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REPRESENTATIONS OF THE SYMMETRIC GROUP S_n ON $\mathbf{K}[x_1, \ldots, x_n]$

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ABSTRACT. The representations of the symmetric group were studied initially by Frobenius, Schur and Young. In more recent work , James ([3] and [4]) describes the irreducible representations of S_n in terms of Specht modules, and Farahat-Peel ([2]) in terms of ideals in the group algebra.
In this work we present a realization of the irreducible representations of the symmetric group S_n in the ring of polynomials in n indeterminates.
The objective of these realizations is to develop the theory of representations of the symmetric group, taking advantage of the structure of the ring of polynomials.
To begin with, we treat the case where $\mathbf{K}$ is a field of characteristic zero. In this case, the politabloids concepts and Specht modules in the language of [4] have a natural realization.
The constructions in the case of characteristic zero, with slight modifications, are used later in section 2, to obtain the irreducible representations of S_n on a field of characteristic different from zero.

1. ORDINARY REPRESENTATIONS.

Let $\mathbf{K}$ be a field and let $\mathbb{N}_0$ be the set of de non negative integers. We consider the polynomial ring $\mathcal{A} = \mathbf{K}[x_1, ..., x_n]$ in the indeterminates $x_1, ..., x_n$.
Given the multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, with x^α we will denote the monomial:

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

We have in $\mathcal{A}$ a bilinear form $< , >$ defined on the generators by :

$$< x^\alpha, x^\beta > = \begin{cases} 1 \text{ si } \alpha = \beta \\ 0 \text{ si } \alpha \neq \beta \end{cases}$$

and extended by linearity.
Let S_n be the symmetric group of $\mathcal{I}_n = \{1, 2, .., n\}$. There is a natural action of S_n on $\mathcal{A}$ and this action preserves the bilinear form $< , >$.
For a set $\mathcal{C}$, such that $\mathcal{C} \subsetneq \mathcal{I}_n$, we denote by $\mathcal{S}(\mathcal{C})$ be the symmetric group of $\mathcal{C}$.

Putting $\mathcal{C} = \{c_1, ..., c_h\}$; $c_1 < c_2 < \cdots < c_h$, we define:

$$\Delta_{\mathcal{C}} = \sum_{\sigma \in \mathcal{S}(\mathcal{C})} sg(\sigma)\,\sigma\,, \quad \mathcal{M}_{\mathcal{C}} = x_{c_1}^0 \cdot x_{c_2}^1 \cdots x_{c_h}^{h-1}, \quad e_{\mathcal{C}} = \Delta_{\mathcal{C}}(\mathcal{M}_{\mathcal{C}})$$

where $sg(\sigma)$ is the sign of σ.
For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, with x^α we write:

$$\mathcal{M} = x^\alpha, \ \mathcal{M}^{\mathcal{C}} = x_{c_1}^{\alpha_{c_1}} x_{c_2}^{\alpha_{c_2}} \cdots x_{c_h}^{\alpha_{c_h}} \text{ and } \mathcal{M}^{\mathcal{C}^*} = \frac{\mathcal{M}}{\mathcal{M}^{\mathcal{C}}}$$

From these definitions and notations, the following proposition is clear:

Proposition 1.1: *i) For each monomial $\mathcal{M}$ we have:*

$$\Delta_{\mathcal{C}}(\mathcal{M}) = \Delta_{\mathcal{C}}\left(\mathcal{M}^{\mathcal{C}}\right) \mathcal{M}^{\mathcal{C}^*}$$

ii) If $\mathcal{M} = x_{c_1}^{\alpha_1} \cdots x_{c_h}^{\alpha_h}$, then:

$$\Delta_{\mathcal{C}}(\mathcal{M}) = \det\left(\left[x_{c_j}^{\alpha_i}\right]_{1 \le i,j \le h}\right)$$

Also, with the same notations from above, it is possible to obtain the following striking identity.

Lemma 1.2. *If $\alpha = (\alpha_1, ..., \alpha_h) \in \mathbb{N}_0^h$ and $\mathcal{C} = \{1, 2, \cdots, h\}$, then:*

$$\det\left(\left[x_j^{\alpha_i}\right]_{1 \le i,j \le h}\right) = \det\left(\left[x_j^{i-1}\right]_{1 \le i,j \le h}\right) \mathcal{P}$$

where $\mathcal{P}$ is a symmetric polynomial for $\mathcal{S}(\mathcal{C})$.

Proof: Let $\phi_1, ..., \phi_h$ be the symmetric elementary polynomials in $x_1, ..., x_h$, that is:

$$\phi_i(x_1, ..., x_h) = \sum_{k_1 < k_2 < \cdots < k_i} x_{k_1} \cdot x_{k_2} \cdots x_{k_i}$$

Then, we have:

$$x_j^h = \phi_1 x_j^{h-1} - \phi_2 x_j^{h-2} + \cdots \pm \phi_h$$

and from this identity it follows that for all $m \in \mathbb{N}_0$ there are symmetric polynomials $\psi_1, ..., \psi_c$ such that:

$$x_j^m = \psi_1 x_j^{h-1} + \psi_2 x_j^{h-2} + \cdots + \psi_h$$

and, therefore there are symmetric polynomials ψ_{ij} such that:

$$\left[x_j^{\alpha_i}\right] = [\psi_{ij}]\left[x_j^{i-1}\right]$$

Now, if we compute determinants on both sides of this equality, we get the lemma. Of course $\mathcal{P} = \det[\psi_{ij}]$. ■

Proposition 1.3: *Let $\mathcal{C} \subseteq \mathcal{I}_n$ be any subset, then:*
i) For $\tau \in \mathcal{S}(\mathcal{C})$ we have:

$$\tau \, \Delta_{\mathcal{C}} = \Delta_{\mathcal{C}} \, \tau = sg(\tau) \, \Delta_{\mathcal{C}}$$

ii) If $\mathcal{M} = x^{\alpha}$ then, $\Delta_{\mathcal{C}}(\mathcal{M}) = 0$, if and only if, there are two elements i, j in $\mathcal{C}$ such that $i \neq j$ with $\alpha_i = \alpha_j$. Futhermore, if $dg\left(\mathcal{M}^{\mathcal{C}}\right) = dg(\mathcal{M}_{\mathcal{C}})$ then $\Delta_{\mathcal{C}}(\mathcal{M}) \neq 0$, if and only if, there is an element $\sigma \in \mathcal{S}(\mathcal{C})$ such that $\mathcal{M}^{\mathcal{C}} = \sigma \, \mathcal{M}_{\mathcal{C}}$.
iii) For each $\mathcal{P}$ in $\mathcal{A}$, $\Delta_{\mathcal{C}}(P) = e_{\mathcal{C}} \, \mathcal{P}^{\star}$ where $\mathcal{P}^{\star}$ is a $\mathcal{S}(\mathcal{C})$-symmetric polynomial .

Proof: i) For $\tau \in \mathcal{S}(\mathcal{C})$, we have:

$$\begin{aligned} \tau \sum_{\sigma \in \mathcal{S}(\mathcal{C})} sg(\sigma) \, \sigma &= \sum_{\sigma \in \mathcal{S}(\mathcal{C})} sg(\sigma) \, \tau\sigma = sg(\tau) \sum_{\sigma \in \mathcal{S}(\mathcal{C})} sg(\tau\sigma) \, \tau\sigma \\ &= sg(\tau) \sum_{\mu \in \mathcal{S}(\mathcal{C})} sg(\mu) \, \mu \end{aligned}$$

but $\tau \, \Delta_{\mathcal{C}} = sg(\tau) \, \Delta_{\mathcal{C}}$ and $\Delta_{\mathcal{C}} \, \tau = sg(\tau) \, \Delta_{\mathcal{C}}$.
ii) If there are indexes $i \neq j$ in $\mathcal{C}$ such that $\alpha_i = \alpha_j$, it is clear that $\Delta_{\mathcal{C}}(\mathcal{M}) = 0$. Conversely, if the images $\sigma\mathcal{M}$ of $\mathcal{M}$ under σ in $\mathcal{S}(\mathcal{C})$ are all different, then they are linearly independent, hence $\Delta_{\mathcal{C}}(\mathcal{M}) = 0$ says that there is an element σ in $\mathcal{S}(\mathcal{C})$ such that $\sigma\mathcal{M} = \mathcal{M}$, consequently, we obtain two elements i, j in $\mathcal{C}$ such that $i \neq j$ with $\alpha_i = \alpha_j$.
iii) Follows from lemma 1.2 and the linearity of $\Delta_{\mathcal{C}}$. ■

Consider σ in $\mathcal{S}_n$ and $\mathcal{O}_1, \mathcal{O}_2, .., \mathcal{O}_m$ the orbits in $\mathcal{I}_n$ of the cyclic group generated by σ . We define Δ_σ in $End(\mathcal{A})$ and $\mathcal{M}_\sigma$, e_σ in $\mathcal{A}$ by:

$$\Delta_\sigma = \prod_{1 \leq i \leq m} \Delta_{\mathcal{O}_i} \ , \ \mathcal{M}_\sigma = \prod_{1 \leq i \leq m} \mathcal{M}_{\mathcal{O}_i} \ , \ e_\sigma = \prod_{1 \leq i \leq m} e_{\mathcal{O}_i}.$$

Let $\mathcal{C}_\sigma$ be denote the group $\mathcal{S}(\mathcal{O}_1) \times \cdots \times \mathcal{S}(\mathcal{O}_m)$ and if γ is a conjugation class of $\mathcal{S}_n$, we will denote $\mathcal{V}_\gamma$ and $\mathcal{W}_\gamma$ the subspaces of $\mathcal{A}$ given by:

$$\mathcal{V}_\gamma = \langle \mathcal{M}_\sigma : \sigma \in \gamma \rangle \quad \text{and} \quad \mathcal{W}_\gamma = \langle e_\sigma : \sigma \in \gamma \rangle$$

If γ_i is the cardinality of $\mathcal{O}_i$, we have that the degree of $\mathcal{M}_\sigma$ is:

$$dg(\mathcal{M}_\sigma) = \sum_{i=1}^{m} \frac{\gamma_i^2 - \gamma_i}{2}$$

and this value is the same for each element in the conjugation class of σ.
If σ belongs to the class of γ, we define the *degree of* γ as $dg(\gamma) = dg(\mathcal{M}_\sigma)$.
From now on, we will use the preceding notations and conventions.

Proposition 1.4: *i) For each class γ, $\mathcal{V}_\gamma$ and $\mathcal{W}_\gamma$ are both invariant under $\mathcal{S}_n$ and $\mathcal{W}_\gamma \subseteq \mathcal{V}_\gamma$.*
ii) If $\gamma \neq \gamma'$, then $\mathcal{V}_\gamma \cap \mathcal{V}_{\gamma'} = 0 = \langle \mathcal{V}_\gamma, \mathcal{V}_{\gamma'} \rangle$. In consequence the maps $\gamma \to \mathcal{V}_\gamma$ and $\gamma \to \mathcal{W}_\gamma$ are one-to- one mappings.

iii) $\tau\,\Delta_\sigma = \Delta_\sigma\,\tau = sg\,(\tau)\,\Delta_\sigma\ \forall\,\tau \in \mathcal{S}_\sigma$.
iv) If $\mathcal{P} \in \mathcal{A}$, *then* $\Delta_\sigma\,(\mathcal{P}) = \mathcal{Q}\,e_\sigma$, *where* $\mathcal{Q}$ *is a* $\mathcal{S}_\sigma$*-symmetric polynomial.*
v) When $\gamma \neq \gamma'$ *and* $dg\,(\gamma') \leq dg\,(\gamma)$ *we have* $\Delta_\sigma\,(\mathcal{V}_{\gamma'}) = 0$.
vi) For $\mathcal{P} \in \mathcal{V}_\gamma$ *and* $\sigma \in \gamma$ *it is valid the identity* $\Delta_\sigma\,(\mathcal{P}) = \langle \mathcal{P}, e_\sigma \rangle\ e_\sigma$.

Proof: i) and *ii)* follow readily from the constructions of $\mathcal{V}_\gamma$ and $\mathcal{W}_\gamma$. Since each monomial $\mathcal{M}$ can be written as:

$$\mathcal{M} = \mathcal{M}^{\mathcal{O}_1} \cdots \mathcal{M}^{\mathcal{O}_m}$$

and the fact that Δ_σ is a linear operator, we obtain *iii)* and *iv)* from the proposition 1.3.
To see *v)* and *vi)* consider $\mathcal{M} = x^\alpha$, and if $\Delta_\sigma\,(\mathcal{M}) \neq 0$, then by *iv)* we must have:

$$dg\,(\mathcal{M}) \geq dg\,(e_\sigma) = dg\,(\mathcal{M}_\sigma) = dg\,(\gamma)$$

If $dg\,(\mathcal{M}) = dg\,(\gamma)$, then putting $\mathcal{M} = \mathcal{M}^{\mathcal{O}_1} \cdots \mathcal{M}^{\mathcal{O}_m}$, we obtain:

$$\Delta_\sigma\,(\mathcal{M}) = \Delta_{\mathcal{O}_1}\left(\mathcal{M}^{\mathcal{O}_1}\right) \cdots \Delta_{\mathcal{O}_m}\left(\mathcal{M}^{\mathcal{O}_m}\right) \neq 0$$

so that $\Delta_{\mathcal{O}_i}\left(\mathcal{M}^{\mathcal{O}_i}\right) \neq 0\ \forall\ i = 1, 2, ..., m$. From part *iii)* of the Proposition 1.3 it follows that:

$$dg\left(\mathcal{M}^{\mathcal{O}_i}\right) \geq dg\,(\mathcal{M}_{\mathcal{O}_i})\quad \forall\ i = 1, 2, .., m$$

and therefore:

$$dg\,(\mathcal{M}) = \sum_i dg\left(\mathcal{M}^{\mathcal{O}_i}\right) \geq \sum dg\,(\mathcal{M}_{\mathcal{O}_i}) = dg\,(\mathcal{M}_\sigma) = dg\,(\gamma)$$

that is, $\mathcal{M}^{\mathcal{O}_i}$ and $\mathcal{M}_{\mathcal{O}_i}$ have the same degree $\forall\ i$. Now, by applying part *ii)* of the Proposition 1.3, we infer that there exists $\tau_i \in (\mathcal{SO}_i)$ such that:

$$\mathcal{M}^{\mathcal{O}_i} = \tau_i\,(\mathcal{M}_{\mathcal{O}_i})$$

Hence, if $\tau = \prod_i \tau_i$, then $\tau \in \mathcal{C}_\sigma$ and:

$$\mathcal{M} = \tau\,\mathcal{M}_\sigma$$

In particular we see that $\mathcal{M} \in \mathcal{V}_\gamma$.
Starting from:

$$\begin{aligned}\Delta_\sigma\,(\mathcal{M}) &= \Delta_\sigma\tau\ (\mathcal{M}_\sigma) = sg\,(\tau)\ \Delta_\sigma\,(\mathcal{M}_\sigma) = sg\,(\tau)\ e_\sigma \\ &= \langle \tau\,\mathcal{M}_\sigma, e_\sigma \rangle\ e_\sigma = \langle \mathcal{M}, e_\sigma \rangle\ e_\sigma\end{aligned}$$

we obtain:

$$\Delta_\sigma\,(\mathcal{P}) = \langle \mathcal{P}, e_\sigma \rangle\ e_\sigma \forall \mathcal{P} \in \mathcal{V}_\gamma$$

because Δ_σ is a linear operator. Besides, if $\tau \in \gamma'$ and $\Delta_\sigma(\mathcal{M}_\tau) \neq 0$, then $\mathcal{M}_\tau \in \mathcal{V}_{\gamma'} \cap \mathcal{V}_\gamma$, and this implies $\gamma\prime = \gamma$. ■
Denoting by ρ_γ the representation of $\mathcal{S}_n$ on $\mathcal{W}_\gamma$, we can establish the following theorem:

Theorem 1.5: *If char(K) = 0 or char(K) > n, then:*
i) ρ_γ is irreducible for every class γ.
ii) $\rho_\gamma \cong \rho_{\gamma'}$ if, and only if $\mathcal{W}_\gamma = \mathcal{W}_{\gamma'}$ if, and only if $\gamma = \gamma\prime$.
iii) The representations ρ_γ are all the absolutely irreducible representations of $\mathcal{S}_n$ up to isomorphism.

Proof: i) Let $\mathcal{W}_\gamma$ be an $\mathcal{S}_n$-invariant subspace . For the assumption on the characteristic of **K**, $\mathcal{W}_\gamma$ is completely reducible, that is, there is an $\mathcal{S}_n$-invariant subspace $\mathcal{T}$ of de $\mathcal{W}_\gamma$, such that:

$$\mathcal{W}_\gamma = \mathcal{S} \oplus \mathcal{T}$$

For $\sigma \in \gamma$, we pick s and t in $\mathcal{S}$ and $\mathcal{T}$ respectively verifying $e_\sigma = s + t$, and we obtain:

$$|\mathcal{C}_\sigma|\ e_\sigma = \Delta_\sigma(e_\alpha) = \Delta_\sigma(s) + \Delta_\sigma(t) = \langle e_\sigma, s\rangle\ e_\sigma + \langle e_\sigma, t\rangle\ e_\sigma$$

By the hypothesis $|\mathcal{C}_\sigma| \neq 0$, and this implies that $\Delta_\sigma(s)$ or $\Delta_\sigma(t)$ can not be both zero, that is to say $\langle e_\sigma, s\rangle$ or $\langle e_\sigma, t\rangle$ are nonzero, hence e_σ is in $\mathcal{S} \cup \mathcal{T}$, so that $\mathcal{S} = \mathcal{W}_\gamma$ or $\mathcal{T} = \mathcal{W}_\gamma$..
ii) Let $\theta : \mathcal{W}_\gamma \to \mathcal{W}_{\gamma'}$ be an $\mathcal{S}_n$-isomorphism. We can suppose $dg(\gamma) \geq dg(\gamma\prime)$. For $\sigma \in \gamma$, we have:

$$\Delta_\sigma(e_\sigma) = \theta\, \Delta_\sigma(e_\sigma) = |\mathcal{S}_\sigma|\ \theta(e_\sigma) \neq 0$$

but now, from *v)* of Proposition 1.3, it follows that $\gamma = \gamma\prime$.
iii) In the hypothesis of the theorem, the number of non isomorphic irreducible representations is equal to the number of conjugate classes of $\mathcal{S}_n$. ■

In what follows, we will do a short analysis of the characters of $\mathcal{S}_n$ assuming that the field **K** has zero characteristic.
Let us consider the partitions $\gamma_1, \gamma_2, ..., \gamma_h$ of n ordered such that:

$$dg(\gamma_i) \leq dg(\gamma_{i+1})$$

and denote with $\mathcal{V}_i$ and $\mathcal{W}_i$ the subspaces $\mathcal{V}_{\gamma_i}$ and $\mathcal{W}_{\gamma_i}$ respectively. Let $\psi_1, ..., \psi_h$ and $\chi_1, ..., \chi_h$ be the characters of $\mathcal{S}_n$ over the spaces $\mathcal{V}_1, ..., \mathcal{V}_h$ and $\mathcal{W}_1, ..., \mathcal{W}_h$ respectively. If t_{ij} is the multiplicity of $\mathcal{W}_i$ in $\mathcal{V}_j$, we have, from Proposition 1.4:

$$t_{ii} = 1 \text{ and } t_{ij} = 0 \text{ if } i > j$$

Let $\mathcal{T} = [t_{ij}]$, then $\mathcal{T}$ is a lower triangular matrix with 1 's in the principal diagonal, and:

$$\psi = \mathcal{T}\chi \text{ where } \psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_h \end{bmatrix} \text{ and } \chi = \begin{bmatrix} \chi_1 \\ \vdots \\ \chi_h \end{bmatrix}$$

Then we may establish:

Theorem 1.6: *i)*

$$\chi_j = \det \begin{bmatrix} \langle\psi_1,\psi_1\rangle & \langle\psi_1,\psi_2\rangle & \cdots & \langle\psi_1,\psi_j\rangle \\ \vdots & \vdots & & \vdots \\ \langle\psi_{j-1},\psi_1\rangle & \langle\psi_{j-1},\psi_2\rangle & \cdots & \langle\psi_{j-1},\psi_j\rangle \\ \psi_1 & \psi_2 & \cdots & \psi_j \end{bmatrix}$$

Here $<,>$ stands for the scalar product for characters.
ii) The multiplicity of $\mathcal{W}_i$ in $\mathcal{V}_j$ is given by:

$$<\psi_i,\chi_j>= \det \begin{bmatrix} \langle\psi_1,\psi_1\rangle & \langle\psi_1,\psi_2\rangle & \cdots & \langle\psi_1,\psi_j\rangle \\ \vdots & \vdots & & \vdots \\ \langle\psi_{j-1},\psi_1\rangle & \langle\psi_{j-1},\psi_2\rangle & \cdots & \langle\psi_{j-1},\psi_j\rangle \\ \langle\psi_i,\psi_1\rangle & \langle\psi_i,\psi_2\rangle & \cdots & \langle\psi_i,\psi_j\rangle \end{bmatrix}$$

iii) $dim(\mathcal{W}_j)= \langle\psi_h,\chi_j\rangle$.

Proof: i) The family of functions φ_k defined by:

$$\varphi_k = \det \begin{bmatrix} \langle\psi_1,\psi_1\rangle & \langle\psi_1,\psi_2\rangle & \cdots & \langle\psi_1,\psi_k\rangle \\ \vdots & \vdots & & \vdots \\ \langle\psi_{k-1},\psi_1\rangle & \langle\psi_{k-1},\psi_2\rangle & \cdots & \langle\psi_{k-1},\psi_k\rangle \\ \psi_1 & \psi_2 & \cdots & \psi_k \end{bmatrix}$$

is an orthonormal system, because:

$$\langle\varphi_k,\varphi_k\rangle = \langle\varphi_k,\psi_k\rangle = \det\left({}^t\mathcal{T}_k\,\mathcal{T}_k\right) = 1$$

where ${}^t\mathcal{T}_k$ is the transpose matrix of the $\mathcal{T}_k$, and $\mathcal{T}_k = [t_{ij}]_{1\leq i,j\leq k}$. In addition $\varphi_1 = \psi_1 = \chi_1$ and φ_k belong to the subspace generated by $\chi_1,...,\chi_k$. Then we conclude that $\chi_k = \varphi_k$ for $k = 1,...,h$.
ii) It is clear from *i)*
iii) Since $\psi_h(\sigma) = 0$ when $\sigma \neq 1$, and $\psi_h(1) = n!$, for each index j we have $\langle\psi_h,\psi_j\rangle = \psi_j(1)$, and now, from *i)* and *ii)* follows that:

$$\langle\psi_h,\chi_j\rangle = \chi_j(1) = \dim(\mathcal{W}_j) \blacksquare$$

Remark: Part *iii),* in Theorem 1.6, is a consequence from the general fact that the multiplicity of a simple module in the left regular representation is precisely this dimension. In our case, it is easy to see that the representation over $\mathcal{V}_h$ is

equivalent to the left regular representation.
Notice that $\psi_j(\sigma)$ is the number of the monomials x^α in $\mathcal{V}_j$ which are fixed by σ.
Moreover, $\mathcal{S}_n$ acts naturally in the tensor product $\mathcal{V}_i \otimes \mathcal{V}_j$ by:

$$\sigma(\mathcal{P} \otimes \mathcal{Q}) = \sigma(\mathcal{P}) \otimes \sigma(\mathcal{Q})$$

Then, $\mathcal{S}_n$ decomposes the canonical base of $\mathcal{V}_i \otimes \mathcal{V}_j$ in a certain number of orbits, and due the Burnside's identity, we can infer that this number is:

$$\langle \psi_i, \psi_j \rangle = \frac{1}{n!} \sum_{\sigma} \psi_i(\sigma)\, \psi_j\left(\sigma^{-1}\right)$$

2. MODULAR REPRESENTATIONS.

Now, let K be a field of characteristic $p > 0$. Let γ be any conjugation class of $\mathcal{S}_n$. Given σ in γ, let $\mathcal{O}_1, \mathcal{O}_2, .., \mathcal{O}_m$ be the orbits of the cyclic group generated by σ in $\mathcal{I}_n$, enumerated such that $|\mathcal{O}_i| \geq |\mathcal{O}_{i+1}|$. Consider $\mathcal{M} = \mathcal{M}_\sigma$ and $\mathcal{C} = \mathcal{C}_\sigma$ as before and denote by $\mathcal{R} = \mathcal{R}_\sigma$ the isotropy group of $\mathcal{M}$. Also, $\mathcal{O}^1, ..., \mathcal{O}^k$ will denote the orbits of $\mathcal{R}$ in $\mathcal{I}_n$, enumerated such that $|\mathcal{O}^i| \geq |\mathcal{O}^{i+1}|$.
Notice that $\mathcal{R} \cap \mathcal{C} = \{1\}$ because $|\mathcal{O}_i \cap \mathcal{O}^j| \leq 1$.
Associated to γ we have the numbers:

$$\gamma_i = |\mathcal{O}_i| \quad , \quad \gamma^i = |\mathcal{O}^i|$$

$$\gamma_{ij} = \sharp \left\{ h / \gamma^h = i + j - 1 \right\} \quad , \quad 1 \leq i, j \leq m$$

It is clear that γ_i, γ^i and γ_{ij} depend only on γ, but not on the election of σ in γ. In particular, we have $\gamma^j = \# \{i : \gamma_i \geq j\}$ and, by the definition of $\mathcal{M}$, $\mathcal{O}_i \cap \mathcal{O}^j$ is non empty if, and only if, $\gamma_i \geq j$. Therefore, $\mathcal{O}_i \cap \mathcal{O}^j$ is non empty if, and only if $1 \leq i \leq \gamma^j$.
We can write:

$$\mathcal{R} = \mathcal{S}\left(\mathcal{O}^1\right) \times \cdots \times \mathcal{S}\left(\mathcal{O}^{\gamma'}\right)$$

Consider $\mathcal{O}_i^j$ defined by:

$$\mathcal{O}_i^j = \bigcup_{\gamma^k = \gamma^j} (\mathcal{O}_i \cap \mathcal{O}_k)$$

The class γ will be called *$p^\star$-regular* if $\gamma_{ij} < p$ for all pairs i, j. The class γ will be *p-regular* if p does not divide γ^i for $i = 1, .., k$.
It is known that the number of non isomorphic irreducible representations of $\mathcal{S}_n$ is the number of p-regular classes. It is possible to establish that the number of p-regular classes and the number of $p^\star$-regular classes are the same.
Let $\mathcal{N}$ be the group formed by the elements of $\mathcal{C}$ which exchange the $\mathcal{R}$-orbits in $\mathcal{I}_n$.
For $\kappa \in \mathcal{R}$ we put:

$$\mathcal{C}^{\kappa} = \kappa\,\mathcal{C}\,\kappa^{-1}, \quad \kappa_{ij} = |\kappa\,\mathcal{O}_i \cap \mathcal{O}_j|$$

that is to say, κ_{ij} is the number of elements in $\mathcal{O}_i$ which κ sends in $\mathcal{O}_j$. With the preceding notations we have:

Lema 2.1: *i) $\mathcal{N}$ is the normalizer of $\mathcal{R}$ in $\mathcal{C}$.*
*ii) The $\mathcal{N}$-*orbits *are the sets $\mathcal{O}_i^j$ which are non empty and $|\mathcal{N}| = \prod_j (\gamma_{1j})!$.*
iii) For $\kappa \in \mathcal{R}$ it holds that $|\mathcal{C} \cap \mathcal{C}^{\kappa}| = \prod_{i,j} (\kappa_{ij})!$.

Proof: i) If $\eta \in \mathcal{N}$, then we write $\eta\,\mathcal{O}^i = \mathcal{O}^{\eta(i)}$ and we have:

$$\eta\,\mathcal{R}\,\eta^{-1} = \times_i \mathcal{S}\left(\mathcal{O}^{\eta(i)}\right) = \mathcal{R}$$

On the other hand, if μ normalizes $\mathcal{R}$ one has:

$$\mathcal{R} = \mu\,\mathcal{R}\,\mu^{-1} = \times_i \mathcal{S}\left(\mu\,\mathcal{O}^i\right)$$

so that μ should exchange these $\mathcal{R}$-orbits.
ii) Given $\mathcal{O}^j$ and $\mathcal{O}^k$ such that $\gamma^j = \gamma^k$, there exists a unique $\pi \in \mathcal{C}$ verifying:

$$\pi\,\mathcal{O}^j = \mathcal{O}^k, \pi\,\mathcal{O}^k = \mathcal{O}^j \quad \text{and} \quad \pi\,\mathcal{O}^l = \mathcal{O}^l \;\; \forall \;\; l \neq j, k$$

so that $\mathcal{N}$ operates as a symmetric group on those $\mathcal{R}$- orbits of the same cardinality. Therefore:

$$|\mathcal{N}| = \prod_j (\gamma_{1j})!.$$

iii) From the identities:

$$\mathcal{C} = \times_i \mathcal{S}\left(\mathcal{O}_i\right), \quad \mathcal{C}^{\kappa} = \times_i \mathcal{S}\left(\kappa\,\mathcal{O}_i\right)$$

it follows that:

$$\times_{i,j} \mathcal{S}\left(\kappa\,\mathcal{O}_i \cap \mathcal{O}_j\right) \subseteq \mathcal{C} \cap \mathcal{C}^{\kappa}$$

Conversely, for $\nu \in \mathcal{C} \cap \mathcal{C}^{\kappa}$ we have:

$$\nu\kappa\,\mathcal{O}_i = \kappa\,\mathcal{O}_i \quad \text{and} \quad \nu\,\mathcal{O}_j = \mathcal{O}_j \;\; \forall i, j$$

then:

$$\nu\;\left(\kappa\,\mathcal{O}_i \cap \mathcal{O}_j\right) = \kappa\,\mathcal{O}_i \cap \mathcal{O}_j$$

so that $v \in \times_{i,j} \mathcal{S}\left(\kappa\,\mathcal{O}_i \cap \mathcal{O}_j\right)$. From this we infer that $|\mathcal{C} \cap \mathcal{C}^{\kappa}| = \prod_{i,j} (\kappa_{ij})!$. ∎

Given i, j , such that $1 \leq i, j \leq m$, consider $\mathcal{O}^{ij}$ defined by:

$$\mathcal{O}^{ij} = \bigcup_{\gamma^k = i+j-1} \mathcal{O}^k$$

Therefore, the collection $\{\mathcal{O}_h \cap \mathcal{O}^{ij} : \mathcal{O}_h \cap \mathcal{O}^{ij} \neq \emptyset\}$ is the set of the $\mathcal{N}$-orbits in $\mathcal{I}_n$.
Let $\tau \in \mathcal{R}$ be the transformation that exchanges the $\mathcal{N}$-orbits:

$$\mathcal{O}_i \cap \mathcal{O}^{ij} \text{ and } \mathcal{O}_j \cap \mathcal{O}^{ij}$$

Notice that τ is well defined, because if $l \in \mathcal{O}_i \cap \mathcal{O}^k$, then putting $j = \gamma^k - i + 1$, we have that $\mathcal{O}_j \cap \mathcal{O}^k$ is non empty, hence $\tau(l)$ is the unique element in $\mathcal{O}_j \cap \mathcal{O}^k$. In addition, we have $\tau_{ij} = \gamma_{ij}$.With this notation we have:

Lema 2.2: *i) Let* $\pi \in \mathcal{S}_n$ *be. If* $< e_\sigma, \pi\, e_\sigma > \neq 0$, *then* $\pi \in \mathcal{CRC}$.
ii) $< e_\mu, e_\nu > \in \mathbb{Z}\,|\mathcal{N}| \;\; \forall\, \mu, \nu \in \gamma$.
iii) If $\kappa \in \mathcal{R}$ *verifies* $\kappa_{ij} = \gamma_{ij} \;\; \forall\, i, j$, *then* $\kappa = \tau$.
iv) $< e_\sigma, \tau\, e_\sigma > = \prod_{i,j} (\gamma_{ij})!$.

Proof: i) The monomials that appear in e_σ are in fact $\mu\,\mathcal{M}$ with $\mu \in \mathcal{C}$, where $\mathcal{M} = \mathcal{M}_\sigma$, which implies that the monomials in $\pi\, e_\sigma$ are $\pi\mu\,\mathcal{M}$. If $< e_\sigma, \pi\, e_\sigma > \neq 0$, then there are μ and ν in $\mathcal{C}$ such that $\pi\mu\,\mathcal{M} = \nu\,\mathcal{M}$,i.e. $\nu^{-1}\pi\mu \in \mathcal{R}$ and therefore, we have $\pi \in \mathcal{CRC}$.
ii) If $\sigma = \varepsilon\mu\varepsilon^{-1} = \lambda\nu\lambda^{-1}$, then $e_\sigma = \varepsilon\, e_\mu = \lambda\, e_\nu$, and hence:

$$< e_\mu, e_\nu > = < e_\sigma, \pi\, e_\sigma > \text{ with } \pi = \varepsilon\lambda^{-1}$$

If $< e_\sigma, \pi\, e_\sigma > \neq 0$, then by *i)* we must have $\pi \in \mathcal{CRC}$ and, writing $\pi = \alpha\eta\beta$ with α,$\beta \in \mathcal{C}$ and $\eta \in \mathcal{R}$, we have:

$$< e_\mu, e_\nu > = < e_\sigma, \alpha\eta\beta\, e_\sigma > = sg\,(\alpha\beta) < e_\sigma, \eta\, e_\sigma >$$

Because $\mathcal{N}$ is the normalizer of $\mathcal{R}$ in $\mathcal{C}$, for ξ in $\mathcal{N}$ and ϑ in $\mathcal{R}$ it holds that:

$$\vartheta\xi\,\mathcal{M} = \xi\left(\xi^{-1}\vartheta\xi\right)\,\mathcal{M} = \xi\,\mathcal{M}$$

Let us write

$$\Delta_{\mathcal{N}} = \sum_{\xi \in \mathcal{N}} sg\,(\xi)\,\xi$$

hence:

$$\vartheta\,\Delta_{\mathcal{N}}\,(\mathcal{M}) = \Delta_{\mathcal{N}}\,(\mathcal{M}) \;\; \forall\, \vartheta \in \mathcal{R}$$

If $\phi_1, ..., \phi_s$ are the representatives of the right cosets of $\mathcal{N}$ in $\mathcal{C}$, one has:

$$e_\sigma = \sum_i sg\,(\phi_i)\,\phi_i\,\Delta_{\mathcal{N}}\,(\mathcal{M}) \;\text{ and }\; \pi e_\sigma = \sum_i sg\,(\phi_i)\,(\pi\phi_i)\,\Delta_{\mathcal{N}}\,(\mathcal{M})$$

hence:

$$< e_\sigma, \pi e_\sigma > = \sum_{i,j} sg\,(\phi_i\phi_j)\,\langle \phi_i\,\Delta_{\mathcal{N}}\,(\mathcal{M})\,, \pi\phi_j\,\Delta_{\mathcal{N}}\,(\mathcal{M})\rangle$$

If $\langle \phi_i\,\Delta_{\mathcal{N}}\,(\mathcal{M})\,, \pi\phi_j\,\Delta_{\mathcal{N}}\,(\mathcal{M})\rangle \neq 0$, then there exists ε and $\delta \in \mathcal{N}$ and $\varphi \in \mathcal{R}$ such that:

$$\phi_i \varepsilon = \pi \phi_j \delta \varphi$$

and from this follows that:

$$sg(\varepsilon)\ \phi_i \Delta_{\mathcal{N}}(\mathcal{M}) = \phi_i \varepsilon\, \Delta_{\mathcal{N}}(\mathcal{M}) = \pi \phi_j \delta \varphi\, \Delta_{\mathcal{N}}(\mathcal{M}) = sg(\delta)\ \pi \phi_j\, \Delta_{\mathcal{N}}(\mathcal{M})$$

and then:

$$\langle \phi_i \Delta_{\mathcal{N}}(\mathcal{M}), \pi \phi_j \Delta_{\mathcal{N}}(\mathcal{M}) \rangle = \pm \langle \Delta_{\mathcal{N}}(\mathcal{M}), \Delta_{\mathcal{N}}(\mathcal{M}) \rangle = \pm |\mathcal{N}|$$

that is, $< e_\mu, e_\nu > \in \mathbf{Z}\, |\mathcal{N}|\ \forall \mu, \nu \in \gamma$.
iii) Let us suppose that $\kappa \in \mathcal{R}$ verifies:

$$\kappa_{ij} = \tau_{ij}\ \forall\, i, j$$

Let O be an $\mathcal{N}$-orbit such that $\mathcal{R}\,O = O_1 \cup \cdots \cup O\gamma^k$, let i, j be indexes with $i < j$, $i + j = \gamma^k + 1$ and write:

$$O_i = \mathcal{R}\,O \cap \mathcal{O}_i \ , \ O_j = \mathcal{R}\,O \cap \mathcal{O}_j$$

If $\kappa\,(O_i) \not\subset O_j$, then let us consider the pair i, j such that $j - i$ is maximum with this condition. Because $\kappa_{ij} = \tau_{ij} = |O|$, there is $h \in \mathcal{O}_i - O$ such that $\kappa\,(h) \in \mathcal{O}_j$. Let O' be the $\mathcal{N}$-orbit of h, hence:

$$\mathcal{R}\,O' = O'_1 \cup \cdots \cup O'_{\gamma^r}$$

with $\eta\,(O'_i) \cap \mathcal{O}_j \neq \emptyset$.

$$O'_i = \mathcal{R}\,O' \cap \mathcal{O}_i \ , \ O'_j = \mathcal{R}\,O' \cap \mathcal{O}_j$$

Let $s = \gamma^r - i + 1$, if $\gamma^r > \gamma^k$, then :

$$s - i = 2i - \gamma^r - 1 > 2i - \gamma^k - 1 = j - i$$

and from this we infer that:

$$\kappa\,(O'_i) = O'_s$$

but this contradicts $\kappa\,(O'_i) \cap \mathcal{O}_j \neq \emptyset$ because $j \neq s$.
If $\gamma^r < \gamma^k$, then let $t = \gamma^r - j + 1$, hence:

$$j - t = 2j - \gamma^r - 1 > 2j - \gamma^k - 1 = j - i$$

so that $\kappa\,(O'_t) = O'_j$, but again, this contradicts $\kappa\,(h) \in O'_j$. We conclude that $\kappa = \tau$.
iv) From *iii)* it follows that $\mathcal{C}\,\tau\,\mathcal{C} \cap \mathcal{R} = \{\tau\}$. Indeed, let $\eta \in \mathcal{R}$, ν , $\mu \in \mathcal{C}$ be such that $\eta = \nu \tau \mu$, then we have:

$$\begin{aligned}\eta_{ij} &= |\eta\, \mathcal{O}_i \cap \mathcal{O}_j| = |\nu\tau\mu\, \mathcal{O}_i \cap \mathcal{O}_j| = |\nu\tau\, \mathcal{O}_i \cap \mathcal{O}_j| \\ &= |\tau\, \mathcal{O}_i \cap \nu^{-1}\mathcal{O}_j| = |\tau\, \mathcal{O}_i \cap \mathcal{O}_j| = \tau_{ij}\end{aligned}$$

Let $\mathcal{U} = \mathcal{C} \cap \mathcal{C}^{\tau}$, since:

$$\Delta_\sigma(\tau\, e_\sigma) = <\tau\, e_\sigma, e_\sigma>\, e_\sigma$$

we infer that:

$$\begin{aligned}<\tau\, e_\sigma, e_\sigma> &= <\Delta_\sigma(\tau\, e_\sigma), \mathcal{M}_\sigma> = <\Delta_\sigma\, \tau\, \Delta_\sigma\, \mathcal{M}_\sigma, \mathcal{M}_\sigma> \\ &= \sum_{\nu,\mu\in\mathcal{C}} sg(\upsilon\mu) < \nu\tau\mu\, \mathcal{M}_\sigma, \mathcal{M}_\sigma>\end{aligned}$$

but $< \nu\tau\mu\, \mathcal{M}_\sigma, \mathcal{M}_\sigma > \neq 0$ if, and only if $\nu\tau\mu \in \mathcal{R}$, that is to say $\nu\tau\mu = \tau$, and this means that ν and μ are both in $\mathcal{U}$ and $\mu = \tau\nu^{-1}\tau^{-1}$. It follow that:

$$<\tau\, e_\sigma, e_\sigma> = |\mathcal{U}| = \prod_{i,j} \tau_{ij}! = \prod_{i,j} \gamma_{ij}! \blacksquare$$

If γ is any conjugation class of S_n and $\sigma \in \gamma$, we define the linear map:

$$f_\sigma : \mathcal{W}_\gamma \to \mathbf{K} \quad \text{by} \quad f_\sigma(x) = \langle e_\sigma, x\rangle$$

Let

$$\widetilde{\mathcal{W}_\gamma} = \langle f_\sigma : \sigma \in \gamma\rangle$$

be the subspace of the dual space $\mathcal{W}_\gamma^*$ generated by the maps f_σ. Let $\tilde{\rho}_\gamma$ denote the natural representation of S_n on $\widetilde{\mathcal{W}_\gamma}$. With this notations we have:

Theorem 2.3: *i)* $\widetilde{\mathcal{W}_\gamma} \neq 0$ *if, and only if,* γ *is* p^**-regular.*
ii) If γ *is* p^**-regular then* $\tilde{\rho}_\gamma$ *is irreducible.*
iii) If γ *and* γ' *are* p^**-regular,* $\gamma \neq \gamma'$*, then* $\tilde{\rho}_\gamma \not\cong \tilde{\rho}_{\gamma'}$.
iv) Every irreducible representation of S_n *is equivalent to* $\tilde{\rho}_\gamma$ *for same* p^**-regular class* γ.

Proof: *i)* is a consequence of lemma 2, *iv)*.
ii) Let $\mathcal{S} \subseteq \widetilde{\mathcal{W}_\gamma}$ be an S_n-invariant subspace. We have:

$$\widetilde{\mathcal{W}_\gamma}^{\mathrm{o}} \subseteq \mathcal{S}^{\mathrm{o}}$$

where $\mathcal{T}^{\mathrm{o}}$ denotes the annihilator space of $\mathcal{T}$. If $\mathcal{S} \neq \widetilde{\mathcal{W}_\gamma}$, there are $x \in \mathcal{S}^{\mathrm{o}}$ and $\sigma \in \gamma$ such that:

$$f_\sigma(x) = \langle e_\sigma, x\rangle \neq 0$$

Since $\mathcal{S}^{\mathrm{o}}$ is S_n-invariant, we have:

$$\Delta_\sigma(x) = \langle e_\sigma, x\rangle\, e_\sigma \in \mathcal{S}^{\mathrm{o}}$$

then $e_\sigma \in \mathcal{S}^{\mathrm{o}}$. It follows that $\mathcal{S}^{\mathrm{o}} = \mathcal{W}_\gamma$, and so, $\mathcal{S} = 0$.
iii) Given σ, $\tau \in \gamma$ we have:

$$(\Delta_\sigma f_T)(e_T) = \langle e_T, \Delta_\sigma(e_t)\rangle = \langle e_\sigma, e_T\rangle^2 = f_\sigma(e_T)^2$$

then $\Delta_\sigma\left(\widetilde{\mathcal{W}_\gamma}\right) = 0$ if, and only if, $f_\sigma = 0$. The proof of *iii)* is now similar to the proof of theorem 1.5, *ii)*.
iv) Because the number of p-regular classes and the number of $p^\star$-regular classes are the same, *iv)* follows. ■

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