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Successive Approximations and Osgood's Theorem II

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Abstract

The Picard's method for solving $y' = f(x,y)$, $y(x_0) = y_0$ is considered here for $|f(x,y_1) - f(x,y_2)| \leq M(x) F(|y_1 - y_2|)$. The method introduced deals mainly with a majorant differential equation. It is shown that for rather general functions M and F, the difference of two consecutive successive approximations converges at exponentially decreasing rate. The main results are an extension of the corresponding one already obtained in C. P. Calderón and V. N. Vera de Serio (1997).

Key Words: *Successive approximations. Theoretical approximation of solutions. Osgood's functions. LaSalle's theorem. Montel's theorem.*

1 Introduction

Recently [1], we have obtained rather general results on the rate of convergence of the successive approximations of the initial value problem

$$y' = f(x,y), \qquad y(0) = 0. \tag{1}$$

for first order ordinary differential equations satisfying an Osgood's condition. The main result there reads as follows:

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If f is a continuous functions on the rectangle $R = [-a, a] \times [-b, b]$ such that

$$|f(x, y_1) - f(x, y_2)| \leq F(|y_1 - y_2|), \tag{2}$$

for $|x| \leq a, |y_i| \leq b, i = 1, 2$, where F verifies the modified Osgood's condition stated below, then the successive approximations

$$y_{n+1}(x) = \int_0^x f(t, y_n(t))\, dt$$

satisfy

$$|y_{n+1}(x) - y_n(x)| \leq Cr^n, \tag{3}$$

for some $r, 0 < r < 1$, some C large enough and $|x| \leq \delta$, for some positive δ, where we have chosen $|y_0(x)| \leq \eta, \eta > 0$ sufficiently small.

A real valued function F is an Osgood's function if the following conditions are met:

1) F is non-negative continuous and monotone non-decreasing on $(0, \delta)$, for some $\delta > 0$.

2) $\lim_{\varepsilon \to 0^+} \int_\varepsilon^x \frac{dt}{F(t)} = \infty$, for any x, $0 < x \leq \delta$.

1) and 2) above imply immediately that $\lim_{x \to 0^+} F(x) = 0$. We set $F(0) = 0$. F is said to be a modified Osgood's function if it is of the following form:

$$F(t) = t \int_t^d \frac{w(s)}{s}\, ds,$$

where $0 < d \leq 1$ is a given constant and w is a non-negative continuous non-decreasing function on $[0, d]$, such that

$$\int_0^d \frac{w(s)}{s}\, ds = \infty.$$

The following functions are the most common modified Osgood's ones:

$$\varphi_k(t) = t\,(\log(1/t))^{\beta_1}\,(\log\log(1/t))^{\beta_2} \ldots (\log\log \ldots \log(1/t))^{\beta_k}$$

for some $0 \leq \beta_i < 1$, $i = 1, \ldots, k-1$, $0 < \beta_k \leq 1, k = 1, 2, \ldots$.

In this paper we extend the above result to a case when a modified Montel or LaSalle's conditions ([4], [3]) are used instead of Osgood's. Namely, let f be Borel measurable and in $L^\infty([-a, a] \times [-b, b])$, $\|f\|_\infty \leq 1$ and

$$|f(x, y_1) - f(x, y_2)| \leq M(|x|) F(|y_1 - y_2|), \tag{4}$$

for $|x| \leq a, |y_i| \leq b, i = 1, 2$, where $M(x) > 0$ is continuous for $x \neq 0$. $F(y) \geq 0$ is continuous for $y \geq 0$, positive for $y > 0$. Finally, they satisfy:

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^a \frac{dt}{F(t)} - \int_\varepsilon^a M(t)\, dt = \infty. \tag{5}$$

The above property is known as LaSalle's condition. Montel's condition in addition requires the finiteness of the integral of M:

$$\int_0^a M(t)\, dt < \infty.$$

The latter case can be reduced to the classical Osgood's one (Remark 2, Section 2).

For the LaSalle's case we will also consider the alternative condition weaker than (5),

$$\lim_{\varepsilon \to 0^+} \int_{v(\varepsilon)}^a \frac{dt}{F(t)} - \int_\varepsilon^a M(t)\, dt = \infty, \tag{6}$$

for any non negative function $v(x) = o(x)$. We will say that F and M satisfy the alternative LaSalle's condition if (6) holds, while $M(x) > 0$ is continuous for $x \neq 0$ and $F(y) \geq 0$ is continuous for $y \geq 0$, positive for $y > 0$. Notice that

$$\int_x^a M(t)\, dt = O\left(\int_{v(x)}^a \frac{dt}{F(t)}\right) \tag{7}$$

holds because of (6). Under the alternative LaSalle's condition the uniqueness of the solution of the initial value problem (1) follows.

In this context we consider a convenient majorizing differential equation, namely:

$$z' = M(x) F(z),$$

where F is Osgood, F and M satisfy LaSalle's condition and

$$\int_0^h M(t) F(t)\, dt \leq h$$

for $h > 0$. Under these assumptions we obtain for the majorizing differential equation:

(I) The successive approximations

$$z_{n+1}(x) = \int_0^x M(t)\, F(z_n(t))\, dt$$

converge uniformly to 0, $0 \leq x \leq \delta$, for some positive δ, provided that the 0-approximation is chosen to satisfy the inequality

$$|z_o(x)| \leq |x|.$$

(II) There exists at least one non-decreasing absolutely continuous function β , $\beta(0) = 0$, such that

$$\int_0^a \beta(t)\, M(t)\, dt < \infty,$$

while

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^c \frac{\beta(t)}{F(t)} dt = \infty.$$

(III) If in addition

$y\beta(y) = o(F(y))$ and

$-y\frac{d}{dy}\left(\frac{1}{y}\frac{F(y)}{\beta(y)}\right)$ is non-negative monotone non-decreasing in $(0, \delta)$,

then the sequence defined in part (I) satisfies the estimate

$$|z_n(x)| < C r^n, \tag{8}$$

$|x| \leq \delta$, for some δ, C and r, $\delta > 0$, C large enough and $0 < r < 1$.

The method introduced here deals mainly with the majorizing differential equation, for which LaSalle's theorem, [3], is not applicable. The use of the majorant differential equation is introduced to have information on the speed of convergence of the successive approximations. On the other hand, this method gives an uniform majorization of the successive approximations for a large family of differential equations, namely for those functions f that satisfy (4) for fixed F and M.

As a by-product of the results obtained for the majorizing differential equation we have similar results for the initial value problem (1). That is, the successive approximations

$$y_{n+1}(x) = \int_0^x f(t, y_n(t))\, dt,$$

$|y_o(x)| \leq x$, converge at least as fast as those of the majorant equation if $f(x, 0) = 0$, for small $\delta > 0$. If in addition (III) above is satisfied, then

$$|y_{n+1}(x) - y_n(x)| < Cr^n$$

for large C, $0 < r < 1$, $|x| < \delta$.

2 Uniqueness and convergence of the successive approximations for the majorizing differential equation.

Given the initial value problem (1), where f satisfies (4), we may reduce the problem of studying the convergence of the successive approximations

$$y_{n+1}(x) = \int_0^x f(t, y_n(t))\, dt,$$

to the study of the following majorizing differential equation:

$$z'(x) = M(x)\, F(z(x)),$$

where we solve by successive approximations the initial value problem $z(0) = 0$. Here the uniqueness and the convergence of the successive approximations are the issue, since LaSalle's theorem is not applicable.

Lemma 1 *Let M and F be the functions on the alternative LaSalle's condition. If F is monotone non-decreasing and v is an absolutely continuous non negative function on $[0, a]$, $v(x) = o(x)$, then the product $M(x)\, F(v(x))$ is integrable on $[0, a]$. Moreover, if $\int_x^a M(t)\, dt = o\left(\int_{v(x)}^a \frac{dt}{F(t)}\right)$ then*

$$\int_0^h M(t)\, F(v(t))\, dt \leq h, \qquad \textit{for } 0 \leq h \leq a. \tag{9}$$

Proof. : Let $0 < h \leq a$ and let $\varepsilon > 0$ be small enough. An integration by parts yields

$$\int_{\varepsilon}^{a} M(t) F(v(t))\, dt = \left(\int_{\varepsilon}^{a} M(s)\, ds\right) F(v(\varepsilon)) + \int_{\varepsilon}^{a} \left(\int_{t}^{a} M(s)\, ds\right) dF(v(t)). \tag{10}$$

From (7), the first term in the right hand side of (10) is bounded by

$$\left(\int_{\varepsilon}^{a} M(s)\, ds\right) F(v(\varepsilon)) \leq K \int_{v(\varepsilon)}^{a} \frac{F(v(\varepsilon))}{F(s)}\, ds \leq Ka, \tag{11}$$

where K is some positive constant, because F is monotone non-decreasing. Similarly, for the second term in (10), one gets

$$\int_{\varepsilon}^{a} \left(\int_{t}^{a} M(s)\, ds\right) dF(v(t)) \leq K \int_{\varepsilon}^{a} \left(\int_{v(t)}^{a} \frac{1}{F(s)}\, ds\right) dF(v(t)) \leq Ka. \tag{12}$$

From (11) and (12), it follows that

$$\int_{0}^{a} M(t) F(v(t))\, dt \leq 2Ka < \infty.$$

Finally, if

$$\int_{x}^{a} M(t)\, dt = o\left(\int_{v(x)}^{a} \frac{dt}{F(t)}\right)$$

is the case, we could take the constant K to be less than $1/2$ and (9) follows. ■

Theorem 1 *Let M and F be the functions on the alternative LaSalle's condition. If v_1 and v_2 are two solutions of the initial value problem (1), where f satisfies (4), then $v_1 = v_2$ on a neighborhood of 0.*

Proof. Let w be the absolutely continuous function defined by $w(x) = v_1(x) - v_2(x)$. Then $w(0) = w'(0) = 0$, which implies that $|w(x)| = o(x)$. Now,

$$w(x) = \int_{0}^{x} \left(f(s, v_1(s)) - f(s, v_2(s))\right) ds,$$

hence,

$$|w(x)| \leq \int_{0}^{x} M(s) F(|v_1(s) - v_2(s)|)\, ds.$$

Suppose that $w \neq 0$ on every small neighborhood of 0, then $\{w \neq 0\} = \cup (a_k, b_k)$, with $0 < a_k < b_k \leq a, w(a_k) = 0$. We may assume with no loss of generality that w is positive on (a_k, b_k), hence

$$\begin{aligned} |w'(x)| &= |f(x, v_1(x)) - f(x, v_2(x))| \\ &\leq M(x) F(|v_1(x) - v_2(x)|) \\ &= M(x) F(|w(x)|) \\ &= M(x) F(w(x)) \end{aligned}$$

for any $x \in (a_k, b_k)$. Take $a_k < b_k' < b_k$ and $\varepsilon > 0$ small enough, then

$$\left| \int_{a_k+\varepsilon}^{b_k'} \frac{w'(s)}{F(w(s))} ds \right| \leq \int_{a_k+\varepsilon}^{b_k'} \frac{|w'(s)|}{F(w(s))} ds \leq \int_{a_k+\varepsilon}^{b_k'} M(s)\, ds.$$

The change of variable $t = w(s)$ in the first integral above yields

$$\left| \int_{w(a_k+\varepsilon)}^{w(b_k')} \frac{1}{F(t)} dt \right| \leq \int_{a_k+\varepsilon}^{b_k'} M(s)\, ds,$$

and so it follows that

$$\lim_{\varepsilon \to 0^+} \int_{w(a_k+\varepsilon)}^{w(b_k')} \frac{1}{F(t)} dt - \int_{a_k+\varepsilon}^{b_k'} M(s)\, ds \leq 0, \tag{13}$$

which contradicts the assumptions that the first integral is not bounded while the second one is finite when $a_k > 0$. If $a_k = 0$, (13) contradicts (6). ∎

From now on, we will assume that M and F are functions that satisfy LaSalle's condition and the following additional one:

$$\int_x^a M(t)\, dt = o\left(\int_x^a \frac{dt}{F(t)} \right). \tag{14}$$

Lemma 2 *If F is monotone non-decreasing then the product $M(x) F(x)$ is integrable on $[0, a]$. Moreover, $\int_0^h M(t) F(t)\, dt \leq h$, for $0 \leq h \leq a$.*

Proof. It is entirely analogous to the proof of Lemma 1. ∎

The following lemma will be used in Section 3.

Lemma 3 *There exists a function β with $\beta(x) > 0$ for $x > 0$, such that*

$$\beta(x) = \int_0^x \alpha(t)\, dt,$$

for some locally integrable function $\alpha(x) > 0$, and such that

$$\int_0^a \beta(t)\, M(t)\, dt < \infty$$

while

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^a \frac{\beta(t)}{F(t)} dt = \infty.$$

Proof. The finiteness of

$$\int_0^a \beta(t)\, M(t)\, dt$$

is equivalent with that of

$$\int_0^a \left(\int_x^a M(t)\, dt \right) \alpha(x)\, dx.$$

Therefore, we shall consider the space $L^1_\mu(0,a)$ where $d\mu = \left(\int_x^a M(t)\, dt\right) dx$. Suppose that the finiteness of

$$\int_0^a \alpha(x)\, d\mu$$

for $\alpha(x) > 0$ implies that of

$$\int_0^a \alpha(x) \left(\int_x^a \frac{1}{F(t)} dt \right) dx,$$

or, equivalently that of

$$\int_0^a \frac{\int_x^a \frac{1}{F(t)} dt}{\int_x^a M(t)\, dt} \alpha(x)\, d\mu.$$

This would indicate that

$$\frac{\int_x^a \frac{1}{F(t)} dt}{\int_x^a M(t)\, dt} \in L^\infty(0,a)$$

which is a contradiction because

$$\int_x^a M(t)\,dt = o\left(\int_x^a \frac{1}{F(t)}dt\right).$$

Hence, there is some $\alpha(x) \in L^1_\psi(0,a)$, $\alpha(x) > 0$, such that

$$\int_0^a \alpha(x)\left(\int_x^a \frac{1}{F(t)}dt\right)dx = \infty,$$

or equivalently

$$\int_0^a \frac{\beta(t)}{F(t)}dt = \infty,$$

for

$$\beta(x) = \int_0^x \alpha(t)\,dt. \qquad \blacksquare$$

Theorem 2 *Let $z' = M(x)F(z)$, $z(0) = 0$ be the majorizing initial value problem for the equation*

$$y' = f(x,y), \qquad y(0) = 0,$$

under the condition

$$|f(x,y_1) - f(x,y_2)| \le M(|x|)\,F(|y_1 - y_2|),$$

for $|x| \le a, x \ne 0, |y_i| \le b, i = 1,2$. Suppose that M and F satisfy LaSalle's condition, (14) above and, furthermore, F is monotone non-decreasing. Then, (i) the successive approximations for the initial value problem of the majorizing differential equation $z' = M(x)F(z)$ converge uniformly in a neighborhood of the origin, provided that the 0-approximation is chosen to satisfy $|z_o(x)| \le |x|$. (ii) This solution is unique.

Proof. We seek solutions z such that $|z(x)| \le |x|$. From Lemma 2,

$$\int_0^h M(t)\,F(t)\,dt \le h,$$

and we could guarantee that

$$z_{n+1}(x) = \int_0^x M(t)\,F(z_n(t))\,dt$$

can be accomplished by functions that satisfy $0 \leq z_n(x) \leq x$, as long as $0 \leq z_o(x) \leq x$, for $0 \leq x \leq \delta$.

For $0 \leq x \leq \delta$, let us set

$$\mu(x) := \limsup_{n\to\infty} z_n(x).$$

Clearly, $0 \leq \mu(x) \leq x$. Notice that μ is continuous since the z_n's are equicontinuous and, by Fatou's Lemma,

$$\mu(x) \leq \int_0^x M(t) F(\mu(t))\, dt < \infty.$$

It will be shown that $\mu(x)$ vanishes on $[0,\delta]$ for some $\delta > 0$. Let us assume that it is not the case, then there is a sequence $x_k \downarrow 0$ such that $\mu(x_k) > 0$. Hence,

$$\int_0^x M(t) F(\mu(t))\, dt > 0$$

for $0 < x \leq \delta$. Let us call $G(x)$ this last integral, then $0 \leq \mu(x) \leq G(x) \leq x$ and

$$G'(x) = M(x) F(\mu(x)) \leq M(x) F(G(x)),$$

for $0 < x \leq \delta$. It is enough to show that G is null on a neighborhood of the origin. If it were not the case, a reasoning similar to the one in the proof of Theorem 1 will show a contradiction: Let $\cup(a_k, b_k)$ be the set where $G > 0$ in $(0,\delta)$. Note that $G(a_k) = 0$ and take $a_k < b'_k < b_k$. Hence, for $\varepsilon > 0$ small enough,

$$\int_{G(a_k+\varepsilon)}^{G(b'_k)} \frac{1}{F(x)}\, dx = \int_{a_k+\varepsilon}^{b'_k} \frac{G'(x)}{F(G(x))}\, dx \leq \int_{a_k+\varepsilon}^{b'_k} M(x)\, dx < \infty,$$

and so

$$\lim_{\varepsilon\to 0^+} \int_{G(a_k+\varepsilon)}^{G(b'_k)} \frac{1}{F(x)}\, dx - \int_{a_k+\varepsilon}^{b'_k} M(x)\, dx \leq 0,$$

which contradicts the hypothesis. Thus, the function μ vanishes in a neighborhood of the origin.

The uniqueness is dealt with in the same manner by considering $\mu(x) = |z(x)|$ for any solution z different from 0. ■

Corollary 1 *Let f be Borel measurable in $L^\infty([-a,a]\times[-b,b])$ such that $\|f\|_\infty \leq 1/2$ and*

$$|f(x,y_1) - f(x,y_2)| \leq M(|x|) F(|y_1 - y_2|),$$

for $|x| \leq a, x \neq 0, |y_i| \leq b, i = 1, 2$. *Suppose that* M *and* F *satisfy LaSalle's condition, (14) above and, furthermore,* F *is monotone non-decreasing. Then, (i) the successive approximations for the integral equation*

$$y(x) = \int_0^x f(t, y(t))\, dt$$

converge uniformly in a neighborhood of the origin, whenever the 0-approximation is chosen to satisfy $|y_o(x)| \leq b$. *(ii) This solution is unique.*

Remark 1 *Notice that in the latter proof the function* M *only needs to be locally integrable on* $(0, a)$. *Moreover, the condition (14) can be replaced by the following one:*

$$\int_0^h M(t)\, F(t)\, dt \leq h, \qquad h > 0.$$

Remark 2 *In this context, Montel's case can be reduced to the classical Osgood's one by a* C^1 *change of variable. Here* M *is integrable and we may assume with no loss of generality that it is positive. By setting :*

$$s(x) = \int_0^x M(t)\, dt,$$

$|x| \leq \delta$, *one gets the equivalent differential equation* $z'(s) = F(z(s))$. *This is no else but a majorant differential equation for the referred Osgood's case. Therefore, if* F *is a modified Osgood's function, the results in [1] apply to yield the following estimate of the successive approximations of the majorizing initial value problem:*

$$|z_n(x)| \leq Cr^n \tag{15}$$

for x *in a neighborhood of the origin, for some* C *and* r, *where* C *is large enough and* $0 < r < 1$.

3 Speed of convergence of the successive approximations

Theorem 3 *If in addition to the assumptions in Theorem 2, it holds that the function* β *from Lemma 3 satisfy* (*i*) $y\beta(y) = o(F(y))$ *and*

(*ii*) $-y\frac{d}{dy}\left(\frac{1}{y}\frac{F(y)}{\beta(y)}\right)$ *is non-negative and monotone non-decreasing in* $(0,\delta)$,

then, the sequence of successive approximations of the initial value problem for the majorizing differential equation, provided that $|z_o(x)| \leq |x|$, *satisfies the estimate*

$$|z_n(x)| < Cr^n,$$

$|x| \leq \delta$, *for some* δ, C *and* r, $\delta > 0$, C *large enough and* $0 < r < 1$.

Proof. Through the aid of the function β, the problem under consideration can be reduced to Montel's case:

$$z' = \tilde{M}(x)\tilde{F}(z).$$

where $\tilde{M}(x) = \beta(x)M(x)$ and $\tilde{F}(y) = F(y)/\beta(y)$. $\tilde{M}(x)$ is integrable on $[0,\delta]$, for some positive δ. Conditions (i) and (ii) imply that $\tilde{F}(y)$ is a modified Osgood's function. By taking into account the Remark above, we may apply the estimate $|z_n(x)| \leq Cr^n$ to the initial value problem $z'(s) = \tilde{F}(z(s))$, $z'(0) = 0$, where we have made the change of variables: $s(x) = \int_0^x \tilde{M}(t)\,dt$. ■

Corollary 2 *Let* f *be Borel measurable and in* $L^\infty([-a,a]\times[-b,b])$ *such that* $\|f\|_\infty \leq 1/2$ *and*

$$|f(x,y_1) - f(x,y_2)| \leq M(|x|)\,F(|y_1 - y_2|),$$

for $|x| \leq a, x \neq 0, |y_i| \leq b, i = 1,2$. *Suppose that* M *and* F *satisfy the conditions on Theorem 2. Then, the successive approximations for the integral equation*

$$y(x) = \int_0^x f(t,y(t))\,dt$$

satisfy the following estimate:

$$|y_{n+1}(x) - y_n(x)| \leq Cr^n,$$

$|x| \leq \delta$, *for some* δ, C *and* r, $\delta > 0$, C *large enough and* $0 < r < 1$.

Example 1 *Let F and M be given by*

$$F(y) = y(\log y)^{\nu}, \qquad 0 < \nu < 1,$$
$$M(x) = \left[x(\log x)^{\delta}\right]^{-1}, \qquad 0 < \delta < 1.$$

Then,

$$\int_{\varepsilon}^{1} \frac{1}{F(y)} dy \simeq (\log \varepsilon)^{1-\nu},$$
$$\int_{\varepsilon}^{1} M(x)\, dx \simeq (\log \varepsilon)^{1-\delta}.$$

Assume that $0 < \nu < \delta$, and let $\gamma > 0$ such that $\nu + \gamma < 1 < \delta + \gamma$. Define

$$\beta(x) = (\log x)^{-\gamma}.$$

It is straightforward to show that these functions M, F and β satisfy the conditions of the above theorem.

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