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NATURALLY REDUCTIVE HOMOGENEOUS STRUCTURES ON 2-STEP NILPOTENT LIE GROUPS

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ABSTRACT. We give an alternative proof of the description of the naturally reductive 2-step nilpotent Lie groups endowed with a left-invariant metric, as an application of the characterization of naturally reductive spaces by homogeneous structures of class $\mathcal{T}_3$. We also prove that if $(N, \langle , \rangle)$ has no euclidean factor, then there exists at most one naturally reductive homogeneous structure (or of class $\mathcal{T}_3$) on $(N, \langle , \rangle)$.

1. INTRODUCTION

In [AS], Ambrose and Singer gave a characterization of the homogeneous Riemannian manifolds by a local condition which is to be satisfied at all points. They proved that a connected, simply connected and complete Riemannian manifold (M, g) is homogeneous if and only if there exists a tensor field T of type (1,2) such that $\tilde{\nabla} g = \tilde{\nabla} R = \tilde{\nabla} T = 0$, where $\tilde{\nabla} = \nabla - T$ and ∇, R denote the Levi-Civita conection and the curvature tensor of (M, g) respectively. Such T is called a *homogeneous structure*.

Afterwards, F. Tricerri and L. Vanhecke characterized the naturally reductive homogeneous spaces as above adding the condition $T_x x = 0$ for all vector field x on M (see [TV1]). In this case T is called a *naturally reductive homogeneous structure* or a *homogeneous structure of class* $\mathcal{T}_3$.

In [K2], A. Kaplan showed that the only naturally reductive H-type groups are the Heisenberg group and its quaternionic analogue. F. Tricerri and L. Vanhecke give in [TV2], as an application of the characterization of naturally reductive spaces by homogeneous structures stated in [TV1], an alternative proof of Kaplan's result. In [Go] C. Gordon studies naturally reductive metrics on homogeneous manifolds. It is proved that if the manifold admits a transitive nilpotent group of isometries then the group is at most 2-step nilpotent. Moreover a necessary and sufficient condition for a 2-step homogeneous nilmanifold is given to be naturally reductive.

In this work we start giving in §3 an alternative proof of the description of naturally reductive 2-step nilpotent Lie groups given in [Go], as another application of the theory of homogeneous structures. We also study in §4 the set of all homogeneous structures of class $\mathcal{T}_3$ in any nilpotent Lie group endowed with a left-invariant metric $(N, \langle , \rangle)$. We shall prove that if $(N, \langle , \rangle)$ has no euclidean factor then there is at most one homogeneous structure of class $\mathcal{T}_3$ on $(N, \langle , \rangle)$.

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2. GEOMETRIC PRELIMINARIES

We consider simply connected real nilpotent Lie groups N endowed with a left-invariant metric, denoted by $(N, \langle , \rangle)$, where $\langle , \rangle$ is the inner product on the Lie algebra $\mathfrak{n}$ of N determined by the metric.

The full group of isometries of a nilpotent Lie group $(N, \langle , \rangle)$ is given by

$$\mathrm{I}(N, \langle , \rangle) = K \ltimes N \qquad \text{(semidirect product)}, \tag{1}$$

where $K = \mathrm{Aut}(\mathfrak{n}) \cap O(\mathfrak{n}, \langle , \rangle)$ is the isotropy subgroup and N acts by left translations (see [W]). Thus the structure of $\mathrm{I}(N, \langle , \rangle)$ is completely determined by K. Note that, since N is simply connected, we make no distinction between automorphisms of N and $\mathfrak{n}$.

Let N be a 2-step nilpotent Lie group and let $\langle , \rangle$ be an inner product on $\mathfrak{n}$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and set $\mathfrak{v} = \mathfrak{z}^{\perp}$. For each $a \in \mathfrak{z}$ we define $J_a : \mathfrak{v} \longrightarrow \mathfrak{v}$ by

$$\langle J_a x, y\rangle = \langle a, [x, y]\rangle, \qquad x, y \in \mathfrak{v}. \tag{2}$$

Note that J_a is skew-symmetric for all $a \in \mathfrak{z}$ and $J : \mathfrak{z} \longrightarrow \mathrm{End}(\mathfrak{v})$ is linear. The maps $\{J_a\}_{a\in\mathfrak{z}}$ give the relationship between the Lie bracket of $\mathfrak{n}$ and the metric $\langle , \rangle$ and thus carry a lot of geometric information about the Riemannian manifold $(N, \langle , \rangle)$ (see for example [K1], [E1], [E2]). It is easy to prove that the isotropy group is given by

$$K = \{(T, \phi) \in O(\mathfrak{v}) \times O(\mathfrak{z}) : T J_a T^{-1} = J_{\phi a}, \quad a \in \mathfrak{z}\}. \tag{3}$$

Let $\mathfrak{k}$ be the Lie algebra of K. Thus $\mathfrak{k} = \mathrm{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n}, \langle , \rangle)$ and

$$\mathfrak{k} = \{(A, B) \in \mathfrak{so}(\mathfrak{v}) \times \mathfrak{so}(\mathfrak{z}) : AJ_a - J_a A = J_{Ba}, \quad a \in \mathfrak{z}\}. \tag{4}$$

If $[\mathfrak{n}, \mathfrak{n}] \neq \mathfrak{z}$ then $N \simeq N_1 \times R^k$, where $N_1 = \exp(\mathfrak{v} \oplus [\mathfrak{n}, \mathfrak{n}])$ and $R^k = \exp([\mathfrak{n}, \mathfrak{n}]^{\perp} \cap \mathfrak{z})$ ($\exp : \mathfrak{n} \longrightarrow N$ is the usual Lie exponential map), thus $(N, \langle , \rangle)$ is isometric to $(N_1, \langle , \rangle|_{\mathfrak{n}_1 \times \mathfrak{n}_1}) \times R^k$. In this case, we will say that $(N, \langle , \rangle)$ *has euclidean factor*. It is easy to see that $(N, \langle , \rangle)$ has Euclidean factor if and only if there exists a nonzero $a \in \mathfrak{z}$ such that $J_a = 0$.

The Levi-Civita conection has been computed in [E1]. We have

$$\begin{cases} \nabla_x y = \frac{1}{2}[x, y] \\ \nabla_a x = \nabla_x a = -\frac{1}{2} J_a x \\ \nabla_a b = 0, \end{cases}$$

where $x, y \in \mathfrak{v}$ and $a, b \in \mathfrak{z}$ are regarded as left-invariant vector fields on N. For arbitrary $x, y \in \mathfrak{n}$ we recall that the Ricci tensor of $(N, \langle , \rangle)$ is defined by $\rho(x, y) = \mathrm{tr}(z \rightarrow R(z, x)y,\ z \in \mathfrak{n})$, where R denotes the curvature tensor defined by $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$. For the Ricci tensor we obtain (see [E1])

(i) $\rho(x, a) = 0 \quad \forall\, x \in \mathfrak{v}, a \in \mathfrak{z}$.

(ii) If $\{a_1, ..., a_m\}$ is an orthonormal basis of $(\mathfrak{z}, \langle , \rangle)$ then

$$\rho(x, y) = \langle (\frac{1}{2} \sum_{i=1}^{m} J_{a_i}^2) x, y\rangle \quad \forall\, x, y \in \mathfrak{v}.$$

(iii) $\rho(a, b) = -\frac{1}{4}\mathrm{tr}(J_a J_b) \quad \forall\, a, b \in \mathfrak{z}$.

In particular ρ is negative definite on $\mathfrak{v} \times \mathfrak{v}$ and ρ is positive semidefinite on $\mathfrak{z} \times \mathfrak{z}$. If $(N, \langle , \rangle)$ has no euclidean factor then ρ is positive definite on $\mathfrak{z} \times \mathfrak{z}$.

3. Naturally reductive 2-step nilpotent Lie groups

Let M be a connected homogeneous manifold. Further let G be a Lie group acting transitively and effectively on the left on M as a group of isometries and denote by K the isotropy subgroup at some point $p \in M$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Suppose $\mathfrak{m}$ is a vector space complement of $\mathfrak{k}$ in $\mathfrak{g}$ such that $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$ (i.e. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a reductive decomposition). Thus we may identify $\mathfrak{m}$ with T_pM via the map $X \longrightarrow \frac{d}{dt}|_0 \exp tX.p$ and we denote by $\langle , \rangle$ the inner product on $\mathfrak{m}$ induced by the Riemannian metric of M.

Definition 3.1. The manifold M is said to be *naturally reductive* if there exists a Lie group G and a subspace $\mathfrak{m}$ with the properties described above and such that

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle = 0 \qquad \forall\, X, Y, Z \in \mathfrak{m}, \tag{5}$$

where $[X, Y]_{\mathfrak{m}}$ denotes the projection of $[X, Y]$ on $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

An important observation is that a Riemannian homogeneous space $M = G/K$ might be naturally reductive although for none reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g}$ the condition (5) holds. It is clear that if we want to find out whether M is naturally reductive or not we first have to determine all transitive isometry groups G of M and then consider all the $\mathrm{Ad}(K)$-invariant complements of $\mathfrak{k}$ in $\mathfrak{g}$.

Because of this ambiguity the following result has been proved by F. Tricerri and L. Vanhecke in [TV2] (see also [TV1]).

Theorem 3.2. [TV1] *Let (M, g) be a connected, simply connected and complete Riemannian manifold. Then (M, g) is a naturally reductive homogeneous space if and only if there exists a tensor field T of type (1,2) such that*

$$(AS)\begin{cases} \text{(i) } g(T_x y, z) + g(y, T_x z) = 0 \\ \text{(ii) } (\nabla_x R)(y, z) = [T_x, R(y, z)] - R(T_x y, z) - R(y, T_x z) \\ \text{(iii) } (\nabla_x T)_y = [T_x, T_y] - T_{T_x y} \end{cases}$$

$$\text{(iv) } T_x x = 0$$

for all $x, y, z \in \chi(M)$, where ∇ denotes the Levi-Civita conection of (M, g) and R is the Riemann curvature tensor defined by $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$.

Note that if $\tilde{\nabla} := \nabla - T$ then the conditions (AS) can be writen as follows

$$(AS)\begin{cases} \text{(i) } \tilde{\nabla} g = 0 \\ \text{(ii) } \tilde{\nabla} R = 0 \\ \text{(iii) } \tilde{\nabla} T = 0. \end{cases}$$

The conditions (AS) are the Ambrose-Singer conditions and the existence of a tensor T satisfying these conditions is equivalent to the homogeneity of the manifold (see [AS]). Note that (AS) is a generalization of Cartan's condition $\nabla R = 0$ for symmetric spaces, in this case the tensor $T \equiv 0$ satisfy (AS).

Definition 3.3. A tensor field T of type (1,2) is said to be a *homogeneous structure* if it satisfies (AS) and is said to be a *naturally reductive homogeneous structure* if T satisfies (AS) and (iv) of Theorem 3.2.

As an application of Theorem 3.2, there is in [TV2] an alternative proof of the classification of naturally reductive H-type groups (see [K2]). Nilpotent Lie groups endowed with left-invariant metrics $(N, \langle,\rangle)$ which are naturally reductive have been studied by C. Gordon. It is proved in [Go] that any naturally reductive nilpotent Lie group $(N, \langle,\rangle)$ is at most 2-step nilpotent. Moreover a characterization of the naturally reductive 2-step nilpotent Lie groups is given (see Theorem 3.4). We next give an alternative proof of this theorem applying naturally reductive homogeneous structures.

Theorem 3.4. [Go] *Let $(N, \langle,\rangle)$ be a 2-step nilpotent Lie group without Euclidean factor. Then $(N, \langle,\rangle)$ is naturally reductive if and only if*

(i) $J_{\mathfrak{z}} = \{J_a\}_{a\in\mathfrak{z}}$ *is a Lie subalgebra of* $\mathfrak{so}(\mathfrak{v}, \langle,\rangle)$.

(ii) $\tau_a \in \mathfrak{so}(\mathfrak{z}, \langle,\rangle)$ *for any* $a \in \mathfrak{z}$, *where* τ_a *is given by* $J_aJ_b - J_bJ_a := J_{\tau_a b}$ *for all* $a, b \in \mathfrak{z}$.

We note that (ii) is equivalent to $(J_a, \tau_a) \in \mathfrak{k}$, i.e. is a skew symmetric derivation of $\mathfrak{n}$ (see (4)).

Alternative proof of Theorem 3.4. Let $(N, \langle,\rangle)$ be a 2-step nilpotent Lie group without euclidean factor. Let ρ denote the Ricci tensor of $(N, \langle,\rangle)$. If $x, y \in \mathfrak{v}$ and $a, b \in \mathfrak{z}$ then (see Section 2)

$$\begin{cases} \nabla_x y = \frac{1}{2}[x,y] \\ \nabla_a x = \nabla_x a = -\frac{1}{2}J_a x \\ \nabla_a b = 0 \end{cases} \qquad \begin{cases} \rho(x,y) = \langle S_1 x, y\rangle \\ \rho(a,x) = 0 \\ \rho(a,b) = \langle S_2 a, b\rangle \end{cases}$$

where S_1, S_2 are negative and positive definite symmetric transformations on $\mathfrak{v}$ and $\mathfrak{z}$ respectively.

Suppose first that $(N, \langle,\rangle)$ is naturally reductive. By Theorem 3.2 there exists a tensor field T of type (1,2) satisfying (AS) and $T_x x = 0$ for any vector field x.

By (AS),(ii) we have that $\nabla R = TR$, it is clear then that $\nabla\rho = T\rho$. If $x \in \mathfrak{v}$ and $a, b \in \mathfrak{z}$ with $S_2 b = \lambda b$ we obtain

$$\begin{aligned} (\nabla_a \rho)(x,b) &= (T_a\rho)(x,b) \\ \rho(\nabla_a x, b) + \rho(x, \nabla_a b) &= \rho(T_a x, b) + \rho(x, T_a b) \\ 0 &= \lambda\langle T_a x, b\rangle + \langle S_1 x, T_a b\rangle \\ 0 &= -\lambda\langle x, T_a b\rangle + \langle S_1 x, T_a b\rangle \\ 0 &= \langle (S_1 - \lambda I)x, T_a b\rangle. \end{aligned}$$

Since S_2 is positive definite and S_1 is negative definite we have that $(S_1 - \lambda I)$ is non-singular, thus $T_a b \in \mathfrak{z}$ for any $a \in \mathfrak{z}$ and any eigenvector $b \in \mathfrak{z}$ of S_2. We choose a basis of $\mathfrak{z}$ of eigenvectors of S_2, and by linearity we obtain

$$(6) \qquad T_a b \in \mathfrak{z} \qquad \forall\, a, b \in \mathfrak{z}$$

It follows from (AS),(i) that $T_a \in \mathfrak{so}(\mathfrak{n}, \langle,\rangle)$, thus

$$(7) \qquad T_a x \in \mathfrak{v} \qquad \forall\, a \in \mathfrak{z},\ x \in \mathfrak{v}.$$

Note that (6) and (7) hold for any homogeneous structure on $(N, \langle,\rangle)$, since we have not used (iv) of Theorem 3.2 yet.

Now, by another application of $\nabla\rho = T\rho$ and using (iv) of Theorem 3.2, for $x, y \in \mathfrak{v}$ and $a \in \mathfrak{z}$ with $S_2 a = \lambda a$ we have

$$\begin{aligned}
(\nabla_x\rho)(y,a) &= (T_x\rho)(y,a)\\
\rho(\nabla_x y, a) + \rho(y, \nabla_x a) &= \rho(T_x y, a) + \rho(y, T_x a)\\
\tfrac{1}{2}\langle [x,y], S_2 a\rangle - \tfrac{1}{2}\langle S_1 y, J_a x\rangle &= \langle T_x y, S_2 a\rangle + \langle S_1 y, T_x a\rangle\\
\tfrac{1}{2}\langle J_{S_2 a} x, y\rangle - \tfrac{1}{2}\langle S_1 J_a x, y\rangle &= -\langle T_x S_2 a, y\rangle + \langle S_1 T_x a, y\rangle\\
\langle \tfrac{1}{2}\lambda J_a x - \tfrac{1}{2} S_1 J_a x, y\rangle &= \langle \lambda T_a x - S_1 T_a x, y\rangle,
\end{aligned}$$

hence

$$\frac{1}{2}(\lambda I - S_1) J_a x = (\lambda I - S_1) T_a x \qquad \forall\, x \in \mathfrak{v},\ a \in \mathfrak{z} \text{ with } S_2 a = \lambda a.$$

Since $(\lambda I - S_1)$ is non-singular, we obtain as before (6) that

$$(8) \qquad T_a x = \frac{1}{2} J_a x \qquad \forall\, a \in \mathfrak{z},\ x \in \mathfrak{v}.$$

It follows from (AS),(iii) that for all $x, y \in \mathfrak{v}$, $a \in \mathfrak{z}$, we have

$$\begin{aligned}
(\nabla_a T)_x y &= (T_a T)_x y\\
\nabla_a T_x y - T_{\nabla_a x} y - T_x \nabla_a y &= T_a T_x y - T_{T_a x} y - T_x T_a y.
\end{aligned}$$

Now, using (6),(7),(8) and (iv) of Theorem 3.2, we obtain for all $b \in \mathfrak{z}$ that

$$\begin{aligned}
\tfrac{1}{2}\langle T_{J_a x} y, b\rangle + \tfrac{1}{2}\langle T_x J_a y, b\rangle &= \langle T_a T_x y, b\rangle - \tfrac{1}{2}\langle T_{J_a x} y, b\rangle - \tfrac{1}{2}\langle T_x J_a y, b\rangle\\
\tfrac{1}{2}\langle y, T_b J_a x\rangle + \tfrac{1}{2}\langle J_a y, T_b x\rangle &= \langle y, T_x T_a b\rangle - \tfrac{1}{2}\langle y, T_b J_a x\rangle - \tfrac{1}{2}\langle J_a y, T_b x\rangle\\
\tfrac{1}{4}\langle J_b J_a x, y\rangle - \tfrac{1}{4}\langle J_a J_b x, y\rangle &= -\tfrac{1}{2}\langle J_{T_a b} x, y\rangle - \tfrac{1}{4}\langle J_b J_a x, y\rangle + \tfrac{1}{4}\langle J_a J_b x, y\rangle.
\end{aligned}$$

This implies that

$$\frac{1}{4} J_b J_a - \frac{1}{4} J_a J_b = -\frac{1}{2} J_{T_a b} - \frac{1}{4} J_b J_a + \frac{1}{4} J_a J_b \qquad \forall\, a, b \in \mathfrak{z},$$

therefore

$$(9) \qquad J_a J_b - J_b J_a = J_{T_a b} \qquad \forall\, a, b \in \mathfrak{z}.$$

So we have that $J_\mathfrak{z}$ is a Lie subalgebra of $\mathfrak{so}(\mathfrak{v}, \langle , \rangle)$ and since $\tau_a b = T_a b$, it follows from (AS),(i) that $\tau_a \in \mathfrak{so}(\mathfrak{z}, \langle , \rangle)$.

Conversely, suppose that (i) and (ii) of the theorem hold. It is not hard to check that the tensor field of type (1,2) defined on left-invariant fields by

$$
\text{(10)} \qquad \begin{cases} T_x y = \frac{1}{2}[x, y] \\ T_a x = -T_x a = \frac{1}{2} J_a x \\ T_a b = \tau_a b \end{cases} \qquad \forall\, x, y \in \mathfrak{v},\ a, b \in \mathfrak{z}
$$

is a naturally reductive homogeneous structure. Thus $(N, \langle , \rangle)$ is naturally reductive by Theorem 3.2. $\square$

We show in the following theorem that in naturally reductive 2-step nilpotent Lie groups we have no ambiguity problem with respect to the different transitive groups and decompositions as observed after Definition 3.1.

Theorem 3.5. *Let $(N, \langle , \rangle)$ be a 2-step nilpotent Lie group without euclidean factor. Then $(N, \langle , \rangle)$ is naturally reductive if and only if it is so with respect to the full isometry group $G = \mathrm{I}(N, \langle , \rangle)$ and the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where*

$$\mathfrak{m} = \{x + a + D_a : x \in \mathfrak{v}, a \in \mathfrak{z}\}$$

and D_a is the element in $\mathfrak{k}$ given by $D_a|_\mathfrak{v} = J_a$, $D_a|_\mathfrak{z} = \tau_a$.

Proof. Suppose that $(N, \langle , \rangle)$ is naturally reductive. Using that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$ (see (1)), it is easy to see that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a direct sum of vector spaces. We note that if $D \in \mathfrak{k}$ then $[D, x] = Dx$ for all $x \in \mathfrak{n}$, and thus $\mathrm{Ad}(\varphi)x = \varphi x$ for all $x \in \mathfrak{n}$, $\varphi \in K$.

We first prove that $\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant. If $\psi \in K$ and $x + a + D_a \in \mathfrak{m}$ then

$$\mathrm{Ad}(\psi)(x + a + D_a) = \psi x + \psi a + \psi D_a \psi^{-1}.$$

Since $\psi J_a \psi^{-1} = J_{\psi a}$ (see (3)) we also have that $\psi \tau_a \psi^{-1} = \tau_{\psi a}$, thus $\psi D_a \psi^{-1} = D_{\psi a}$ and hence $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$.

We now prove (5) for the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Let $\langle , \rangle_e$ denote the inner product on $\mathfrak{m}$ determined by the Riemannian metric of $(N, \langle , \rangle)$. It is easy to see that $\langle x + D, y + D' \rangle_e = \langle x, y \rangle$ for all $x, y \in \mathfrak{n}$, $D, D' \in \mathfrak{k}$. Thus, for any $x, y, z \in \mathfrak{v}$, $a, b, c \in \mathfrak{z}$, we have

$$
\begin{aligned}
&\langle [x + a + D_a, y + b + D_b]_\mathfrak{m}, z + c + D_c \rangle_e = \\
&= \langle ([x, y] - J_b x - \tau_b a + J_a y + \tau_a b + [D_a, D_b])_\mathfrak{m}, z + c + D_c \rangle_e \\
&= \langle [x, y] - J_b x + J_a y + 2\tau_a b, z + c \rangle \\
&= \langle J_c x, y \rangle - \langle J_b x, z \rangle + \langle J_a y, z \rangle + 2\langle \tau_a b, c \rangle \\
&= \langle y, J_c x \rangle - \langle y, J_a z \rangle - \langle J_b x, z \rangle - 2\langle b, \tau_a c \rangle \\
&= -\langle y + b, [x, z] - J_c x + J_a z + 2\tau_a c \rangle \\
&= -\langle y + b + D_b, ([x, z] - J_c x + J_a z + 2\tau_a c + [D_a, D_c])_\mathfrak{m} \rangle_e \\
&= -\langle y + b + D_b, [x + a + D_a, z + c + D_c] \rangle_e.
\end{aligned}
$$

This proves (5), concluding the proof, since the converse is obvious. □

4. Homogeneous structures of class $\mathcal{T}_3$ on 2-step nilpotent Lie groups

The naturally reductive homogeneous structures (see Definition 3.3) are also called *homogeneous structures of class* $\mathcal{T}_3$ (see [TV1],[P],[ChG1],[ChG2] for further information about the classes of homogeneous structures).

F. Tricerri and L. Vanhecke determined in [TV1] all the homogeneous structures on the 3-dimensional Heisenberg group, obtaining that they are parametrized by $\{T(\mu) : \mu \in R\}$. Further, $T(\mu)$ is of class $\mathcal{T}_3$ if and only if $\mu = \frac{1}{2}$.

In [ChG2] the homogeneous structures on the $(2p+1)$-dimensional Heisenberg group H_p are characterized. They gave a large class of explicit examples

$$\{T(r, s, t_1, ..., t_p) : r, s, t_i \in R\}.$$

Analogously to the case $p = 1$ one has that $T(r, s, t_1, ..., t_p)$ is of class $\mathcal{T}_3$ if and only if $r = s = 0$ and $t_1 = ... = t_p = \frac{1}{2}$.

The homogeneous structures on the generalized Heisenberg group $H(1, r)$ are also characterized (see [ChG1]), and this group does not admit any homogeneous structure of class $\mathcal{T}_3$.

In this section we study the set of all homogeneous structures of class $\mathcal{T}_3$ in any nilpotent Lie group endowed with a left-invariant metric $(N, \langle , \rangle)$. We shall prove that if $(N, \langle , \rangle)$ has no euclidean factor then there is at most one homogeneous structure of class $\mathcal{T}_3$.

It follows from Theorem 3.2 that $(N, \langle , \rangle)$ is naturally reductive if and only if there is on $(N, \langle , \rangle)$ a homogeneous structure of class $\mathcal{T}_3$. Thus, if a nilpotent Lie group $(N, \langle , \rangle)$ has a homogeneous structure of class $\mathcal{T}_3$ then N is at most 2-step nilpotent.

Theorem 4.1. *Let $(N, \langle , \rangle)$ be a naturally reductive 2-step nilpotent Lie group without euclidean factor. The tensor field T of type* (1,2) *defined in* (10) *is the unique homogeneous structure of class $\mathcal{T}_3$ on $(N, \langle , \rangle)$.*

Proof. Let T' be a homogeneous structure of class $\mathcal{T}_3$ on $(N, \langle , \rangle)$. In the proof of Theorem 3.4 (see (8) and (9)) we have obtained

$$\begin{cases} T'_a x = \frac{1}{2} J_a x \\ & \forall\, x \in \mathfrak{v},\ a, b \in \mathfrak{z} \\ T'_a b = \tau_a b. \end{cases}$$

Using that $\nabla \rho = T' \rho$, for $x, y \in \mathfrak{v}$, $a \in \mathfrak{z}$ with $S_2 a = \lambda a$, we have

$$\begin{aligned}
(\nabla_x \rho)(y, a) &= (T'_x \rho)(y, a) \\
\rho(\nabla_x y, a) + \rho(y, \nabla_x a) &= \rho(T'_x y, a) + \rho(y, T'_x a) \\
\tfrac{1}{2}\langle [x, y], S_2 a\rangle - \tfrac{1}{2}\langle S_1 y, J_a x\rangle &= \langle T'_x y, S_2 a\rangle + \langle S_1 y, T'_x a\rangle \\
\lambda\langle \tfrac{1}{2}[x, y], a\rangle - \tfrac{1}{2}\langle S_1 y, J_a x\rangle &= \lambda\langle T'_x y, a\rangle - \tfrac{1}{2}\langle S_1 y, J_a x\rangle \\
\lambda\langle \tfrac{1}{2}[x, y], a\rangle &= \lambda\langle T'_x y, a\rangle \\
\lambda\langle \tfrac{1}{2}[x, y], a\rangle &= \lambda\langle T'_x y, a\rangle,
\end{aligned}$$

hence

(11) $$(T'_x y)_\mathfrak{z} = \frac{1}{2}[x,y] \qquad \forall\, x, y \in \mathfrak{v}.$$

We then obtain that

$$\begin{cases} T'_x y = T_x y + T^1_x y \\ T'_a x = T_a x \\ T'_a b = T_a b, \end{cases} \qquad \forall\, x, y \in \mathfrak{v},\ \forall a, b \in \mathfrak{z}$$

where $T^1 : \mathfrak{v} \times \mathfrak{v} \longrightarrow \mathfrak{v}$ is a skew-symmetric bilinear form. Thus, we should prove that $T^1 \equiv 0$.

We can see T^1 as a field tensor of type (1,2) putting $T^1_a x = T^1_x a = T^1_a b = 0$ for all $a, b \in \mathfrak{z}$, $x \in \mathfrak{v}$. So we have $T' = T + T^1$. It follows from $\nabla T' = T'T'$ (see (AS),(iii)) that

$$\nabla T + \nabla T^1 = TT + TT^1 + T^1 T + T^1 T^1.$$

Since $\nabla T = TT$, for all $x, y \in \mathfrak{v}$, $a \in \mathfrak{z}$, we have

$$\begin{aligned} (\nabla_x T^1)_y a &= (T_x T^1)_y a + (T^1_x T)_y a + (T^1_x T^1)_y a \\ -T^1_y \nabla_x a &= -T^1_y T_x a + T^1_x T_y a - T_{T^1_x y} a \\ \tfrac{1}{2} T^1_y J_a x &= \tfrac{1}{2} T^1_y J_a x - \tfrac{1}{2} T^1_x J_a y + \tfrac{1}{2} J_a T^1_x y, \end{aligned}$$

and this implies that

(12) $$T^1_x J_a = J_a T^1_x \qquad \forall\, x \in \mathfrak{v},\ a \in \mathfrak{z}.$$

Now, for any $x, y, z \in \mathfrak{v}$ we obtain

$$\begin{aligned} (\nabla_x T^1)_y z &= (T_x T^1)_y z + (T^1_x T)_y z + (T^1_x T^1)_y z \\ \nabla_x T^1_y z &= T_x T^1_y z - T_{T^1_x y} z - T_y T^1_x z + T^1_x T^1_y z - T^1_{T^1_x y} z - T^1_y T^1_x z \\ \tfrac{1}{2}[x, T^1_y z] &= \tfrac{1}{2}[x, T^1_y z] - \tfrac{1}{2}[T^1_x y, z] - \tfrac{1}{2}[y, T^1_x z] + T^1_x T^1_y z - T^1_{T^1_x y} z - T^1_y T^1_x z, \end{aligned}$$

and since all brackets are in $\mathfrak{z}$ then

(13) $$T^1_{T^1_x y} = T^1_x T^1_y - T^1_y T^1_x \qquad \forall\, x, y \in \mathfrak{v}.$$

Since T^1 is bilinear and skew-symmetric it follows from (13) that $[x,y]_1 := T^1_x y$ defines a Lie algebra structure on $\mathfrak{v}$. Furthermore, we have that $T^1_x = \operatorname{ad} x$, where ad denotes the adjoint representation of the Lie algebra $(\mathfrak{v}, [\,,\,]_1)$.

From (AS),(i) we obtain that $\operatorname{ad} x \in \mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle)$ for all $x \in \mathfrak{v}$. This implies that there exists an orthogonal decomposition $\mathfrak{v} = \mathfrak{v}' \oplus \mathfrak{c}$ with $\mathfrak{v}'$ a compact semisimple ideal of $(\mathfrak{v}, [\,,\,]_1)$ and $\mathfrak{c}$ the center of $(\mathfrak{v}, [\,,\,]_1)$. Denote by $\mathfrak{v}' = \mathfrak{v}_1 \oplus \ldots \oplus \mathfrak{v}_r$ the orthogonal decomposition of $\mathfrak{v}'$ in simple ideals.

By (12) we have that $J_a \operatorname{ad} x = \operatorname{ad} x J_a$ for all $x \in \mathfrak{v}$, $a \in \mathfrak{z}$. Thus J_a must preserve the ideal $\mathfrak{v}_1$ for all $a \in \mathfrak{z}$, since it is skew-symmetric and $\mathfrak{v}_2 \oplus \ldots \mathfrak{v}_r \oplus \mathfrak{c} = \bigcap_{x \in \mathfrak{v}_1} \operatorname{Ker}(\operatorname{ad} x)$.

We now consider the complexification of $\mathfrak{v}_1$, i.e. the complex vector space $(\mathfrak{v}_1)_C := \mathfrak{v}_1 \otimes_R C$. The maps $\operatorname{ad} x$ and J_a extend naturally to $(\mathfrak{v}_1)_C$, and clearly they still commute with each other. Further, since $\mathfrak{v}_1$ is a compact simple Lie algebra, we have that $(\mathfrak{v}_1)_C$ is a simple complex Lie algebra. Hence the maps $\{\operatorname{ad} x\}_{x \in \mathfrak{v}_1}$ act irreducibly on $(\mathfrak{v}_1)_C$.

We then obtain by Schur's Lemma that $J_a = cI$ with $c \in C$ on $(\mathfrak{v}_1)_C$. Since $J_a\mathfrak{v}_1 \subset \mathfrak{v}_1$ we have that $c \in R$. Using now that J_a is skew-symmetric we obtain that $J_a|_{\mathfrak{v}_1} \equiv 0$ for all $a \in \mathfrak{z}$. This implies that $\mathfrak{v}_1 \subset \mathfrak{z}$ (see (2)), which is a contradiction.

Therefore $\mathfrak{v}' = 0$, i.e. $(\mathfrak{v}, [,]_1)$ is an abelian Lie algebra and thus $T^1 \equiv 0$. □

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