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SMOOTH PARAMETRIZATION OF SUBSPACES IN A BANACH SPACE

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ABSTRACT: Studying smooth families of certain subspaces of a Banach space X entails a construction of a Grassmann manifold defined over the similarity class of a projection in a Banach space. Standard principles of fiber bundle theory can be adapted to describe these families in terms of smooth maps from a possibly infinite dimensional paracompact manifold to the Grassmannian.

1 INTRODUCTION

Let X be a Banach space and let $\mathrm{Proj}(X)$ be the set of all bounded (nontrivial) projections in X. For a given $P_0 \in \mathrm{Proj}(X)$, take $\overline{P_0} = \mathrm{Sim}(P_0)$ to be the similarity class of P_0 in X. In this paper we outline an *explicit* construction of the Grassmannian $\mathrm{Gr}(\overline{P_0}, X)$

of images of projections in $\overline{P_0}$. It is shown to be a Banach analytic manifold modelled on the Banach space $\mathcal{L}(\mathrm{Im}\, P_0, \mathrm{Ker}\, P_0)$ (cf. [CPR 1]). It is worth stressing that such an explicit description of the manifold structure of $\mathrm{Gr}(\overline{P_0}, X)$ is needed in applications which necessitate computing over coordinate charts.

An outline of the paper is as follows. After stating several background results in §2, we proceed in §3 to give the explicit details of constructing $\mathrm{Gr}(\overline{P_0}, X)$ along with a core-sponding principal bundle $\mathcal{S}(P_0, X)$. Accordingly, our approach is quite different in spirit from the more abstract theory as pursued by other authors. In §4, we give an important application by showing that there exist decompositions of X *smoothly* parametrized by points in M, the latter taken to be a smooth contractible manifold which is a subset of some Hilbert space. It entails implementing an extension of the well-known smoothing lemma of Steenrod [St] to the case of fiber bundles with Banach spaces as fibers over M . Since to the best of our knowledge, the precise result does not appear to be known in the literature, it is necessary for us to establish this as one of our main results (Theorem 4.3). Consequently, the results of §4 can be used to generalize some known results and applications for matrix-valued functions over finite dimensional vector spaces (for the latter, see e.g. [FGP]). The authors express their gratitude to Professors G. Corach and H. Porta for informative discussions on this subject.

2 COMPLEMENTED SUBSPACES AND PROJECTIONS

Firstly, some remarks concerning a notation which is used frequently. If f is a mapping from a set A to a set B, and if C is a subset of A, then $f|_C$ denotes the ordinary restriction of f to C. If D is any set containing the image of f, then $f|^D$ denotes the mapping from A into D defined by $f|^D(x) = f(x)$ for all $x \in A$. Combining these two operations, if E is any set containing $f(C)$, then we denote by $f|_C^E$ the mapping from C into E defined by $f|_C^E(x) = f(x)$ for all $x \in C$.

Henceforth X and Y denote Banach spaces. Let $\mathcal{L}(X,Y)$ denote the Banach space of all bounded linear operators from X into Y with $\mathcal{L}(X) = \mathcal{L}(X,X)$. The set of isomorphisms $\mathrm{Isom}(X,Y)$ from X to Y is an open subset of $\mathcal{L}(X,Y)$; in particular $\mathrm{GL}(X)$ is open in $\mathcal{L}(X)$ (see e.g. [Ha, 4.4.5.8]). If X is the algebraic direct sum of nontrivial vector subspaces X_1 and X_2, then the pair (X_1, X_2) is an *algebraic complementary pair* of subspaces of X . Further $X = X_1 \oplus X_2$ is a *topological direct sum* if in addition X_1 and X_2 are closed. Consequently, X_1 and X_2 are complemented in X and (X_1, X_2) is said to be a *topological complementary pair.* Note that in X it is always possible to select a complement X_2 of each complemented subspace X_1 such that X_2 depends continuously on X_1 [PR 2]. If X is the algebraic sum of non-trivial vector subspaces X_1 and X_2, then the projection of X onto X_1 along X_2 is denoted by $P_{X_1}^{X_2}$ and is given by

$$P_{X_1}^{X_2}(x_1 + x_2) = x_1, \qquad \forall x_1 \in X_1, x_2 \in X_2 .$$

The following lemma is elementary.

Lemma 2.1 *Let (X_1, X_2) be a topological complementary pair of subspaces of X . Let Y_1, Y_2 be closed vector subspaces of Y. For $\Phi_1 \in \mathcal{L}(X_1, Y_1)$ and $\Phi_2 \in \mathcal{L}(X_2, Y_2)$ the direct sum*

$$\Phi_1 \oplus \Phi_2 = \Phi_1|^Y \circ P_{X_1}^{X_2} + \Phi_2|^Y \circ P_{X_2}^{X_1} ,$$

satisfies $\Phi_1 \oplus \Phi_2 \in \mathcal{L}(X,Y)$.

Let $\mathrm{Proj}(X) = \{P \in \mathcal{L}(X,X) : P^2 = P\}$ be the set of all bounded (non-trivial) projections in X. As a straightforward consequence of the definitions we obtain the following:

Proposition 2.2 *Let* (X_1, X_2) *be an algebraic complementary pair of vector subspaces of* X. *Then*

$$(P^{X_2}_{X_1})^2 = P^{X_2}_{X_1}, \ \mathrm{Im}\, P^{X_2}_{X_1} = X_1, \ \mathrm{Ker}\, P^{X_2}_{X_1} = X_2 \ .$$

Further, $P^{X_2}_{X_1}$ *is nontrivial,* $P^{X_1}_{X_2} = \mathrm{id}_X - P^{X_2}_{X_1}$, *and* $P^{X_2}_{X_1} \circ P^{X_1}_{X_2} = 0 = P^{X_1}_{X_2} \circ P^{X_2}_{X_1}$.

Conversely, if P *is a nontrivial projection in* X, *then* $(\mathrm{Im}\, P, \mathrm{Ker}\, P)$ *is an algebraic complementary pair of vector subspaces of* X, *and* $P = P^{\mathrm{Ker}\, P}_{\mathrm{Im}\, P}$. *If* (X_3, X_2) *is another algebraic complementary pair of vector subspaces of* X, *then*

$$P^{X_2}_{X_3} \circ P^{X_2}_{X_1} = P^{X_2}_{X_3} \ .$$

Theorem 2.3 (see e.g. [TL Theorems IV.12.1 and IV.12.2]) *Let* (X_1, X_2) *be an algebraic complementary pair of vector subspaces of* X. *Then* (X_1, X_2) *is a topological complementary pair of vector subspaces of* X *if and only if the linear operator* $P^{X_2}_{X_1}$ *is bounded.*

Corollary 2.4 *We have* $X = \mathrm{Im}\, P \oplus \mathrm{Ker}\, P$ *for all* $P \in \mathrm{Proj}(X)$.

Lemma 2.5 *Let* (L_0, L_0') *be a topological complementary pair of subspaces of* X *and* L *a vector subspace of* X . *Then* L *is a topological complementary subspace to* L_0' *if and only if* $P^{L_0'}_{L_0}\big|^{L_0}_{L} \in \mathrm{Isom}(L, L_0)$ *whereby we have* $\left(P^{L_0'}_{L_0}\big|^{L_0}_{L}\right)^{-1} = P^{L_0'}_{L}\big|^{L}_{L_0}$.

Proof. Let $P_0 = P^{L_0'}_{L_0}$. Then $P_0 \in \mathcal{L}(X)$ by Theorem 2.3. If L is a topological complementary subspace to L_0' then again from Theorem 2.3 we have $P^{L_0'}_{L} \in \mathcal{L}(X)$. Now for $x_0 \in L_0$, $x_L = P^{L_0'}_{L}(x_0)$ and $x_0' = P^{L}_{L_0'}(x_0)$, we have $x_0 = x_L + x_0'$ by Proposition 2.2. It follows that

$$P_0\big|^{L_0}_{L} \circ P^{L_0'}_{L}\Big|^{L}_{L_0}(x_0) = P_0\left(P^{L_0'}_{L}(x_0)\right) = P_0(x_L) = P_0(x_0 - x_0') = x_0 \ ,$$

and so $P_0\big|^{L_0}_{L} \circ P^{L_0'}_{L}\Big|^{L}_{L_0} = \mathrm{id}_{L_0}$. By interchanging the roles of L_0 and L, we obtain $P^{L_0'}_{L}\Big|^{L}_{L_0} \circ P_0\big|^{L_0}_{L} = \mathrm{id}_L$. Thus $P_0\big|^{L_0}_{L} \in \mathrm{Isom}(L, L_0)$ and $\left(P_0\big|^{L_0}_{L}\right)^{-1} = P^{L_0'}_{L}\Big|^{L}_{L_0}$.

Conversely, suppose that $P_0\big|^{L_0}_{L} \in \mathrm{Isom}(L, L_0)$. Let $S = \left(P_0\big|^{L_0}_{L}\right)^{-1}$ and take $P = S\big|^{X} \circ P_0$. Then it follows that $P \in \mathcal{L}(X)$ and

$$P^2 = S\big|^{X} \circ P_0 \circ S\big|^{X} \circ P_0 = S\big|^{X} \circ \left(P_0\big|^{L_0}_{L} \circ S\right) \circ P_0 = S\big|^{X} \circ \mathrm{id}_{L_0} \circ P_0 = S\big|^{X} \circ P_0 = P \ ,$$

so that $P \in \mathrm{Proj}(X)$. Clearly, $\mathrm{Im}\, P = S(P_0(X)) = S(L_0) = L$, and the relation $P = S\big|^{X} \circ P_0$ implies that $\mathrm{Ker}\, P_0 \subseteq \mathrm{Ker}\, P$. Furthermore,

$$P_0 P = P_0 \circ S\big|^{X} \circ P_0 = \mathrm{id}_{L_0}\big|^{X} \circ (P_0\big|^{L_0}_{L} \circ S) \circ P_0 = \mathrm{id}_{L_0}\big|^{X} \circ \mathrm{id}_{L_0} \circ P_0 = P_0 \ ,$$

which implies that $\operatorname{Ker} P \subseteq \operatorname{Ker} P_0$. Thus $\operatorname{Ker} P = \operatorname{Ker} P_0$, in other words we have $\operatorname{Ker} P = L_0'$. Therefore, by Proposition 2.2, we have $P_L^{L_0'} = P_{\operatorname{Im} P}^{\operatorname{Ker} P} = P \in \operatorname{Proj}(X)$ which implies by Corollary 2.4 that $X = L \oplus L_0'$. ■

We say that A, $B \in \mathcal{L}(X)$ are *similar* in $\mathcal{L}(X)$ if and only if there exists $S \in \operatorname{GL}(X)$ such that $A = SBS^{-1}$. Given $A \in \mathcal{L}(X)$, we denote by $\operatorname{Sim}(A)$ the set of all operators in $\mathcal{L}(X)$ that are similar to A .

Lemma 2.6

(a) *Let* P_1, $P_2 \in \mathcal{L}(X)$ *be similar. Then* $P_1 \in \operatorname{Proj}(X)$ *if and only if* $P_2 \in \operatorname{Proj}(X)$.

(b) *Let* P_1, $P_2 \in \operatorname{Proj}(X)$. *For every* $i \in \{1,2\}$, *let* $L_i = \operatorname{Im} P_i$ *and* $L_i' = \operatorname{Ker} P_i$. *Then* P_1 *and* P_2 *are similar if and only if the subspaces* L_1 *and* L_2 *are isomorphic and the subspaces* L_1' *and* L_2' *are isomorphic.*

Proof. (a) This follows immediately from the properties of P_1 and P_2 . For part (b), first suppose that P_1 and P_2 are similar. Then there exists $S \in \operatorname{GL}(X)$ such that $P_2 = SP_1S^{-1}$. Consequently,

$$L_2 = \operatorname{Im} P_2 = P_2(X) = SP_1S^{-1}(X) = SP_1(X) = S(\operatorname{Im} P_1) = S(L_1) ,$$

and since $S \in \operatorname{GL}(X)$, it follows that $S|_{L_1}^{L_2} \in \operatorname{Isom}(L_1, L_2)$. Furthermore,

$$P_2(S(L_1')) = P_2S(L_1') = SP_1(L_1') = S(P_1(\operatorname{Ker} P_1)) = \{0\} ,$$

which implies that $S(L_1') \subseteq \operatorname{Ker} P_2 = L_2'$. On the other hand,

$$P_1(S^{-1}(L_2')) = P_1S^{-1}(L_2') = S^{-1}P_2(L_2') = S^{-1}P_2(\operatorname{Ker} P_2) = \{0\} .$$

Consequently, $S^{-1}(L_2') \subseteq \operatorname{Ker} P_1 = L_1'$. Hence $L_2' \subseteq S(L_1') \subseteq L_2'$ and so $L_2' = S(L_1')$. Since $S \in \operatorname{GL}(X)$, it follows that $S|_{L_1'}^{L_2'} \in \operatorname{Isom}(L_1', L_2')$.

Conversely, assume that there exist $\Phi \in \operatorname{Isom}(L_1, L_2)$ and $\Phi' \in \operatorname{Isom}(L_1', L_2')$. Then by Theorem 2.3, we have $X = L_i \oplus L_i'$ for each $i \in \{1,2\}$. Let $S = \Phi \oplus \Phi'$. Then S is bijective with $S^{-1} = \Phi^{-1} \oplus \Phi'^{-1}$ and by Lemma 2.1, $S \in \operatorname{GL}(X)$. It follows from Proposition 2.2 that $P_i = P_{L_i}^{L_i'}$ for each $i \in \{1,2\}$. Let $Q_1 = P_{L_1'}^{L_1}$, we have then

$$\begin{aligned} P_2S &= P_2(\Phi \oplus \Phi') = P_2(\Phi P_1 + \Phi' Q_1) = P_2 \Phi P_1 = \Phi|^X \circ P_1 \\ &= \Phi|^X \circ P_1 + \Phi'|^X \circ Q_1 P_1 = (\Phi|^X \circ P_1 + \Phi'|^X \circ Q_1)P_1 = SP_1 , \end{aligned}$$

and since $S \in \operatorname{GL}(X)$, it follows that P_1 and P_2 are similar. ■

3 THE BANACH GRASSMANNIAN $\operatorname{Gr}(\overline{P}, X)$ AND ITS PRINCIPAL BUNDLE

Definition 3.1 Taking $P \in \operatorname{Proj}(X)$ and $\overline{P} = \operatorname{Sim}(P)$, we denote by $\operatorname{Gr}(\overline{P}, X)$ the *Grassmannian of images of projections in* $\overline{P}$ where

$$\operatorname{Gr}(\overline{P}, X) := \{\operatorname{Im} Q \mid Q \in \overline{P}\} .$$

In the following we take $L = \operatorname{Im} P$ and $L' = \operatorname{Ker} P$ and denote by U_P the set of all topological complementary subspaces to L' in X . Further, let

$$\mathcal{L}^*(P,X) = \{T \in \mathcal{L}(L,X) : \ \operatorname{Im} T \in \operatorname{Gr}(\overline{P},X), \ T\big|^{\operatorname{Im} T} \in \operatorname{Isom}(L, \operatorname{Im} T)\} .$$

As $\mathcal{L}^*(P,X) \subseteq \mathcal{L}(L,X)$, we grant $\mathcal{L}^*(P,X)$ the topology induced by the topology of $\mathcal{L}(L,X)$. Further, let π_P the mapping from $\mathcal{L}^*(P,X)$ into $\operatorname{Gr}(\overline{P},X)$ defined by $\pi_P(T) = \operatorname{Im} T$ for all $T \in \mathcal{L}^*(P,X)$.

Lemma 3.2 *Let $P \in \operatorname{Proj}(X)$ and $\overline{P} = \operatorname{Sim}(P)$. For P_1, $P_2 \in \overline{P}$ and for each $i \in \{1,2\}$, let $\mathcal{T}_i$ denote the final topology of $\operatorname{Gr}(\overline{P},X)$ associated to π_{P_i} . Then $\mathcal{T}_1 = \mathcal{T}_2$.*

Proof. Let $L_i = \operatorname{Im} P_i$ for each $i \in \{1,2\}$ and take $(i,j) \in \{(1,2),(2,1)\}$. Since P_i and P_j are similar, there exists an isomorphism $\Phi \in \operatorname{Isom}(L_i, L_j)$ given by Lemma 2.6 (b). Let $\widetilde{\Psi} : \mathcal{L}(L_j,X) \to \mathcal{L}(L_i,X)$ be defined by $\widetilde{\Psi}(T_j) = T_j \circ \Phi$ for all $T_j \in \mathcal{L}(L_j,X)$. Then clearly, $\widetilde{\Psi}(T_j) \in \mathcal{L}(L_i,X)$ for all $T_j \in \mathcal{L}(L_j,X)$, $\widetilde{\Psi}$ is linear and $\|\widetilde{\Psi}\| \le \|\Phi\| < \infty$. So it follows that $\widetilde{\Psi} \in \mathcal{L}(\mathcal{L}(L_j,X),\mathcal{L}(L_i,X))$.

Let $T_j \in \mathcal{L}^*(P_j,X)$ and $T_i = \widetilde{\Psi}(T_j)$. Then

$$\operatorname{Im} T_i = T_j(\Phi(L_i)) = T_j(L_j) = \operatorname{Im} T_j \in \operatorname{Gr}(\overline{P},X) ,$$

because $T_j \in \mathcal{L}^*(P_j,X)$. Therefore,

$$T_i\big|^{\operatorname{Im} T_i} = (T_j \circ \Phi)\big|^{\operatorname{Im} T_j} = T_j\big|^{\operatorname{Im} T_j} \circ \Phi \in \operatorname{Isom}(L_i, \operatorname{Im} T_j) = \operatorname{Isom}(L_i, \operatorname{Im} T_i) .$$

This implies that $\widetilde{\Psi}(T_j) = T_i \in \mathcal{L}^*(P_i,X)$ and so $\widetilde{\Psi}(\mathcal{L}^*(P_j,X)) \subseteq \mathcal{L}^*(P_i,X)$. In this way we obtain a well-defined map

$$\Psi = \widetilde{\Psi}\Big|_{\mathcal{L}^*(P_j,X)}^{\mathcal{L}^*(P_i,X)} ,$$

satisfying

$$\pi_i \circ \Psi(T_j) = T_j(\Phi(L_i)) = T_j(L_j) = \pi_j(T_j) , \qquad \forall T_j \in \mathcal{L}^*(P_j,X) ,$$

in other words $\pi_i \circ \Psi = \pi_j$. If $U \in \mathcal{T}_i$, then $\pi_i^{-1}(U)$ is open in $\mathcal{L}^*(P_i,X)$ and it is straightforward to see that $\mathcal{T}_i \subseteq \mathcal{T}_j$ for all $(i,j) \in \{(1,2),(2,1)\}$. Hence it follows that $\mathcal{T}_1 = \mathcal{T}_2$. ■

Let $P \in \operatorname{Proj}(X)$ and $\overline{P} = \operatorname{Sim}(P)$. We grant $\operatorname{Gr}(\overline{P},X)$ the final topology associated with the mapping π_Q for any $Q \in \overline{P}$. By Lemma 3.2, this topology does not depend on the choice of $Q \in \overline{P}$.

Lemma 3.3 *Let $P \in \operatorname{Proj}(X)$, $\overline{P} = \operatorname{Sim}(P)$ and $P_0 \in \overline{P}$. Then U_{P_0} is an open neighborhood of $\operatorname{Im} P_0$ in $\operatorname{Gr}(\overline{P},X)$.*

Proof. Let $L_0 = \operatorname{Im} P_0$ and $L_0' = \operatorname{Ker} P_0$. By Corollary 2.4, $L_0 \in U_{P_0}$. If $L \in U_{P_0}$, then L is isomorphic to L_0 by Lemma 2.5 and therefore $P_L^{L_0'} \in \overline{P_0} = \overline{P}$ by Lemma 2.6 (b). Since $L = \operatorname{Im} P_L^{L_0'}$ by Proposition 2.2, it follows that $L \in \operatorname{Gr}(\overline{P},X)$. Thus $U_{P_0} \subseteq \operatorname{Gr}(\overline{P},X)$. Consider the map $\widetilde{\Psi} : \mathcal{L}(L_0,X) \to \mathcal{L}(L_0)$ defined by

$$\widetilde{\Psi}(T) = P_0\big|^{L_0} \circ T , \qquad \forall T \in \mathcal{L}(L_0,X) .$$

Then it is clear that $\widetilde{\Psi}$ is linear, $\|\widetilde{\Psi}\| \leq \|P_0\| < \infty$ and $\widetilde{\Psi} \in \mathcal{L}(\mathcal{L}(L_0, X), \mathcal{L}(L_0))$

Setting $\Psi = \widetilde{\Psi}\big|_{\mathcal{L}^*(P_0,X)}$, we proceed to show that $\pi_{P_0}^{-1}(U_{P_0}) = \Psi^{-1}(\mathrm{GL}(L_0))$. First of all take $T \in \pi_{P_0}^{-1}(U_{P_0})$ and $L = \pi_{P_0}(T)$, following which $L \in U_{P_0}$. We have $P_0 = P_{L_0}^{L'_0}$ by Proposition 2.2. and hence $P_0\big|_L^{L_0} \in \mathrm{Isom}(L, L_0)$ by Lemma 2.5. Since we have $T \in \mathcal{L}^*(P_0, X)$, it follows that $T\big|^L \in \mathrm{Isom}(L_0, L)$ and

$$\Psi(T) = P_0\big|_L^{L_0} \circ T\big|^L \in \mathrm{GL}(L_0) .$$

In other words, $T \in \Psi^{-1}(\mathrm{GL}(L_0))$. Furthermore, $\pi_{P_0}^{-1}(U_{P_0}) \subseteq \Psi^{-1}(\mathrm{GL}(L_0))$.

Conversely, if $T \in \Psi^{-1}(\mathrm{GL}(L_0))$, then $\Psi(T) \in \mathrm{GL}(L_0)$. Let $L = \mathrm{Im}\, T$. By definition of Ψ, we have $T \in \mathcal{L}^*(P_0, X)$ and so $T\big|^L \in \mathrm{Isom}(L_0, L)$. Consequently, the relation $P_0\big|_L^{L_0} \circ T\big|^L = \Psi(T)$ implies that

$$P_0\big|_L^{L_0} = \Psi(T) \circ (T\big|^L)^{-1} \in \mathrm{Isom}(L, L_0) .$$

Therefore $L \in U_{P_0}$ by Lemma 2.5, that is, $\pi_{P_0}(T) \in U_{P_0}$ and hence $T \in \pi_{P_0}^{-1}(U_{P_0})$. This shows that $\pi_{P_0}^{-1}(U_{P_0}) = \Psi^{-1}(\mathrm{GL}(L_0))$. Since $\mathrm{GL}(L_0)$ is an open subset of $\mathcal{L}(L_0)$ and Ψ is continuous, then $\pi_{P_0}^{-1}(U_{P_0})$ is an open subset of $\mathcal{L}^*(P_0, X)$ and consequently U_{P_0} is an open subset of $\mathrm{Gr}(\overline{P}, X)$. ■

Let $P \in \mathrm{Proj}(X)$, $\overline{P} = \mathrm{Sim}(P)$ and $P_0 \in \overline{P}$. As $U_{P_0} \subseteq \mathrm{Gr}(\overline{P}, X)$ by Lemma 3.3, we can grant U_{P_0} the subspace topology induced by the topology of $\mathrm{Gr}(\overline{P}, X)$.

Lemma 3.4 *Let $P_0 \in \overline{P}$, $L'_0 = \mathrm{Ker}\, P_0$ and Φ denote the mapping from $U_{P_0} \subseteq \mathrm{Gr}(\overline{P}, X)$ into $\mathcal{L}(X)$ defined by $\Phi(L) = P_L^{L'_0}$ for $L \in U_{P_0}$. Then Φ is continuous.*

Proof. Theorem 2.3 guarantees that Φ takes its values in $\mathcal{L}(X)$. By Lemma 3.3, U_{P_0} is an open subset of $\mathrm{Gr}(\overline{P}, X)$. Let $L_0 = \mathrm{Im}\, P_0$ and $P_0 = P_{L_0}^{L'_0}$. By Lemma 3.3., U_{P_0} is an open subset of $\mathrm{Gr}(\overline{P_0}, X)$. If $V_0 = \pi_{P_0}^{-1}(U_{P_0})$, then by definition of the final topology V_0 is an open subset of $\mathcal{L}^*(P_0, X)$ and it will be sufficient to show that $(\Phi \circ \pi_{P_0})\big|_{V_0}$ is continuous. Let $T_1 \in V_0$, $L_1 = \pi_{P_0}(T_1)$ and $\widetilde{T}_1 = T_1\big|^{L_1}$. For $\varepsilon > 0$, set

$$\delta = \min\left\{ \frac{\varepsilon}{2\|P_{L_1}^{L'_0}\|\|P_{L'_0}^{L_1}\|\|\widetilde{T}_1^{-1}\|} , \frac{1}{2\|P_{L_1}^{L'_0}\|\|\widetilde{T}_1^{-1}\|} \right\} .$$

Let $T \in B_{\mathcal{L}^*(P_0,X)}(T_1, \delta)$, $L = \pi_{P_0}(T)$ and $\widetilde{T} = P_{L_1}^{L'_0}\Big|_L^{L_1} \circ T$. Since $L \in U_{P_0}$ it follows from Proposition 2.2 that we have $\widetilde{T} \in \mathrm{Isom}(L_0, L_1)$. Furthermore,

$$\|\widetilde{T} - \widetilde{T}_1\| \leq \|P_{L_1}^{L'_0}\|\|T - T_1\| < \|P_{L_1}^{L'_0}\|\delta \leq \frac{1}{2\|\widetilde{T}_1^{-1}\|} .$$

It follows by [Ha, proof of Theorem 3.1.4] that

$$\|\widetilde{T}^{-1} - \widetilde{T}_1^{-1}\| \leq 2\|\widetilde{T}_1^{-1}\|^2\|\widetilde{T} - \widetilde{T}_1\| \leq \|\widetilde{T}_1^{-1}\| ,$$

which in turn implies

$$\|\widetilde{T}^{-1}\| \leq \|\widetilde{T}^{-1} - \widetilde{T}_1^{-1}\| + \|\widetilde{T}_1^{-1}\| \leq 2\|\widetilde{T}_1^{-1}\| .$$

Let $x \in B_X(0,1)$ and $y = \widetilde{T}^{-1}(P_{L_1}^{L_0'}(x))$. Then

$$\|y\| \leq \|P_{L_1}^{L_0'}\|\|\widetilde{T}^{-1}\| \leq 2\|P_{L_1}^{L_0'}\|\|\widetilde{T}_1^{-1}\| .$$

By Proposition 2.2, we have

$$T(y) = (P_L^{L_0'} \circ P_{L_1}^{L_0'} \circ T)(y) = P_L^{L_0'}(\widetilde{T}(y)) = P_L^{L_0'}(P_{L_1}^{L_0'}(x)) = P_L^{L_0'}(x) .$$

Since $(P_L^{L_0'} - P_{L_1}^{L_0'})(x) \in L_0'$, it is straightforward to show that

$$(\Phi \circ \pi_{P_0}(T) - \Phi \circ \pi_{P_0}(T_1))(x) = P_{L_0'}^{L_1}((T - T_1)(y)) .$$

Consequently, for every $x \in B_X(0,1)$, we have

$$\begin{aligned}\|(\Phi \circ \pi_{P_0}(T) - \Phi \circ \pi_{P_0}(T_1))(x)\| &\leq \|P_{L_0'}^{L_1}\|\|T - T_1\|\|y\| \\ &\leq 2\|P_{L_1}^{L_0'}\|\|P_{L_0'}^{L_1}\|\|\widetilde{T}_1^{-1}\|\delta < \varepsilon .\end{aligned}$$

Therefore $\|\Phi \circ \pi_{P_0}(T) - \Phi \circ \pi_{P_0}(T_1)\| < \varepsilon$ for all $T \in B_{\mathcal{L}^\bullet(P_0,X)}(T_1,\delta)$. Thus $\Phi \circ \pi_{P_0}$ is continuous which implies that Φ is continuous. ∎

Theorem 3.5 *Let $P_0 \in \mathrm{Proj}(X)$. Then $\mathrm{Gr}(\mathrm{Sim}(P_0), X)$ is an analytic Banach manifold modelled on the Banach space $\mathcal{L}(\mathrm{Im}\, P_0, \mathrm{Ker}\, P_0)$.*

Proof. Let $L_0 = \mathrm{Im}\, P_0$, $L_0' = \mathrm{Ker}\, P_0$ and $\overline{P}_0 = \mathrm{Sim}(P_0)$. In order to establish the theorem we need an analytic $\mathcal{L}(L_0, L_0')$-atlas $((U_P)_{P\in\overline{P}_0}, (\varphi_P)_{P\in\overline{P}_0})$ for $\mathrm{Gr}(\overline{P}_0, X)$. For every $P_1 \in \overline{P}_0$, let

$$U_{P_1} = \{\mathrm{Im}\, P \ : \ P \in \overline{P}_0 \text{ and } \mathrm{Im}\, P \oplus \mathrm{Ker}\, P_1 = X\} .$$

By Lemma 3.3, for every $P \in \overline{P}_0$, U_P is an open neighborhood of $\mathrm{Im}\, P$, and so $(U_P)_{P\in\overline{P}_0}$ is an open covering of $\mathrm{Gr}(\overline{P}_0, X)$.

Let $P_1 \in \overline{P}_0$, $L_1 = \mathrm{Im}\, P_1$, $L_1' = \mathrm{Ker}\, P_1$ and $\widetilde{\varphi}_{P_1} : U_{P_1} \to \mathcal{L}(L_1, L_1')$, given by

$$\widetilde{\varphi}_{P_1} = (P_{L_1'}^{L_1} \circ P_L^{L_1'})\Big|_{L_1}^{L_1'} , \qquad \forall L \in U_{P_1} .$$

We shall show that $\widetilde{\varphi}_{P_1}$ is continuous. Let $L_2 \in U_{P_1}$ and $\varepsilon > 0$. Taking ρ to denote the mapping from U_{P_1} into $\mathcal{L}(X)$ defined by $\rho(L) = P_L^{L_1'}$, it follows from Lemma 3.4 that ρ is continuous. Hence we can find an open neighborhood N_{L_2} of L_2 in U_{P_1} such that

$$\|\rho(L) - \rho(L_2)\| < \frac{\varepsilon}{\|P_{L_1'}^{L_1}\|} , \qquad \forall L \in N_{L_2} .$$

Since $P_{L_1'}^{L_1}$ is linear, it follows that for every $L \in N_{L_2}$,

$$\|\widetilde{\varphi}_{P_1}(L) - \widetilde{\varphi}_{P_1}(L_2)\| = \|P_{L_1'}^{L_1}(P_L^{L_1'} - P_{L_2}^{L_1'})\| \leq \|P_{L_1'}^{L_1}\| \|\rho(L) - \rho(L_2)\| < \varepsilon ,$$

hence showing that $\tilde{\varphi}_{P_1}$ is continuous.

Now we proceed to construct the inverse $\widetilde{\psi}_{P_1}$ of $\widetilde{\varphi}_{P_1}$. First of all consider the map

$$\lambda_{P_1} : \mathcal{L}(L_1, L_1') \longrightarrow \mathcal{L}(L_1, X) ,$$

defined as follows. For every $\alpha \in \mathcal{L}(L_1, L_1')$, let $\lambda_{P_1}(\alpha) \in \mathcal{L}(L_1, X)$ be defined by

$$\lambda_{P_1}(\alpha)(x_1) = x_1 + \alpha(x_1) , \qquad \forall x_1 \in L_1 ,$$

otherwise expressed by

$$\lambda_{P_1}(\alpha) = \mathrm{id}_{L_1}|^X + \alpha|^X .$$

Since λ_{P_1} is an affine mapping from $\mathcal{L}(L_1, L_1')$ into $\mathcal{L}(L_1, X)$ it is analytic. Let $\alpha \in \mathcal{L}(L_1, L_1')$ and $\widetilde{\psi}_{P_1}(\alpha) = \mathrm{Im}\, \lambda_{P_1}(\alpha)$. We shall show that $\widetilde{\psi}_{P_1}(\alpha) \in U_{P_1}$. Take then $L_2 = \widetilde{\psi}_{P_1}(\alpha)$ and let $(y_n)_{n \in \mathbb{N}}$ be a sequence of points of L_2 converging to a point $y \in X$. By definition of L_2, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of points of L_1 such that $y_n = \lambda_{P_1}(\alpha)(x_n)$ for all $n \in \mathbb{N}$. If $x = P_1(y)$, then $x \in L_1$ and we have

$$x = P_1(y) = P_1\big(\lim_{n\to\infty} y_n\big) = \lim_{n\to\infty} P_1(y_n) = \lim_{n\to\infty} P_1\big(x_n + \alpha(n_n)\big) = \lim_{n\to\infty} x_n .$$

Hence

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \lambda_{P_1}(\alpha)(x_n) = \lambda_{P_1}(\alpha)\big(\lim_{n\to\infty} x_n\big) = \lambda_{P_1}(\alpha)(x) \in L_2 ,$$

which in turn shows that L_2 is closed. It is straightforward to show that $L_2 \cap L_1' = \{0\}$.

Take $x \in X$, $x_1 = P_1(x)$, $x_2 = \lambda_{P_1}(\alpha)(x_1)$ and $x_1' = x - x_2$. Then $x = x_2 + x_1'$, $x_2 \in L_2$ and by Proposition 4.2 we have

$$x_1' = x - x_2 = x - x_1 - \alpha(x_1) = (\mathrm{id}_X - P_{L_1}^{L_1'})(x) - \alpha(x_1) = P_{L_1'}^{L_1}(x) - \alpha(x_1) \in \mathrm{L}_1' .$$

It follows that $X = L_2 \oplus L_1'$. Therefore L_1 and L_2 are isomorphic by Lemma 2.5. Since $P_1 \in \overline{P}_0$, we know that $L_1 \cong L_0$ and $L_1' \cong L_0'$ by Lemma 2.6 (b). Also from this Lemma, $P_{L_1}^{L_1'} \in \overline{P_0}$ and hence $\widetilde{\psi}_{P_1}(\alpha) \in U_{P_1}$, thus showing that $\widetilde{\psi}_{P_1}$ is a mapping from $\mathcal{L}(L_1, L_1')$ into U_{P_1} . Since π_{P_1} is continuous and λ_{P_1} is analytic, we find that $\widetilde{\psi}_{P_1} = \pi_{P_1} \circ \lambda_{P_1} \in \mathcal{C}(\mathcal{L}(L_1, L_1'), U_{P_1})$.

Now we show that $\widetilde{\psi}_{P_1}$ is the inverse of $\widetilde{\varphi}_{P_1}$. For $L \in U_{P_1}$ we have

$$(P_{L_1}^{L_1'} \circ P_L^{L_1'})\Big|_{L_1}^{L_1} = \mathrm{id}_{L_1} ,$$

by Lemma 2.5. and so obtain

$$\begin{aligned}\lambda_{P_1}(\widetilde{\varphi}_{P_1}(L)) &= \mathrm{id}_{L_1}|^X + \widetilde{\varphi}_{P_1}(L)|^X = (P_{L_1}^{L_1'} \circ P_L^{L_1'})\Big|_{L_1} + (P_{L_1'}^{L_1} \circ P_L^{L_1'})\Big|_{L_1} \\ &= (P_{L_1}^{L_1'} + P_{L_1'}^{L_1}) \circ P_L^{L_1'}\Big|_{L_1} = P_L^{L_1'}\Big|_{L_1} .\end{aligned}$$

It follows that $\widetilde{\psi}_{P_1}(\widetilde{\varphi}_{P_1}(L)) = \operatorname{Im} \lambda_{P_1}(\widetilde{\varphi}_{P_1}(L)) = \operatorname{Im} P_L^{L_1'}\Big|_{L_1} = L$ and hence $\widetilde{\psi}_{P_1} \circ \widetilde{\varphi}_{P_1} = \mathrm{id}_{U_{P_1}}$.

Conversely, consider $\alpha \in \mathcal{L}(L_1, L_1')$. Let $L = \psi_{P_1}(\alpha)$ and $x_1 \in L_1$. Then by definition of $\lambda_{P_1}(\alpha)$, we have $x_1 = \lambda_{P_1}(\alpha)(x_1) - \alpha(x_1)$ with $\lambda_{P_1}(\alpha)(x_1) \in L$ and $-\alpha(x_1) \in L_1'$. Hence by definition of $P_L^{L_1'}$, we have $P_L^{L_1'}(x_1) = \lambda_{P_1}(\alpha)(x_1)$. Thus $P_L^{L_1'}\Big|_{L_1} = \lambda_{P_1}(\alpha)$. Using the definition of $\widetilde{\varphi}$, a routine claculation shows that $\widetilde{\varphi}_{P_1}(\widetilde{\psi}_{P_1}(\alpha)) = \alpha$ and so $\widetilde{\varphi}_{P_1} \circ \widetilde{\psi}_{P_1} = \mathrm{id}_{(L_1, L_1')}$. Hence we obtain $\widetilde{\psi}_{P_1} = \widetilde{\varphi}_{P_1}^{-1}$.

Since $P_1 \in \mathrm{Sim}(P_0)$ there exist by Lemma 2.6 (b) corresponding isomorphisms $\Phi \in \mathrm{Isom}(L_1, L_0)$ and $\Phi' \in \mathrm{Isom}(L_1', L_0')$. Let $S : \mathcal{L}(L_1, L_1') \to \mathcal{L}(L_0, L_0')$ be the map defined by

$$S(\alpha) = \Phi' \circ \alpha \circ \Phi^{-1} , \qquad \forall \alpha \in \mathcal{L}(L_1, L_1') .$$

Then S is invertible with inverse determined by

$$S^{-1}(\alpha) = \Phi'^{-1} \circ \alpha \circ \Phi, \qquad \forall \alpha \in \mathcal{L}(L_0, L_0') .$$

Clearly, S and S^{-1} are linear, $\|S\| \leq \|\Phi'\| \|\Phi^{-1}\|$ and $\|S^{-1}\| \leq \|\Phi'^{-1}\| \|\Phi\|$, so that $S \in \mathrm{Isom}(\mathcal{L}(L_1, L_1'), \mathcal{L}(L_0, L_0'))$. Let $\varphi_{P_1} = S \circ \widetilde{\varphi}_{P_1}$. Since $\widetilde{\varphi}_{P_1}$ is an homeomorphism from U_{P_1} onto $\mathcal{L}(L_1, L_1')$, it follows that φ_{P_1} is an homeomorphism from U_{P_1} onto $\mathcal{L}(L_0, L_0')$.

Let P_1, $P_2 \in \overline{P_0}$ be such that $U_{P_1} \cap U_{P_2} \neq \emptyset$. Then by Lemma 2.6 (a) we have P_1, $P_2 \in \mathrm{Proj}(X)$. Our next step will be to show that the change of chart

$$(\varphi_{P_2} \circ \varphi_{P_1}^{-1})\Big|_{\varphi_{P_1}(U_{P_1} \cap U_{P_2})} ,$$

is analytic. For each $i \in \{1,2\}$, let $L_i = \operatorname{Im} P_i$, $L_i' = \operatorname{Ker} P_i$ and $L \in U_{P_1} \cap U_{P_2}$. Since $L \oplus L_i' = X$ and $L_i \oplus L_i' = X$ for each $i \in \{1,2\}$, we deduce from Lemma 2.5 that $P_L^{L_1'}\Big|_{L_1}^{L} \in \mathrm{Isom}(L_1, L)$ and $P_2|_L^{L_2} = P_{L_2}^{L_2'}\Big|_L^{L_2} \in \mathrm{Isom}(L, L_2)$. Let $\alpha = \varphi_{P_1}(L)$. Just as before, we have $P_L^{L_1'}\Big|_{L_1} = \lambda_{P_1}(\alpha)$, since $L = \varphi_{P_1}^{-1}(\alpha)$. A routine calculation shows :

$$(\varphi_{P_2} \circ \varphi_{P_1}^{-1})(\alpha) = \varphi_{P_2}(L) = \left(P_{L_2}^{L_2}\Big|^{L_2'} \circ \lambda_{P_1}(\alpha)\right) \circ \left(P_2|^{L_2} \circ \lambda_{P_1}(\alpha)\right)^{-1} .$$

Since λ_{P_1} is analytic, the above change of chart $\varphi_{P_2} \circ \varphi_{P_1}^{-1}$ is also analytic. ■

Consider now:

$$\mathcal{S}(P_0, X) := \{T \in \mathcal{L}^*(P_0, X) : T(X) = Q(X),\ Q \in \overline{P_0} ,\ T$$

Let $\pi = \pi_P|_{\mathcal{S}(P_0,X)} : \mathcal{S}(P_0, X) \longrightarrow \mathrm{Gr}(\overline{P_0}, X)$ be given by $\pi(T) = \operatorname{Im} T$. The following two results follow directly from the properties of a typical orbit of similarity in $\mathrm{Proj}(X)$ as outlined in [CPR 1], [CPR 2] and [Ma] :

Proposition 3.6 *The fiber space $\{\mathcal{S}(P_0, X), \pi, \mathrm{Gr}(\overline{P_0}, X), \mathrm{GL}(X_0)\}$ defines a smooth locally trivial principal $\mathrm{GL}(X_0)$-bundle.*

Let $L_0 \in \mathrm{Gr}(\overline{P_0}, X)$. If we take $T \in \mathcal{S}(P_0, X)$ and $A \in \mathrm{GL}(X)$, then we observe that $\pi(AT)$ does not depend on the choice of A . Define the map

$$\tilde{\pi} : \mathrm{GL}(X) \longrightarrow \mathrm{Gr}(\overline{P_0}, X) ,$$

by $\tilde{\pi}(A) = AL_0$. Consider now the subgroup $H \subset \mathrm{GL}(X_0)$ defined by

$$H = \tilde{\pi}^{-1}(L_0) = \{h \in \mathrm{GL}(X) : \ hL_0 = L_0\} .$$

Proposition 3.7 *The homogeneous fibration*

$$\tilde{\pi} : \ \mathrm{GL}(X) \longrightarrow \mathrm{GL}(X)/H \cong \mathrm{Gr}(\overline{P_0}, X) ,$$

is a smooth locally trivial principal H-bundle.

A more extensive account of infinite dimensional Banach homogeneous spaces such as $\mathrm{Gr}(\overline{P_0}, X) \cong \mathrm{GL}(X_0)/H$ that are modelled on Banach algebras, can be found in e.g. [PR 1], [Ra], [Ma], [MR] and [Wi] (see also references therein).

4 APPLICATION TO SMOOTH FAMILIES OF SUBSPACES

We commence with the following well–known result between smooth maps and principal bundles :

Lemma 4.1 (see e.g. [Hu]) *Let $f \in C^\infty(M, \ \mathrm{Gr}(\overline{P}_0, X))$. Then with respect to the diagram*

$$\begin{array}{ccc} f^*(\mathcal{S}(P_0, X)) & \xrightarrow{\ \tilde{f}\ } & \mathcal{S}(P_0, X) \\ {\scriptstyle f^*(\pi)}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\ f\ } & \mathrm{Gr}(\overline{P}_0, X) \end{array}$$

the bundle $f^*(\pi) : f^*(\mathcal{S}(P_0, X)) \to M$, is a smooth principal $\mathrm{GL}(X_0)$–bundle and $\tilde{f}$ is an equivariant bundle morphism.

Theorem 4.2 *Let M be a smooth manifold and let $f : M \to \mathrm{Gr}(\overline{P}_0, X)$ be a smooth family of subspaces parametrized by M. Then we have the following equivalent statements:*

(1) $f \in C^\infty(M, \mathrm{Gr}(\overline{P}_0, X))$.

(2) *For every $x_0 \in M$ there exists an open neighborhood V_{x_0} of x_0 together with a family of maps $\zeta_i \in C^\infty(V_{x_0}, X)$ such that $\{\zeta_i(x)\}_{i \in I}$ is a smooth subspace of $f(x)$,for all $x \in V_{x_0}$.*

(3) *For every $x_0 \in M$, there exists an open neighborhood V_{x_0} of x_0 in M together with a map $\tilde{f}_{x_0} \in C^\infty(V_{x_0}, \mathrm{GL}(X))$ such that*

$$f(x) = \tilde{f}_{x_0}(x) \cdot f(x_0) ,$$

for all $x \in V_{x_0}$.

Proof. Firstly we recall that if $L_0 \in \mathrm{Gr}(\overline{P}_0, X)$ and L_0' is a complementary subspace to L_0 in X, and U_0 is the set of all complementary subspaces of L_0', then we have seen that U_0 is an open neighborhood of L_0 . To establish the equivalence of (1) and (2), we first of all suppose that (1) holds. Let $V_{x_0} = f^{-1}(U_0)$ and $x \in V_0$. We take the $\{\zeta_i(x_0)\}_{i \in I}$ as a smooth subspace $f(x_0)$. Then for $L = f(x_0)$ we define $\zeta_i(x) = P_L^{L'_0}(\zeta_i(x_0))$. Conversely, let $A(x) \in \mathcal{S}(P_0, X)$ be the projection to the image of $\{\zeta_i(x)\}$. Then we may define an element $f \in C^\infty(M,\ \mathrm{Gr}(\overline{P}_0, X))$ such that $f = \pi \circ A$ on V_{x_0} .

To prove the equivalence of (1) and (3), let $f \in C^\infty(M, \mathrm{Gr}(\overline{P}_0, X)$. Let U_0 be such that for $x_0 \in M$, $f(x_0) \in U_0$ and let $V_{x_0} = f^{-1}(U_0)$. Then with regards to the principal H-bundle $\tilde{\pi}$: $\mathrm{GL}(X) \to \mathrm{Gr}(\overline{P}_0, X)$, we take a local smooth section $\tilde{s}_0$: $U_0 \to \mathrm{GL}(X_0)$ and for each $x \in V_{x_0}$, set $\bar{f}(x) = \tilde{s}_0(f(x))$. We define $\tilde{f}_{x_0} \in C^\infty(V_{x_0}, \mathrm{GL}(X))$ by

$$\tilde{f}_{x_0}(x) = \bar{f}(x) \cdot [\bar{f}(x_0)]^{-1} ,$$

such that $f(x) = \tilde{f}_{x_0}(x) \cdot f(x_0)$. Let $L_0 = f(x_0)$. If (3) holds, namely

$$f(x) = \tilde{f}_{x_0}(x) \cdot L_0 ,$$

then the pointwise evaluation of the image of f is smooth and from this we can conclude $f \in C^\infty(M, \mathrm{Gr}(\overline{P}_0, X))$. ■

In order to obtain the global version of Theorem 4.2, we will need the following *smoothing approximbtion* :

Theorem 4.3 *Let M be a paracompact smooth (Hausdorff) manifold which is taken to be a subset of a Hilbert space. Let $(\mathcal{P}, \pi, M)$ be a smooth locally trivial fiber bundle with fiber F a Banach manifold. Assume that there exists a global continuous section $s_0 : M \to \mathcal{P}$. Then there exists a global smooth section $s : M \to \mathcal{P}$.*

Proof. This entails modifying the smooth approximation theorem of [St 6.7] (see also [Ho V.4.1]) where the local compactness property is to be replaced by the paracompactness of M .

Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of M where for each $\alpha \in I$, we have a smooth local trvialization $U_\alpha \times F \to U_\alpha$ that we elect to call property S. As M is paracompact and Hausdorff we obtain a σ-discrete refinement $\{V_\alpha\}_{\alpha \in J}$ of this cover and as property S is hereditary on open sets, then each member of this new open cover has this same property. By forming the union of each of the discrete subfamilies of which there are only countably many, we obtain a countable open cover of M in which each member has property S. Since this cover admits a locally finite refinement $\{A_\alpha\}_{\alpha \in K}$ it follows from [D, VIII] that it must have a precise open locally finite cover $\{B_\alpha\}_{\alpha \in L}$ of M (i.e. $K = L$ and $B_\alpha \subset A_\alpha$ for each α). In turn we obtain a countable locally finite open cover of M in which each member has property S.

Now the same σ-discrete argument allows us to assume that we have a corresponding cover of the continuous section $s_0 : M \to \mathcal{P}$ by open sets with typical fiber diffeomorphic to the Banach space on which the fiber F is modelled. The approximating arguments of [St] then can be applied verbatim by replacing the (ϵ,δ)-inequalities by open sets. The compactness property required in [St] enables the Stone–Weierstrass theorem to be implemented locally so as to achieve the smooth approximation. Here it is replaced by the existence of local smooth partitions of unity that is ensured by taking M to be a subset of a Hilbert space (see [La II,3]). ■

Theorem 4.3 is in the spirit of the neighborhood (NEP) and section (SEP) extension property of sections for more general classes of fibrations as considered in [Du 1] and [Du 2].

Theorem 4.4 *Let M be as in Theorem 4.3 and let M be contractible. Take a smooth map $f \in C^\infty(M, \mathrm{Gr}(\overline{P}_0, X))$. Then*

(1) *There exists a $Q \in \mathrm{Proj}(X)$ such that $f(x) = \operatorname{Im} Q$ and $X = \operatorname{Im} Q \oplus \ker Q$ is a decomposition of X depending smoothly on x, for all $x \in M$.*

(2) *There exists a map $\hat{f} \in C^\infty(M, \mathrm{GL}(X))$ and a subspace $L_0 \in \mathrm{Gr}(\overline{P}_0, X)$ such that $f(x) = \hat{f}(x)L_0$, for all $x \in M$. In particular, we have*

$$f(x) = \hat{f}(x) \cdot \hat{f}(x_0)^{-1} \cdot f(x_0) \ ,$$

for all $x, x_0 \in M$.

Proof. From Lemma 4.1, the smooth $\mathrm{GL}(X_0)$-bundle $f^*(\mathcal{S}(P_0, X)) \to M$ admits a global continuous section by the contractibility assumption on M . In turn, by Theorem 4.3, there exists a global smooth section $s : M \to f^*(\mathcal{S}(P_0, X))$. From the definition of $\mathcal{S}(P_0, X)$, for all $x \in M$, $(\tilde{f} \circ s)(x)$ defines an injective linear map T which corresponds to a projection $Q \in \mathrm{Proj}(X)$ such that $T(X) = Q(X)$. Further, Q is continuous as a bounded linear map and has closed image in X given by

$$(\pi \circ \tilde{f} \circ s)(x) = f(x) \ .$$

Setting $f(x) = \operatorname{Im} Q$, part (1) follows.

For (2) we apply essentially the same principle as we did in proving (1) in view of the diagram

$$\begin{array}{ccc} f^*(\mathrm{GL}(X)) & \xrightarrow{\tilde{f}'} & \mathrm{GL}(X) \\ {\scriptstyle s'}\uparrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ M & \xrightarrow{f} & \mathrm{Gr}(\overline{P}_0, X) \end{array}$$

where s' is a global smooth section. We take $\hat{f} = \tilde{f}' \circ s'$ and $L_0 = \pi(T)$ where $T \in \mathcal{S}(P_0, X)$, so that by the definition of $\tilde{\pi}$, we obtain the global version of Theorem 4.2 (3):

$$f(x) = (\tilde{\pi} \circ \hat{f})(x) = \hat{f}(x_0) \cdot L_0 \ .$$

■

The finite dimensional versions of Theorems 4.2 and 4.4 are [FGP, IV-1-2] and [FGP, IV-2-3] respectively.

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