquad h(0)=g(0) \ , \ |h(x)| \le c_o \exp(c_1 x^{2-\epsilon}) \ , \ c_o > 0 \ , \ c_1 > 0 \ , \ \epsilon > 0 \ , \text{ for all } x > 0 \ ;$$

(H2) Φ is uniformly Hölder continuous in x for each compact subset of $\mathbb{R}^+$;

(H3) There exists

$$(5) \quad \left| \begin{array}{l} M = M(w,t) > 0 \ / \ \left| \mathcal{F}_{(t_2)}(V(.),.) - \mathcal{F}_{(t_1)}(V(.),.) \right| \le M\,|t_2 - t_1| \ , \ \forall\, V \in S_w(0,t) \\ \text{for all } 0 < t_1,\, t_2 < t \ , \end{array} \right.$$

where $S_w(0,t)$ is the set of piecewise continuous functions on the interval [0,t] such that

$$(6) \qquad \|V\|_t = \sup_{\tau \in [0,t]} |V(\tau)| \le w \ .$$

Roughly speaking, we say that $\mathcal{F}_{(t)}$ is uniformly Lipschitz continuous for any subset of $S_w(0,t) \times [0,t]$ for each $t > 0$;

(H4) There exists a positive and locally bounded function $L=L(t)$, defined for $t > 0$, such that

(7) $$\left|\mathcal{F}_{(t)}(V_2(.),.)-\mathcal{F}_{(t)}(V_1(.),.)\right| \leq L(t)\,\|V_2-V_1\|_t\,, \text{ for all } V_1\,,\,V_2\in S_w(0,t)\ ;$$

(H5) $$\mathcal{F}_{(t)}(0,.)=0 \quad , \quad \forall\, t>0;$$

(H6) Function Φ is such that there exists a positive monotone increasing function $A=A(t)$, defined for $t>0$, which verifies

(8) $$\int_{t_1}^{t_2} R(t_2-\tau)\,L(\tau)\,d\tau \;\leq\; A(t_2-t_1) \quad , \quad t_2>t_1>0$$

with

(9) $$\lim_{t\to 0^+} A(t)=0$$

where R is defined, in function of Φ, by (14) (see below).

Under the preceding assumptions the theory developed in chapter 20 of [Ca] can be generalized to obtain the following representation for the solution of problem (P)

(10) $$u(x,t)=-2\int_0^t K_x(x,t;0,\tau)\,g(\tau)\,d\tau+\int_0^{+\infty} G(x,t;\xi,0)\,h(\xi)\,d\xi\,+$$
$$+\int_0^t\left(\int_0^{+\infty} G(x,t;\xi,\tau)\,\Phi(\xi)\,d\xi\right)\mathcal{F}_{(\tau)}(V(.),.)\,d\tau$$

where the function $V=V(t)$ defined by

(11) $$V(t)=u_x(0,t) \quad , \quad t>0$$

must satisfy the following functional integral equation of Volterra type

(12) $$V(t)=f(t)+\int_0^t R(t-\tau)\,\mathcal{F}_{(\tau)}(V(.),.)\,d\tau$$

where

(13) $$f(t)=\frac{1}{\sqrt{\pi}}\left(\int_0^{+\infty}\frac{1}{\sqrt{t}}\exp\!\left(\frac{-\xi^2}{4t}\right)h'(\xi)\,d\xi-\int_0^t\frac{\dot g(\tau)}{\sqrt{t-\tau}}\,d\tau\right)$$

(14) $$R(z)=\frac{1}{2\sqrt{\pi}\,z^{3/2}}\int_0^{+\infty}\xi\,\exp\!\left(\frac{-\xi^2}{4z}\right)\Phi(\xi)\,d\xi\,.$$

THEOREM 1.– Assume (H1) to (H6), then there exists a unique piecewise function V solution of the functional integral equation (12). Therefore, u given by (10) is the unique solution of the problem (P) in the class of functions which satisfies a growth condition of the form

(15) $$|u(x,t)| \le C_2 \exp(C_3 x^2)$$

where C_2 and C_3 are positive constants.

Before to prove the theorem 1 we shall give some sufficient conditions on data in order to clarify the above hypothese.

Remark 1.– (H4) and (H5) imply that $\mathcal{F}_{(t)}$ is bounded for bounded t and $V \in S_w(0,t)$, that is

(16) $$\exists\, C = C(w,t) \,/\, \left|\mathcal{F}_{(t)}(V(.),.)\right| \le C\,, \text{ for all V such that } \|V\|_t \le w\,, \quad \text{with } w > 0\,,$$

with

(17) $$C = L(t)\, w\,.$$ □

Remark 2.– If function Φ verifies the inequality

(18) $$0 < \Phi(x) \le C_o \exp(C_1 x^2)\,, \quad \forall x \ge 0 \text{ with } C_o > 0\,, C_1 > 0$$

then

(19)
$$\text{(i)} \quad R(t) \le \frac{4C_o}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{t(1-4C_1 t)}} \qquad \text{if} \qquad t < \frac{1}{4C_1}$$

$$\text{(ii)} \quad \int_{t_1}^{t_2} R(t_2-\tau)\, d\tau \le \frac{2C_o}{\sqrt{\pi C_1}} f_o(t_2-t_1) \qquad \text{if} \qquad C_1 < \frac{1}{4(t_2-t_1)}$$

where $f_o = f_o(t)$, defined by

(20) $$f_o(t) = \log\left(\frac{1+2\sqrt{C_2 t}}{1-2\sqrt{C_1 t}}\right)$$

is a monotone increasing function over the domain $[0, \frac{1}{4C_1})$ with $\lim_{t \to 0^+} f_o(t) = 0$. □

Remark 3.– If Φ verifies condition (18) and

(21) $$L(t) \le L_o t^n \qquad , \qquad n = 0, 1, \ldots$$

then

(22) $$\int_{t_1}^{t_2} R(t_2-\tau)\, L(\tau)\, d\tau \le \frac{2C_o L_o t_2^n}{\sqrt{\pi C_1}} f_o(t_2-t_1)\,.$$ □

Remark 4.– If function Φ is given by

(23) $$\Phi(x)=\Phi_o \,.\, x^n, \qquad \Phi_o=\text{const.} > 0, \qquad n=1,2,\dots$$

then

(24) $$R(t)=\frac{\Phi_o}{\sqrt{\pi}}\, a_n\, t^{\frac{n-1}{2}}$$

where

(25) $$a_{2m}=4^m\, m!\ , \qquad a_{2m-1}=2^{m-1}\,(2m-1)!!\ , \qquad m=1,2,\dots$$

Moreover

(26) $$\int_{t_1}^{t_2} R(t_2-\tau)\, d\tau = \frac{2\,\Phi_o}{\sqrt{\pi}}\,\frac{a_n}{n+1}\,(t_2-t_1)^{\frac{n+1}{2}}\ .$$ □

Proof of Theorem 1.– From (7) we deduce

(27) $$\left|R(t-\tau)\,\mathcal{F}_\tau(V_2(.),.)-R(t-\tau)\,\mathcal{F}_\tau(V_1(.),.)\right| \le \left|R(t-\tau)\right| L(\tau)\, \|V_2-V_1\|_\tau\ ,$$

where V_1 , $V_2 \in S_w(0,t)$ and $0<\tau<t$. Taking into account (H5), (8), (9) and (27), the conclusion of the Theorem follows applying Th.8.2.1 of [Ca]. □

Remark 5.– (i) If the functional $\mathcal{F}_{(t)}$ is given by (1–4) then (5) and (7) must be replaced respectively by

(28) $$\left|F(V,t_2)-F(V,t_1)\right| \le M(V)\left|t_2-t_1\right|\ , \quad \text{for all } 0<t_1\,,\, t_2<t\ ,$$

(29) $$\left|F(V_2,t)-F(V_1,t)\right| \le L(t)\left|V_2-V_1\right|\ , \quad \text{for all } t>0$$

for all V, V_1 and V_2 in compact sets into $\mathbb{R}$, and similarly condition (H5) must be replaced by

(30) $$F(0,t)=0\ , \quad \forall\, t>0\ .$$

(ii) If the functional $\mathcal{F}_{(t)}$ is given by (1–5) then (5) and (7) are verified whenever

(31) $$\left|F(V_2,t)-F(V_1,t)\right| \le L_o(t)\left|V_2-V_1\right|$$

(32) $$\left|F(V,t_2)-F(V,t_1)\right| \le L_1(V)\left|t_2-t_1\right|$$

(33) $$L_1\!\left(\int_0^t V(\tau)\, d\tau\right) \le L_2(t)\,\|V\|_t$$

with

(34) $$M=[L_o(t_1)+L_2(t_2)]\,.\,\|V\|_{\max(t_1,t_2)} \quad \text{in (5)}$$

(35) $$L(t)=t\, L_o(t) \quad \text{in (7)}\ .$$ □

LEMMA 2.— Under the assumptions (H1), (H2), $\Phi \in L^\infty(\mathbb{R}^+)$ and

(H7) There exist $\int_0^t L(\tau)\, d\tau$ and $\int_0^t \frac{L(\tau)}{\sqrt{t-\tau}}\, d\tau$ for each $t > 0$

h, g, h' and $\dot{g} \in L^\infty(\mathbb{R}^+)$

the following estimates for the solution $V = V(t)$ of the integral equation (12) and the function $u = u(x,t)$, given by (10), are obtained

$$\|V\|_t \le \|f\|_t \exp\left(\frac{\|\Phi\|_\infty}{\sqrt{\pi}} \int_0^t \frac{L(\tau)}{\sqrt{t-\tau}}\, d\tau\right) \tag{36}$$

$$|u(x,t)| \le \|g\|_t + \|h\|_\infty + \|\Phi\|_\infty \left(\int_0^t L(\tau)\, d\tau\right) \|V\|_t \tag{37}$$

where

$$\|f\|_t \le \|h'\|_\infty + 2\, \|\dot{g}\|_t \sqrt{\tfrac{t}{\pi}}\ . \tag{38}$$

Proof. Taking into account (H7), (38) follows from (13). On the other hand, by (7), (H5) and (H7), from (12) we find

$$\|V\|_t \le \|f\|_t + \frac{\|\Phi\|_\infty}{\sqrt{\pi}} \int_0^t \frac{L(\tau)}{\sqrt{t-\tau}} \|V\|_\tau\, d\tau \tag{39}$$

and (36) follows from (39) by using a Gronwall's inequality. The inequality (37) follows from (10), taking into account (H7) and the fact that

$$\int_0^t \frac{\partial K}{\partial x}(x,t;0,\tau)\, d\tau = -\frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right) \tag{40}$$

$$\int_0^{+\infty} G(x,t;\xi,\tau)\, d\xi = \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)\ , \tag{41}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\, du\ , \tag{42}$$

is the error function. □

III. QUALITATIVE ANALYSIS

In this section we shall consider the case (1-4), that is

$$\mathcal{F}_{(t)}(V(.),.) = F(V(t),t)\ , \qquad t > 0 \tag{1}$$

where F satisfies the aditional condition

$$(2)\quad \begin{cases} V.F(V,t) > 0 \quad , \quad \forall V \neq 0 \ , \ \forall t > 0 \ , \\ F(0,t) = 0 \ , \quad \forall t > 0 \ . \end{cases}$$

Moreover we suppose the following hypothese for data in problem (P)

(H8) $h'(0^+) > 0$, $h'(x) \geq 0$, $\forall x > 0$,

(H9) $\dot{g}(t) \leq 0$, $\forall t > 0$,

(H10) $\Phi'(x) \geq 0$, $\forall x > 0$.

LEMMA 3.– Under assumptions (H1), (H2), (H6), (II-28), (II-29), (II-30), (2), (H8), (H9) and (H10) we have that

(3) $V(t) > 0 \quad , \ t > 0$

(4) $u_x(x,t) \geq 0 \quad , \quad x \geq 0 \ , \ t \geq 0$.

Proof.– We have $V(0) = h'(0^+) > 0$. On the other hand, the function $v = v(x,t)$ defined by

(5) $v(x,t) = u_x(x,t)$

satisfies the following heat conduction problem

$$(6)\quad \begin{cases} v_t - v_{xx} = \Phi'(x)\ F[v(0,t),t] \ , \ x > 0 \ , t > 0 \\ v_x(0,t) = \dot{g}(t) - \Phi(0)\ F(v(0,t),t) \ , \ t > 0 \\ v(x,0) = h'(x) \ , \ x > 0 \ . \end{cases}$$

Then, we suppose that there exists a time $t_1 > 0$ such that

(7) $V(t_1) = v(0,t_1) = 0 \ , \ V(t) = v(0,t) > 0 \ , \ 0 < t < t_1$.

Therefore, we have

(8) $v_t(x,t) - v_{xx}(x,t) = \Phi'(x)\ F(v(0,t),t) \geq 0 \ , \ x > 0 \ , \ 0 < t < t_1$

and, from the maximum principle it follows

(9) $v(x,t) \geq 0 \ , \ x \geq 0 \ , \ 0 \leq t \leq t_1$.

Taking into account (7), the maximum principle implies that $v_x(0,t_1) > 0$ which is contradictory with

(10) $$v_x(0,t_1)=\dot{g}(t_1)-\Phi(0)\ F(V(t_1),t_1) = \dot{g}(t_1) \leq 0.$$

Hence, we conclude that (3) and (4) hold. □

Now , we shall consider the continuous dependence of the functions $V=V(t)$ and $u=u(x,t)$ given by (II-12) and (II-10) respectively upon the data h, g, Φ and F.

Let us denote by $V_i=V_i(t)$ $(i=1,2)$ the solution of (II-12) and $u_i=u_i(x,t)$ given by (II-10) respectively for data h_i, g_i, Φ_i and F $(i=1,2)$ in problem (P). Then we obtain the following results.

THEOREM 4.– Let us consider the problem (P) under assumptions (H1), (H2), (H6), (II-28) – (II-30), (2) and (H7) for $L = L(t)$ and furthermore h_i' , Φ_i , $\dot{g}_i \in L^\infty(\mathbb{R}^+)$ $(i=1,2)$. Then, we obtain

(11) $$|V_2(t)-V_1(t)| \leq P(t)\exp\left(\left(\|\Phi_1\|_{L^\infty(\mathbb{R}^+)}+\|\Phi_2\|_{L^\infty(\mathbb{R}^+)}\right)\frac{1}{2\sqrt{\pi}}\int_0^t \frac{L(\tau)}{\sqrt{t-\tau}}\,d\tau\right),\ 0\leq t\leq T$$

where

(12) $$P(t) = \|h_2'-h_1'\|_{L^\infty(\mathbb{R}^+)} + \frac{2}{\sqrt{\pi}}\ \|\dot{g}_2-\dot{g}_1\|_{L^\infty(\mathbb{R}^+)}\ \sqrt{t} +$$

$$+\frac{1}{2}\ \frac{\|\Phi_2-\Phi_1\|_{L^\infty(\mathbb{R}^+)}}{\sqrt{\pi}}\ \left(\|V_1\|_t+\|V_2\|_t\right)\int_0^t \frac{L(\tau)}{\sqrt{t-\tau}}\,d\tau\ ,$$

(13) $$|u_2(x,t)-u_1(x,t)| \leq \|g_2-g_1\|_{L^\infty(\mathbb{R}^+)}+\|h_2-h_1\|_{L^\infty(\mathbb{R}^+)}+$$

$$+\frac{1}{2}\left\{\|\Phi_1+\Phi_2\|_{L^\infty(\mathbb{R}^+)}\ \|V_2-V_1\|_t+\|\Phi_2-\Phi_1\|_{L^\infty(\mathbb{R}^+)}\left(\|V_1\|_t+\|V_2\|_t\right)\right\}\int_0^t L(\tau)\,d\tau\ .$$

We recall that estimates for $\|V_2-V_1\|_t$, $\|V_1\|_t$, $\|V_2\|_t$ can be obtained from (11) and (II-36) to be inserted in (13).

Proof.– From (II-12)–(II-14) we can write

(14) $$V_2(t)-V_1(t)=f_2(t)-f_1(t)+\int_0^t \left(R_2(t-\tau)\ F(V_2(\tau),\tau)-R_1(t-\tau)\ F(V_1(\tau),\tau)\right)d\tau$$

and using the equality given by

$$R_2(t-\tau)\, F(V_2(\tau),\tau) - R_1(t-\tau)\, F(V_1(\tau),\tau) = \tfrac{1}{2}\Big\{ R_2(t-\tau)\Big(F(V_2(\tau),\tau) - F(V_1(\tau),\tau)\Big) +$$

(15) $$+ R_1(t-\tau)\Big(F(V_2(\tau),\tau) - F(V_1(\tau),\tau)\Big) + F(V_2(\tau),\tau)\Big(R_2(t-\tau) - R_1(t-\tau)\Big) +$$

$$+ F(V_1(\tau),\tau)\Big(R_2(t-\tau) - R_1(t-\tau)\Big)\Big\}\ , \qquad 0 < \tau < t\ ,$$

and the fact that

(16) $$|f_2(t) - f_1(t)| \le \|h_2' - h_1'\|_{L^\infty(\mathbb{R}^+)} + \frac{2}{\sqrt{\pi}} \|\dot{g}_2 - \dot{g}_1\|_{L^\infty(\mathbb{R}^+)} \sqrt{t}\ ,$$

from (14) we find

(17) $$|V_2(t) - V_1(t)| \le \|h_2' - h_1'\|_{L^\infty(\mathbb{R}^+)} + \frac{2}{\sqrt{\pi}} \|\dot{g}_2 - \dot{g}_1\|_{L^\infty(\mathbb{R}^+)} \sqrt{t} +$$

$$+ \tfrac{1}{2} \|\Phi_2 - \Phi_1\|_{L^\infty(\mathbb{R}^+)} \Big(\|V_1\|_t + \|V_2\|_t\Big) \int_0^t \frac{L(\tau)}{\sqrt{t-\tau}}\, d\tau +$$

$$+ \tfrac{1}{2}\Big(\|\Phi_1\|_{L^\infty(\mathbb{R}^+)} + \|\Phi_2\|_{L^\infty(\mathbb{R}^+)}\Big) \int_0^t \frac{L(\tau)}{\sqrt{t-\tau}}\, |V_2(\tau) - V_1(\tau)|\, d\tau\ .$$

Then, (11) follows from (17) using a Gronwall's inequality. To obtain (13) we note that from (II-10) we can write

(18) $$u_2(x,t) - u_1(x,t) = -2\int_0^t K_x(x,t;0,\tau)\Big(g_2(\tau) - g_1(\tau)\Big)d\tau + \int_0^{+\infty} G(x,t;\xi,0)\Big(h_2(\xi) - h_1(\xi)\Big)d\xi +$$

$$+ \int_0^t \int_0^{+\infty} G(x,t;\xi,\tau)\Big(\Phi_2(\xi)\, F(V_2(\tau),\tau) - \Phi_1(\xi)\, F(V_1(\tau),\tau)\Big)d\xi\, d\tau\ .$$

Using the inequality

(19) $$|\Phi_2(\xi)\, F(V_2(\tau),\tau) - \Phi_1(\xi)\, F(V_1(\tau),\tau)| \le \tfrac{1}{2}\Big(|\Phi_1(\xi) + \Phi_2(\xi)|\ |F(V_2(\tau),\tau) - F(V_1(\tau),\tau)| +$$

$$+ |F(V_2(\tau),\tau) + F(V_1(\tau),\tau)|\ |\Phi_2(\xi) - \Phi_1(\xi)|\Big)$$

and equalities (II-40) and (II-41), (13) follows from (18). This complete the proof of Theorem 3□

Now, we shall consider a monotone property of the functions $V_i = V_i(t)$ and $u_i = u_i(x,t)$ for data h_i, g_i, Φ_i $(i=1,2)$ and F in problem (P). Then, we obtain the following results.

THEOREM 5. – Let us suppose the assumptions of Lemma 3 for data h_i, g_i, Φ_i (i=1, 2) and F. Moreover, if F is an increasing function in the first variable, i.e.

(20) $$V_2 \le V_1 \quad \text{implies} \quad F(V_2,t) \le F(V_1,t) \quad , \quad \forall\, t > 0$$

and data verify the relations

(21) $$\left| \begin{array}{l} 0 \le \Phi_2(0) \le \Phi_1(0) \;\; , \;\; 0 \le \Phi_2'(x) \le \Phi_1'(x) \;\; , \;\; \forall\, x > 0 \, , \\ 0 < h_2'(0) \le h_1'(0) \;\; , \;\; 0 \le h_2'(x) \le h_1'(x) \;\; , \;\; \forall\, x > 0 \, , \\ \dot g_1(t) \le \dot g_2(t) \le 0 \;\; , \;\; \forall\, t > 0 \end{array} \right.$$

then we have the following monotonicity properties

(22) $$\left| \begin{array}{l} 0 < V_2(t) \le V_1(t) \;\; , \;\; \forall\, t > 0 \, , \\ u_{2_x}(x,t) \le u_{1_x}(x,t) \;\; , \;\; \forall\, x \ge 0 \;\; , \;\; \forall\, t > 0 \, . \end{array} \right.$$

Proof. – We define

(23) $$W(x,t) = u_{2_x}(x,t) - u_{1_x}(x,t)$$

and we suppose that

(24) $$0 < V_2(t) \le V_1(t) \quad , \quad \forall\, t \in (0,t_2) \quad , \text{ and } \; V_2(t_2) = V_1(t_2).$$

Then, for $x > 0$ and $0 < t < t_2$, we obtain:

(25) $$W_t(x,t) - W_{xx}(x,t) = \Phi_2'(x)\, F(V_2(t),t) - \Phi_1'(x)\, F(V_1(t),t) \le$$
$$\le \Phi_1'(x)\, [F(V_2(t),t) - F(V_1(t),t)] \le 0 \, ,$$

(26) $$W(x,0) = h_2'(x) - h_1'(x) \le 0 \quad ,$$

(27) $$W(0,t) = V_2(t) - V_1(t) \le 0 \;\text{ and }\; W(0,t_2) = 0 \quad ,$$

(28) $$W_x(0,t) = \dot g_2(t) - \dot g_1(t) + \Phi_1(0)\, F(V_1(t),t) - \Phi_2(0)\, F(V_2(t),t) \ge$$
$$\ge \Phi_1(0)\, F(V_1(t),t) - \Phi_2(0)\, F(V_2(t),t) \ge [\Phi_1(0) - \Phi_2(0)]\, F(V_2(t),t) .$$

Therefore $W \le 0$ and $W(0,t_2) = 0$. From the maximum principle we obtain that $W_x(0,t_2) < 0$ which is contradictory with

(29) $$W_x(0,t_2) \ge [\Phi_1(0) - \Phi_2(0)]\, F(V_1(t_2),t_2) \ge 0 \, .$$

Then, we have (22). □

COROLLARY 6. – If the relation

(30) $$g_2(t) \le g_1(t) \, , \, t > 0$$

is also assumed in Theorem 5 then we obtain the inequality

(31) $$u_2(x,t) \le u_1(x,t)\ ,\ x \ge 0,\ t > 0\ . \qquad \square$$

Now, let $u_i = u_i(x,t)$, $V_i = V_i(t)$ $(i=1,2)$ be the functions given respectively by (II-10) and (II – 12) for data F_1 and F_2.

THEOREM 7.– If we consider the problem (P) under assumption (H1), (H2), (H6), (H7), (II-28) – (II-30) and furthermore assume $\Phi \in L^\infty(\mathbb{R}^+)$; then, the following continuous dependence for u is obtained

(32) $$\left| u_2(x,t) - u_1(x,t) \right| \le E\ \| F_2 - F_1 \|_{t,M} \quad ,\ x \ge 0\ ,\ 0 \le t \le T$$

with E is a positive constant which depends on

(33) $$E = E\left(\| \Phi \|_{L^\infty(\mathbb{R}^+)}, T \right).$$

The number M and the expression $\| F_2 - F_1 \|_{t,M}$ are defined by

(34) $$M = \left(\| h' \|_{L^\infty(\mathbb{R}^+)} + 2 \| \dot{g} \|_{L^\infty(\mathbb{R}^+)} \sqrt{\frac{T}{\pi}} \right) \exp\left(\frac{\| \Phi \|_{L^\infty(\mathbb{R}^+)}}{\sqrt{\pi}} \int_0^T \frac{L(\tau)}{\sqrt{t-\tau}}\, d\tau \right)$$

and

(35) $$\| F_2 - F_1 \|_{t,M} = \sup_{\substack{\| z \|_t \le M \\ 0 \le \tau \le t}} \left| F_2(z(\tau),\tau) - F_1(z(\tau),\tau) \right| .$$

Proof.– From (II-12) we have

(36) $$V_2(t) - V_1(t) = \frac{1}{2\sqrt{\pi}} \int_0^t \left(\int_0^{+\infty} \xi \exp\left(-\frac{\xi^2}{4(t-\tau)} \right) \Phi(\xi)\, d\xi \right) \left(F_2(V_2(\tau),\tau) - F_1(V_1(\tau),\tau) \right) d\tau.$$

Taking into account the assumption on Φ, (34), (II-29) and the inequality

(37) $$\left| F_2(V_2(\tau),\tau) - F_1(V_1(\tau),\tau) \right| \le \left| F_2(V_2(\tau),\tau) - F_2(V_1(\tau),\tau) \right| + \left| F_2(V_1(\tau),\tau) - F_1(V_1(\tau),\tau) \right|$$

from (36) we obtain

(38) $$\left| V_2(t) - V_1(t) \right| \le \frac{2\ \| \Phi \|_{L^\infty(\mathbb{R}^+)}}{\sqrt{\pi}} \| F_2 - F_1 \|_{t,M}\ \sqrt{t} + \frac{\| \Phi \|_{L^\infty(\mathbb{R}^+)}}{\sqrt{\pi}} \int_0^t \frac{L(\tau)}{\sqrt{t-\tau}} \left| V_2(\tau) - V_1(\tau) \right| d\tau$$

and using a Gronwall's inequality it follows that

(39)
$$\left|V_2(t)-V_1(t)\right| \leq \frac{2\,\|\Phi\|_{L^\infty(\mathbb{R}^+)}}{\sqrt{\pi}}\,\|F_2-F_1\|_{t,M}\sqrt{t}\,\exp\left(\frac{\|\Phi\|_{L^\infty(\mathbb{R}^+)}}{\sqrt{\pi}}\int_0^t \frac{L(\tau)}{\sqrt{t-\tau}}\,d\tau\right)$$
$$0 \leq t \leq T$$

On the other hand, in view of (35), (37) and (II-29), from (II-10) we find

(40)
$$\left|u_2(x,t)-u_1(x,t)\right| \leq \|\Phi\|_{L^\infty(\mathbb{R}^+)}\,\|F_2-F_1\|_{t,M}\;t+\int_0^t L(\tau)\left|V_2(\tau)-V_1(\tau)\right|d\tau$$
$$0 \leq t \leq T$$

and (32) follows from (39) and (40). □

Now, we shall give a result on the asymptotic behavior of the solution $u(x,t)$ when t goes to infinity.

THEOREM 8. – Let us suppose the assumptions of Lemma 3 and

(41) $$h(x) > 0\,,\qquad \Phi(x) \leq 0 \quad \text{in } \mathbb{R}^+ .$$

We obtain the following results :

(i) If $g \equiv 0$ in $\mathbb{R}^+$, then

(42) $$0 \leq u(x,t) \leq u_0(x,t)\quad,\quad x \geq 0\ ,\ t \geq 0$$

where

(43) $$u_0(x,t)=\int_0^{+\infty} G(x,t;\xi,0)\,h(\xi)\,d\xi\,.$$

Moreover, we have

(44) $$\lim_{t\to\infty} u(x,t)=0\ ,\ \text{uniformly for } x \geq 0.$$

(ii) If

(45) $$\lim_{t\to\infty} g(t)=0\,,\qquad g(t)>0\,,\qquad 0 \leq -\dot{g}(t) \leq K\exp(-\alpha t)\ \ (\text{with } K>0,\ \alpha>0),\ t>0\,,$$

then the aymptotic behavior of the solution u is also given by (44) for compact sets in $\mathbb{R}^+$.

Proof. – (i) The Lemma 3, conditions (41) and the maximum principle imply inequalities (42). Then (44) follows from (42) and (43).

(ii) From (II-10) we have

(46) $$g(t) \leq u(x,t)=u_0(x,t)+g(t)-g(0)\ \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)-\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)\dot{g}(\tau)\,d\tau\ +$$

$$+ \int_0^t \left(\int_0^{+\infty} G(x,t;\xi,\tau)\, \Phi(\xi)\, d\xi \right) F(V(\tau),\tau)\, d\tau \leq$$

$$\leq u_o(x,t) + g(t) - g(0)\ \mathrm{erf}\Big(\frac{x}{2\sqrt{t}}\Big) + K \frac{x^2}{2} \exp(-\alpha t) \int_{\frac{x}{2\sqrt{t}}}^{+\infty} \exp\Big(\frac{\alpha x^2}{4 z^2}\Big) \frac{\mathrm{erf}(z)}{z^3}\, dz .$$

Taking into account that the following limit

$$(47) \qquad \lim_{t\to+\infty} \frac{x^2}{2} \exp(-\alpha t) \int_{\frac{x}{2\sqrt{t}}}^{+\infty} \exp\Big(\frac{\alpha x^2}{4 z^2}\Big) \frac{\mathrm{erf}(z)}{z^3}\, dz = \lim_{t\to+\infty} \frac{1}{\alpha} \mathrm{erf}\Big(\frac{x}{2\sqrt{t}}\Big) = 0 ,$$

then we obtain (44) for compact sets in $\mathbb{R}^+$. □

Remark 6. – Some results concerning the fact that the control process (P) (with the source term) is asymptotically faster than the corresponding heat conduction problem (without the source term) for the case g=0, that is

$$(48) \qquad \lim_{t\to+\infty} \frac{u(x,t)}{u_o(x,t)} = 0 , \quad x > 0 ,$$

are given in [BeTaVi]. □

Now, we shall present two examples with explicit solution for problem (P) in order to illustrate cases in which the source "controls" the difference $|u(x,t) - h(x)|$ in time.

Example 1.– We consider

$$(49) \qquad \mathcal{F}_{(t)}(V(\tau),\tau) = \int_0^t V(\sigma)\, d\sigma \quad , \quad h(x) = \frac{x^3}{3} \ , \quad g=0 \quad , \quad \Phi(x) = -x \ .$$

From (II-10), (II-12) we find

$$(50) \qquad V(t) = 2\,\mathrm{sen}\, t \qquad , \qquad u(x,t) = \frac{x^3}{3} + 2x\,\mathrm{sen}\, t \ .$$

Hence

$$(51) \qquad |u(x,t) - h(x)| \leq 2x\,|\sin t| \leq 2x \ , \quad \forall t > 0 \quad , \quad \forall x > 0 \ .$$ □

Example 2.– We consider

$$(52) \qquad \mathcal{F}_{(t)}(V(\tau),\tau) = d\ V(t)\ , \ d > 0 \quad , \quad h(x) = \frac{x^3}{3} \ , \quad g=0 \quad , \quad \Phi(x) = -x \ .$$

In this case, we find

(53) $$V(t)=\frac{2}{d}(1-\exp(-dt)) \quad , \qquad u(x,t)=\frac{x^3}{3}+\frac{2}{d}x\,(1-\exp(-dt))\,.$$

Hence

(54) $$|u(x,t)-h(x)| \le \frac{2}{d}x\left(1-\exp(-dt)\right) \le \frac{2}{d}x \quad , \quad \forall t>0 \quad , \quad \forall x>0\,.$$

Moreover, we see that there is a monotonicity dependence of $u(x,t)$ upon the parameter $d>0$ in the sense

(55) $$0<d_2\le d_1 \quad \text{implies} \quad 0<u_{(d_1)}(x,t)\le u_{(d_2)}(x,t) \quad , \quad x>0 \quad , \quad t\ge 0\,. \qquad \square$$

IV. A GENERAL REMARK

A this point it is opportune to note that the procedure outlined in previous sections to study problem (P) also applies to the analysis of problem (P′).

In fact, problem (P′) can be reduced to problem (P), defining the function $u=u(x,t)$ by

(1) $$u(x,t)=\int_0^x v(z,t)\,dz+\int_0^t g_o(\tau)\,d\tau \;,\; x>0 \;,\; t>0\,.$$

LEMMA 9. – If v is a solution of problem (P′) then u, defined by (1), is a solution of problem (P) with the following relations among data

(2) $$\Phi(x)=\int_0^x \phi(z)\,dz \quad , \quad h(x)=\int_0^x h_o(z)\,dz \quad , \quad g(t)=\int_0^t g_o(\tau)\,d\tau\,.$$

Conversely, if u is a solution of problem (P) then v defined by

(3) $$v(x,t)=u_x(x,t)$$

is a solution of problem (P′) with the following relations among data

(4) $$\phi(x)=\Phi'(x) \;, \qquad h_o(x)=h'(x), \qquad g_o(t)=\dot g(t)-\Phi(0)\,\mathcal{F}_{(t)}(u_x(0,.),.) \;\square$$

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