Revista de la
Unión Matemática Argentina
Volumen 41, 1, 1998.

A priori estimates, Lagrangian coordinates and the regularity of nonlinear diffusion equations *

Juan Luis Vazquez †

dedicated to the memory of our friend Julio Bouillet

1 Introduction

The second half of the XXth century has seen enormous progress in the application of the techniques of functional analysis to investigate in a mathematically rigorous way the properties of nonlinear partial differential equations and systems which appear in the different branches of continuum physics. Among the variety of problems which received attention in this time from engineers, physicists, applied mathematicians and analysts are the problems of nonlinear diffusion and nonlinear heat propagation which can be expressed in mathematical terms as nonlinear parabolic equations. Some of these equations are degenerate parabolic since the diffusivity matrix, which depends on the solution u, vanishes for some values of u. The most typical example of such behaviour is maybe the *porous medium equation*

(1) $$u_t = \Delta u^m, \quad m > 1.$$

We can also consider a more general nonlinearity and then we get the so-called *filtration equation*

(2) $$u_t = \Delta\Phi(u),$$

where Φ is for instance a continuous increasing real function. In this generality it encompasses the famous *Stefan problem*, which as is well-known represents the simplest model of heat propagation involving two phases and the presence of a latent heat in the process of phase change.

In the spring of 1983 I met Julio in Minneapolis, a rather happy time when both of us were visiting the School of Mathematics of the Univ. of Minnesota, a major institution in the world of nonlinear PDEs, and problems like the above were a large part of our frequent conversations on mathematics, an interest that both of us kept and which in later years we shared with a number of friends. Julio was always enthusiastic (in his rather pessimistic way) about these problems and in my opinion he deserves quite a share

*Partially supported by HCM Project CHRX-CT94-0618.
†Depto. Matemáticas, Univ. Autónoma de Madrid, 28049 Madrid, Spain.

of the credit for the creation of an active interest in Buenos Aires around such problems in particular, and more generally in the mathematics of continuous media.

Before proceeding with the mathematical contribution I would like to add a personal comment on what I recall about the work on the filtration equations at that time and also on some of the people who influenced or were connected with us in that respect. Even though such equations were already posed by Boussinesq at the beginning of the century in the study of ground water infiltration, cf. [4], progress was slow for a long time. In the 50's the Moscow school contributed the first serious attack and basic results about existence and uniqueness of weak solutions as well as the property of finite speed of propagation, with the corresponding existence of a free boundary, were established after the work of such personalities as P. Polubarinova, Ya. Zeldovich, A. Kompaneets, G. Barenblatt, O. Oleinik and collaborators. A program of investigation was gradually organized around the problems of existence and uniqueness of suitable solutions for general data, regularity (even continuity was a serious problem) of such solutions, behaviour and regularity of the free boundary, asymptotic behaviour,... Such problems turned out to be far from trivial. It was only around 1970 that the West became involved in the study through the work of D. Aronson. A great activity ensued in the 70's and early 80's with beautiful analytical results by a number of distinguished people, among them two Argentinian researchers, professors in Minnesota at the time, L. Caffarelli and C. Kenig. The above-mentioned program was almost completed in the 80's, we were involved in it with a number of other colleagues, among them N. Wolanski, and I will let the recollection at this point, not without recalling that the preceding lines do not pretend to be a historical survey and many important names and facts are omitted. The interested reader is referred to works like [1], [14], [20] for more details on the authors involved and the most recent mathematics. Let me just say that Julio and I continued to keep a certain subject affinity and I will give an example: several years ago we touched the subject of large-time behaviour for solutions of the PME with changing sign, a subject that had been settled by A. Friedman and S. Kamin, [12], for nonnegative solutions. There was a paper by Kamin and myself [15] and then another one by Ph. Bénilan and J. Bouillet [5], where they improve considerably our results. On the other hand, I would like to mention that Julio was always interested in travelling waves, finite propagation, shocks and particularly in the work of J. M. Burgers. I was attracted to those subjects years later.

Let us now turn to the contents of the present paper. It deals with the regularity theory of the PME and other similar equations. The solutions of such equations have a limited regularity and one of the classical approaches to the qualitative theory proceeds through obtaining suitable a priori estimates. In the case of the PME Aronson and Bénilan [2] discovered a type of pointwise estimate with surprising properties: it is simple-looking, it is universal for the class of nonnegative solutions and it has far-reaching implications in domains as different as existence, asymptotic behaviour or free boundary regularity. Unfortunately, the usual derivations of such estimates seem rather ad hoc. This is why it is interesting to find a method to derive them from general and basic properties of the equations under consideration. A will propose below one such method whose main ingredients are the homogeneity of the equations, the transformation into Lagrangian coordinates and the Maximum Principle for the transformed equation. The method can be applied to a number of equations sharing a few basic properties, and some of them will be discussed below. Unfortunately, the argument *does not* work in more than one space dimension and the failure is not only technical: the simplest corresponding result that the same method would produce in several dimensions is false. Since the type of estimates holds in several dimensions, an open problem worth considering is to find similar or other methods to prove the n-dimensional results.

2 Pointwise a priori estimates

We start with a continuous and nonnegative function $u(x,t)$ defined in $Q = \mathbf{R} \times (0,\infty)$ which solves in distribution sense the PME equation

(3) $$u_t = (u^m)_{xx}$$

and takes initial data

(4) $$u(x,0) = u_0(x) \geq 0, \quad x \in \mathbf{R},$$

in some weak sense. Because of known density results we may assume for our purposes that u_0 is continuous and then the initial data are taken continuously. We may also assume that $u_0 \in L^1 \cap L^\infty$.

Our purpose at this stage is to review the a priori estimates on which the current regularity theory for the PME is based. One of the first results (1950) in the theory of the PME was the explicit construction of a class of solutions which came to be known as the source type solutions and also *Barenblatt* solutions and are given by the formula (valid in n dimensions)

(5) $$U(x,t) = t^{-k}\left(C - D\frac{x^2}{t^{2k/n}}\right)_+^{1/(m-1)},$$

where

$$k = (m-1+2/n)^{-1}, \quad D = k(m-1)/2mn, \quad C > 0 \text{ arbitrary.}$$

For $n = 1$ we have $k = 1/(m+1)$. These solutions turned out to be in many respects the nonlinear equivalent of the fundamental solutions of the linear heat equation and as such fundamental in the ensuing theory, particularly in the work of Caffarelli. But at the same time they exposed one of the main difficulties of such theory: these solutions are not smooth. In fact, for $m > 2$ they are not even C^1 in space or time! Thus, the theory had to call upon the concept of weak solutions. The problem is then to investigate which is the maximal regularity that general solutions satisfy.

In this respect it can be observed in (5) that the function U^{m-1} has a Lipschitz smoothness independent of the expoenent m. Indeed, the use of (1) to model the flow of an isentropic gas through a porous medium suggests considering the function

(6) $$\pi = \frac{m}{m-1}u^{m-1},$$

which represents the pressure of the flow and satisfies the equation

(7) $$\pi_t = (m-1)\pi\Delta\pi + |\nabla\pi|^2.$$

Then it is easily seen that the pressure Π of the source type solution (5) has an inverse parabolic shape as a function of x and more precisely it satisfies in the positivity set the estimate

$$\Delta\Pi = -\frac{k}{t}. \tag{8}$$

Since Π has Laplacian 0 on the null set $\{U = 0\}$ and since on the free boundary the Laplacian is (in the distribution sense) a positive measure, we finally obtain the inequality

$$\Delta\Pi \geq -\frac{k}{t} \qquad \text{in } \mathcal{D}'(Q). \tag{9}$$

Establishing that this one-sided estimate holds for *every* nonnegative weak solution of the PME was a turning point in the theory and the basis of many later developments. This was done by Aronson and Bénilan in a justly famous 3-page Note to the Comptes Rendus de l'Académie des Sciences de Paris in 1979, [2], which extends even to the range $0 < m < 1$ (called the fast diffusion equation).

THEOREM 2.1. *For every classical positive solution of the porous medium equation the pressure v satisfies*

$$\Delta\pi \geq -\frac{k}{t} \qquad \text{in } \mathcal{D}'(Q). \tag{10}$$

By density the result is true for all nonnegative weak solutions. Actually, the result holds for $m > (n-2)/n$.

Using (as an amusement) the sophisticated language borrowed from the theory of Hamilton-Jacobi equations, we may say that the weak nonnegative solutions of the PME belong to the class of *semi-subharmonic* functions. Unfortunately, general solutions need not be semi-convex, though the source solution is because of its radial symmetry. Of course, in 1D all solutions are semi-convex.

An important consequence of this fundamental estimate is the following useful estimate which says that u is at least *Lipschitz from below* with respect to the time variable.

COROLLARY 2.1. *For every solution as above we have*

$$u_t \geq -\frac{u}{kt}, \quad \pi_t \geq -\frac{(m-1)\pi}{kt}. \tag{11}$$

with $k = (m - 1 + (2/n))^{-1}$ as defined before.

The proof is based on the inequality $\pi_t \geq (m-1)\pi\Delta\pi$ that follows immediately from (7). (Let us mention for the record that in 1D π is actually Lipschitz, but the proof is not so immediate. The result is false in several dimensions). In order to grasp the immediate consequences of such estimates for the size and regularity of the solution the reader could prove without much effort the following result in 1D.

COROLLARY 2.2. *Let u be a solution of the PME with finite mass, i.e. $\int u(x,t)dx = M < \infty$. Then it is bounded and its pressure has bounded space gradient for $t \geq \tau > 0$. More precisely, we have the estimates*

$$u(x,t) \leq cM^{2/(m+1)}t^{-1/(m+1)} \tag{12}$$

$$|\pi_x| \leq cM^{\alpha}t^{-\beta}, \quad \alpha = \frac{m-1}{m+1}, \ \beta = \frac{m}{m+1}. \tag{13}$$

It is remarkable that these are precisely the decay rates of the source-type solution, hence they are optimal. Estimate (10) was also a key tool in the proof by Caffarelli-Friedman [8] of the C^1-regularity of the interface in 1D after the wating time. The consequences in more space dimensions are not as immediate but, in the hands of Bénilan-Crandall-Pierre [7] and Aronson-Caffarelli [3], estimate (10) allowed to produce a theory of existence and uniqueness of solutions with optimal initial data.

So the problem is posed: such a simple formula necessarily should have a derivation based on a few basic and general facts. We will present below one such general argument that works as we said in one space dimension. We will also show that it can be applied to other equations with similar power-like structure.

3 The homogeneity argument of Bénilan and Crandall

Our proof of the a priori estimates is based on the formulation of the equation in Lagrangian coordinates and the application of a remarkable argument found by Ph. Bénilan and M. Crandall [6] in 1981 that allows to derive a certain regularity for the solutions of evolution equations associated to a homogeneous operator, and precisely from the homogeneity plus the Maximum Principle. Morever, the argument is extremely easy. Let us review it here.

(i) We consider an evolution equation of the form

$$u_t = \mathcal{A}(u), \tag{14}$$

where $\mathcal{A}$ is an operator acting on u in a homogeneous way, i.e., there is a k such that for all $\lambda > 0$

$$\mathcal{A}(\lambda u) = \lambda^r A(u). \tag{15}$$

We will need to assume that $\mathcal{A}$ is nonlinear, $r \neq 1$. Typically $\mathcal{A}$ is a differential operator involving only space derivatives.

(ii) We assume the initial-value problem for equation (14) admits a unique solution for data $u(0) = u_0$ in a certain linear space X of real functions or a convex subspace thereof. Such solution can be classical or generalized. Let us denote by $u(t)$ the solution at time t. A semigroup S_t is then generated by the rule

$$S_t : u_0 \mapsto u(t). \tag{16}$$

We also assume that $S_t(0) = 0$ for every t.

(iii) We assume the Maximum Principle which is equivalent to say that the semigroup is ordered, i.e., $u_0 \geq v_0$ implies $S_t(u_0) \geq S_t(v_0)$.

(iv) Finally, the class of solutions must be invariant under the natural scaling in the sense that for every solution $u(t)$ (with initial data u_0) also the function

$$u_\lambda(t) = \lambda u(\lambda^\alpha t), \quad \alpha = r - 1 \tag{17}$$

is a solution (precisely the one with initial data λu_0). In other words,

$$S_t(\lambda u_0) = \lambda S_{\lambda^\alpha t}(u_0). \tag{18}$$

Assume now that $u_0 \geq 0$ and $\lambda > 1$. Then $\lambda u_0 \geq 0$, so that by the Maximum Principle $S_t(\lambda u_0) \geq S_t(u_0)$. This can be written as

$$\lambda(S_{\lambda^\alpha t}(u_0) - S_t(u_0)) + (\lambda - 1)S_t(u_0) \geq 0.$$

Let now $\lambda \neq 1$ and divide everything by $(\lambda - 1)t$. We get

$$\frac{u(\lambda^\alpha t) - u(t)}{(\lambda - 1)t} \geq -\frac{1}{t}u(t).$$

Let now $\lambda \to 1$ for fixed t. Since $\alpha = r - 1$ we get

THEOREM 3.1. *Under the above assumptions*

(19) $$(r - 1)u_t \geq -\frac{u}{t}$$

The last step of the proof and the formula offer a difficulty of interpretation when we work in an abstract setting. Below we will either take it as pointwise when we work with classical solutions, or in the distribution sense for generalized solutions of nonlinear diffusion equations.

We want to apply the above result to the Cauchy problem for the PME. When doing a direct application we observe that the conditions on the semigroup listed above are satisfied, therefore we get estimate (19) with $r = m$, $m \neq 1$. This has the same form as (11) but the constant is *not optimal* since for $m > 1$ k is always greater than $1/(m - 1)$ (the optimal constant plays a role in some of the consequences), while for $m < 1$ we find different sings in the inequalities. Moreover, there is no way of obtaining the stronger estimate (10), which is **the result we are heading for.**

4 Lagrangian coordinates

We need now to consider the method of Lagrangian coordinates. Briefly stated, it amounts to the following: we think of the PME as the equation of mass conservation for the density u of a certain continuous substance which moves in time. We thus write (1) in the form

(20) $$u_t + \nabla \cdot (uv) = 0,$$

and in this way we identify the *velocity* of the flow as

(21) $$v = -mu^{m-2}\nabla u = -\nabla \pi.$$

Given a solution with velocity v we may consider the movement with equations

(22) $$\frac{dx}{dt} = v(x(\eta, t), t), \quad x(\eta, 0) = \eta.$$

This allows to follow in time the particle with initial position η. In this way we may obtain a transformation of space-time

(23) $$(\eta, t) \mapsto (x; t).$$

It is a standard result (proved in all texts of continuum mechanics) that this transformation has Jacobian matrix u_0/u. In 1D it is convenient to introduce a new spatial Lagrangian coordinate by means of the formula

$$d\xi = u_0(\eta)d\eta. \tag{24}$$

This is called the *mass variable.* The inverse transformation allows us to pass from the original, so-called Euler variables (x,t) to the Lagrange variables (ξ,t). Let f be a function of (x,t) and thanks to (23) of (ξ,t). It is usual to denote by $D_t f$ the derivative of a function f with respect to t for fixed ξ while keeping the sign f_t for the Eulerian time derivative. The relation between both time derivatives is given by

$$D_t f = f_t + v \cdot \nabla f, \tag{25}$$

while in 1D

$$f_\xi = \frac{1}{u} f_x. \tag{26}$$

The change of space variables in several dimensions is more involved.

Lagrangian coordinates have been sucessfully applied to the PME and other nonlinear parabolic equations by several authors after the work of M. Gurtin, R. McCamy and E. Socolovski [13] and the Novosibirsk school, cf. [17]. It is a very useful tool in the study of free boundaries, since in the new coordinates the free boundaries become straight lines, ξ =const, [18]. Recently, S. Shmarev and the author have shown how to adapt the method to reaction-diffusion equations where the basic conservation equation (20) is violated, cf. [19]. This allows the authors to obtain the regularity of the interfaces.

5 The estimates for the PME

We now write the PME in Lagrangian coordinates. We work in 1D. We start from the equation

$$u_t = a(u^m)_{xx} = am(u^{m-1}u_x)_x. \tag{27}$$

The exponent m is larger than 1 in the PME but we can admit $m = 1$ (classical heat equation) and even less than 1 (fast diffusion). The latter case will be reviewed for convenience in the next section. The inessential parameter $a > 0$ will be useful later. We write (27) in the form

$$u_t + (uv)_x = 0, \quad v = -amu^{m-2}u_x = -\pi_x, \tag{28}$$

or, equivalently

$$u_t + vu_x = -uv_x = amu(u^{m-2}u_x)_x, \tag{29}$$

i.e.,

$$D_t u = -amu^2(u^{m-1}u_\xi)_\xi. \tag{30}$$

The existence of classical solutions of this equation for positive data comes from transforming the solutions of the PME. By virtue of the results of Section 3 we have

THEOREM 5.1. *For all nonnegative solutions of Cauchy problem for the PME in 1D we have*

$$D_t u \geq -\frac{u}{(m+1)t} \quad \text{in } \mathcal{D}'(Q). \tag{31}$$

Before giving the proof let us observe that

$$u_t = D_t u - v u_x = D_t u + a m u^{m-2} u_x^2 \geq D_t u, \tag{32}$$

so that (31) implies estimate (11) with best constant. On the other hand, we can write (29) as

$$D_t u = u \pi_{xx} \tag{33}$$

which shows that (31) is just **equivalent** to the fundamental estimate (10).

Proof. The second member is a nonlinear second-order differential operator, homogeneous of degree $r = m + 2$. We need to verify the assumptions listed in Section 3. It is well-known that in Eulerian coordinates the PME generates a semigroup of contractions in $L^1(\mathbf{R})$, that this semigroup admits the scaling law, that the Maximum Principle applies and that positive solutions are smooth and dense in the set of all nonnegative solutions. This allows us to perform the change to Lagrangian coordinates for smooth positive initial data and obtain thus the desired properties for a dense set of solutions of equation (30). Let us point out that the Maximum Principle **has to be checked** directly for equation (30), but it offers no difficulty.

6 Fast diffusion equations

Actually, nothing in the proof forces m to be larger than 1. We only need to assume that $m + 2 \neq 1$, i.e., $m \neq -1$. This opens the way for the application to the linear heat equation ($m = 1$), and to the *fast-diffusion* range $m < 1$. In the sub-range $0 < m < 1$ no novelty arises and Theorem 3.1 holds verbatim for the solutions of equation (3). When $-1 < m \leq 0$ this formula does not produce any more a parabolic equation. The way out is use the variant consisting in formally putting $am = 1$ so that

$$u_t = a(u^{m-1} u_x)_x, \quad a > 0. \tag{34}$$

In the case $m = 0$ this can further be written as $u_t = a(\log u)_{xx}$. Cf. the details of [9], [23].

The situation is different for $m < -1$. The application of Thorem (3.1) gives a *uniform upper estimate* for $t\, D_t u/u$ in the form

$$D_t u \leq -\frac{u}{(m+1)t}. \tag{35}$$

which holds for every positive solution of equation (34) with $m < -1$. In this case however, the estimate is worse than the one obtained from the application of Theorem 3.1 to the direct Euler formulation, since this one reads

$$u_t \leq \frac{u}{(1-m)t}. \tag{36}$$

and, as we have seen in (32) $D_t u \leq u_t$. Information about the theory of this type of fast diffusion problems can be found in [21] and its references.

7 The p-Laplacian equation

The same arguments can be applied *mutatis mutandis* to the other popular model of nonlinear diffusion, the so-called p-Laplacian equation (PLE)

$$u_t = \Delta_p u = \nabla\cdot(|\nabla u|^{p-2}\nabla u), p \neq 2, \tag{37}$$

which for $p = 2$ becomes the classical heat equation. Here ∇u denotes the spacial gradient of u. It is well-known that this equation generates a semigroup of contractions in all L^q spaces, $q \geq 1$, it is a scaling law with exponent $r = p - 1$ and the Maximum Principle applies.

To apply the above ideas we can write (37) in the form (20) with velocity

$$v = -\frac{1}{u}|\nabla u|^{p-2}\nabla u, \tag{38}$$

i.e., $v = -|\nabla\pi|^{p-2}\nabla\pi$, where

$$\pi = \frac{p-1}{p-2}u^{(p-2)/(p-1)} \tag{39}$$

is the so-called nonlinear potential, cf. [10], [22]. Let us continue from now on in 1D. The passage to Lagrangian coordinates produces the equation

$$D_t u = -uv_x = u\Delta_p\pi, \tag{40}$$

or in fully Lagrangian terms

$$D_t u = -u^2 v_\xi = u^2(u^{p-2}|u_\xi|^{p-2}u_\xi)_\xi. \tag{41}$$

We immediately get the conclusion

THEOREM 7.1. *For all nonnegative solutions of the PLE we have the estimate*

$$D_t u \geq -\frac{u}{2(p-1)t}, \qquad \Delta_p\pi \geq -\frac{1}{2(p-1)t}. \tag{42}$$

As a consequence,

$$u_t \geq -\frac{u}{2(p-1)t}. \tag{43}$$

These are again sharp estimates satisfied with equality sign by the source-type or Barenblatt solutions. They were originally proved in [10]. They admit a several-dimension version which was proved in [11] and which we cannot obtain by the present methods.

Again, the restriction to $p > 2$ plays no role in the argument and we can consider the fast-diffusion p-Laplacian version with $1 < p < 2$ with same results. The case $p < 1$ has never been studied. Once the appropriate qualitative results are verified, the application of the theory gives (as in the PME) a uniform upper estimate for $tD_t u/u$, instead of the uniform lower estimate obtained in (42).

8 The doubly nonlinear equation

We can combine both types of nonlinearities into the so-called *doubly nonlinear heat equation* (DNL)

(44) $$u_t = c(u^s|u_x|^{p-2}u_x)_x,$$

where $s \geq 0$, $p > 1$, $c > 0$. We can write it also as

(45) $$u_t = a(|(u^m)_x|^{p-2}(u^m)_x)_x$$

with $(m-1)(p-1) = s$, $am^{p-1} = c$. The velocity is now

(46) $$v = -cu^{s-1}|u_x|^{p-2}u_x,$$

and the potential

(47) $$\pi = \frac{d(p-1)}{s+p-2}u^{(s+p-2)/(p-1)}, \quad d = c^{1/(p-1)}.$$

We get the equations

(48) $$D_t u = -uv_x = u\Delta_p \pi,$$

or in fully Lagrangian terms:

(49) $$D_t u = -u^2 v_\xi = cu^2(u^{s+p-2}|u_\xi|^{p-2}u_\xi)_\xi.$$

Since the second-member operator has homogeneity $r = s + 2p - 1 = (m+1)(p-1) + 1$ we get

THEOREM 8.1. *For all nonnegative solutions of the DNL equation with $(m+1)(p-1) > 0$ we have the estimates*

(50) $$D_t u \geq -\frac{u}{(m+1)(p-1)t}, \qquad \Delta_p \pi \geq -\frac{1}{(m+1)(p-1)t}.$$

As a consenquence,

(51) $$u_t \geq -\frac{u}{(m+1)(p-1)t}.$$

This was precisely the result proved in [10]. The estimates are sharp. Corresponding inequalities with reversed sign hold for $(m+1)(p-1) < 0$. We can also consider cases where $s \leq 0$.

9 Convection models

Though homogeneity is a strong restriction it is not difficult to find examples of equations similar to the PME for which the method we have outlined works. We will explain two of them which will allow us to see a bit deeper into the technique.

9.1. We can consider the following diffusion-convection

(52) $$u_t = a(u^m)_{xx} + b(u^n)_x,$$

for some constants $a, b, m, n > 0$. Working as before we get a conservation law with velocity

(53)
$$v = -amu^{m-2}u_x - bu^{n-1}.$$

Now we do not know how to get a potential, but we discover that such a concept, though convenient, is not essential. Proceeding further we get the Lagrangian equation

(54)
$$D_t u = -uv_x = -u^2 v_\xi = u^2(amu^{m-1}u_\xi + bu^{n-1})_\xi.$$

In order to get a homogeneous nonlinear operator we need the conditions

(55)
$$m \neq 1, \quad n = m + 1.$$

In this case we arrive at the following estimate valid for general nonnegative solutions,

(56)
$$D_t u \geq -\frac{u}{(m+1)t},$$

which in terms of the velocity gives

(57)
$$v_x \leq \frac{1}{(m+1)t},$$

that is exactly equivalent to the π_{xx} estimate of the PME, and shows that the pressure is not necessary. Of course, (56) immediately implies the Eulerian u_t-estimate (11).

Let us remark that equation (52) admits for the $n = m + 1$ a source type solution of the selfsimilar form

(58)
$$u(x,t) = t^{-\alpha} f(y), \quad y = xt^{-\alpha},$$

with $\alpha = 1/(m+1)$. The profile $f \geq 0$ must then satisfy the equation

$$a(f^m)_{yy} + b(f^n)_y + \alpha(f + yf') = 0.$$

Integrating one gives

(59)
$$a(f^m)_y + bf^n + \alpha yf = 0.$$

Dividing now by f we get in the set $\{f > 0\}$ the equation

$$amf^{m-2}f_y + bf^{n-1} + \alpha y = 0,$$

which upon differentiation gives

(60)
$$(amf^{m-2}f_y + bf^{n-1})_y = -\frac{1}{m+1}.$$

This is equivalent to saying that estimates (57) and (56) are exact on the positivity set of such solution. Hence they are sharp.

9.2. It is interesting to revisit the purely convective case,

(61)
$$u_t = b(u^n)_x,$$

corresponding to putting $a = 0$ in equation (52). Let us take $b \neq 0$. A well-posed is obtained when we consider the class of entropy solutions which are usually known as *Kruzhkov solutions*. The litteral reproduction of the calculations just performed leads, under the restriction $n \neq 1$, to an estimate of the form

$$D_t u \geq -\frac{u}{nt}, \tag{62}$$

valid for all nonegative entropy solutions. In terms of the velocity $v = -bu^{n-1}$ gives $v_x \leq 1/(nt)$, i.e.,

$$b(u^{n-1})_x \geq -\frac{1}{nt}. \tag{63}$$

Observe that for $b < 0$ this is an upper estimate, while for $b > 0$ it is a lower estimate. This estimate is exactly satisfied by the explicit solutions of the problem which start at $t = 0$ from a Dirac delta and which as is well-known have the formula (we put for simplicity $b = 1$)

$$u(x,t)^{n-1} = \frac{x}{nt} \quad \text{for } -r(t) \leq x \leq 0, \tag{64}$$

being zero otherwise. The radius $r(t)$ is given by

$$r(t) = Ct^{\frac{1}{n}}. \tag{65}$$

and $C > 0$ is a free constant. These solutions with triangular shape are fundamental in the theory, for instance they represent the large-time asymptotics, cf. [LP]. The reader will easily check that the homogeneity argument in Eulerian coordinates leads to an estimate similar to (62), but with non-optimal constant.

10 A comment on several dimensions

The method of proof that we have proposed fails in several space dimensions because the coordinete transformation cannot be reduced to an equation. We then have to deal with a system where the Maximum Principle fails. On the other hand, the method as described above would automatically lead to an estimate of the form

$$\nabla \cdot v \leq \frac{1}{(r-1)t},$$

where r is the homogeneity degree of the second-member operator in Lagrangian formulation. Now, it is difficult to see how such a number can fit the actual optimal estimate that for instance for the PME gives

$$-\Delta\pi = \nabla \cdot v \leq \frac{C}{t}, \quad C = \frac{1}{m - 1 + (2/n)}, \tag{66}$$

which depends on n. Let us finally recall that the direct application of the homogeneity result of [6] to the original PME gives the valid estimate

$$u_t \geq -\frac{u}{(m-1)t}, \tag{67}$$

cf [6], which is neither optimal for the Cauchy problem nor implies in any way a $\Delta\pi$ estimate. On the other hand, it is to be remarked that such estimate is actually optimal for the Cauchy-Dirichlet problem with zero boundary data, as can be checked from the separable solutions

$$u(x,t) = t^{-\beta} f(x), \quad \beta = \frac{1}{m-1}. \tag{68}$$

Similar remarks apply to the PLE and the DNL.

11 Final comments

The rigorous proof of the above statements is easy in the case of the PME where approximation by smooth solutions can be done by merely smoothing and making positive the data. In the other cases there is a technical work which we postpone for a specific publication.

The author is grateful to Ph. Bénilan for his critical reading of the mathematical contents of this work.

References

[1] D. G. Aronson, *The porous medium equation* , in "Nonlinear Diffusion Problems", Lecture Notes in Math. **1224**, A. Fasano y M. Primicerio eds., Springer Verlag, New York, 1986, pp. 12-46.

[2] D. G. Aronson, Ph. Bénilan, *Régularité des solutions de l'équation des milieux poreux dans* $\mathbf{R}^n$, C.R. Acad. Sci. Paris. Ser. I, v. 228 (1979), pp. 103-105.

[3] D. G. Aronson, L. Caffarelli *The initial trace of a solution of the porous medium equation*, Trans. Amer. Math. Soc., **280** (1983), pp. 351-366.

[4] J. Bear, "Dynamics of Fluids in Porous Media", Dover, New York, 1972.

[5] Ph. Bénilan, J. Bouillet, *On a parabolic equation with slow and fast diffusions*, Nonlinear Anal. TMA, **26** (1996), pp. 813-822.

[6] Ph. Bénilan, M. G. Crandall, *Regularizing effects of homogeneous evolution equations*, in Contributions to Analysis and Geometry, suppl. to Amer. Jour. Math., Baltimore (1981), pp. 23-39.

[7] Ph. Bénilan, M. G. Crandall, M. Pierre, *Solutions of the PME in* $\mathbf{R}^{\mathbf{N}}$ *with optimal conditions on the initial data*, Indiana Univ. Math. Jour., **33** (1984) pp. 51-87.

[8] L. Caffarelli, A. Friedman, *Regularity of the free boundary for the one-dimensional flow o gas in a porous medium*, Amer. J. Math., **101** (1979) pp. 1193-1218.

[9] J. R. Esteban, A. Rodriguez, J. L. Vazquez, *A nonlinear heat equation with singualr diffusivity*, Comm. Partial Diff. Equations, **13** (1988), pp. 985-1039.

[10] J. R. Esteban, J. L. Vazquez, *Homogeneous diffusion in* $\mathbf{R}$ *with power-like diffusivity*, Archive Rat. Mech. Anal., **103** (1988), pp. 39-88.

[11] J. R. Esteban, J. L. Vazquez, *Régularité des solutions positives de l' équation parabolique p-laplacienne*, Comptes Rendus Acad. Sciences Paris I, **310** (1990), pp. 105-110.

[12] A. Friedman & S. Kamin, *The asymptotic behaviour of gas in an n-dimensional porous medium*, Trans. Amer. Math. Soc., **262** (1980), pp. 551-563.

[13] M. E. Gurtin, R. C. MacCamy, E. A. Socolovsky, *A coordinate transformation for the porous media equation that renders the free boundary stationary*, Quart. Appl. Math., **42** (1984), pp. 345-357.

[14] A. S. Kalashnikov, *Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations*, Uspekhi Matem. Nauk, **42** (1987), pp. 135-176 (in Russian); English translation: Russian Math. Surveys **42** (1987), 169-222.

[15] S. Kamin, J. L. Vazquez, *Asymptotic behaviour of the solutions of the porous medium equation with changing sign*, SIAM Jour. Math. Anal., **42** (1991), pp. 34-45.

[16] T. P. Liu, M. Pierre, *Source solutions and asymptotic behaviour in conservation laws*, Jour. Diff. Equations, **51** (1984), pp. 419-441.

[17] A. M. Meirmanov, V. V. Pukhnachov, *Lagrangian coordinates in the Stefan problem*, Dinamika sploshnoy sredy , bf 47 (1980), pp. 90-111, (in Russian).

[18] S. I. Shmarev, *On a degenerate parabolic equation in filtration theory: monotonicity and C^∞-regularity of interface*, Advances in Mathematical Sciences and Applications, to appear.

[19] S. Shmarev, J. L. Vazquez, *The regularity of solutions of reaction-diffusion equations via Lagrangian coordinates*, Nonlinear Differential Equations, to appear.

[20] J. L. Vazquez, *An introduction to the mathematical theory of the porous medium equation*, in "Shape Optimization and Free Boundaries", M. C. Delfour ed., Math. and Phys. Sciences, Series C, vol. **380**, Kluwer Acad. Publ., Dordrecht, Boston, Leiden; 1002. Pp. 347-389.

[21] J. L. Vazquez, *Nonexistence of solutions for nonlinear heat equations of fast-diffusion type*, Jour. Math. Pures Appl.,**71** (1992), pp. 503-526.

[22] J. L. Vazquez, *Two nonlinear diffusion equations with finite speed of propagation*, in Problems Involving Change of Phase, K. Kirchgässner ed., Springer Lecture Notes in Phys. 359, Springer Verlag, Berlin, 1990, pp. 197-206.

[23] J. L. Vazquez, J. R. Esteban, A. Rodriguez, *The fast diffusion equation with logarithmic nonlinearity*, Advances Diff. Equations, **1** (1996), pp. 21-50.