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MULTIFRACTAL FORMALISM OF THE FAREY PARTITION.†

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ABSTRACT: The Farey-Brocot partition P of the unit segment I induces a probability measure μ on a universal class of fractal sets Ω that occur in Physics and other disciplines. Key properties of the Multifractal formalism $(\alpha, f(\alpha))$ of (Ω, μ) can be derived from the Multifractal formalism of (I, P). In this paper we study some properties of the latter. We find a significant discrepancy between the Theory of Multifractal spectra $(\alpha, f(\alpha))$ of general sets Ω and the spectrum of our concrete example (I, P). The proofs in this paper include some interesting generalizations of classical results in Number Theory.

1. INTRODUCTION.

1.1 THE PHYSICAL MOTIVATION FOR STUDYING THE α-INDEX.

In order to construct a Cantor set K, we depart from a unit segment [0,1], take away its central third, then take away the central thirds of the smaller segments remaining,... and iterate this procedure *ad infinitum.* This is an example of what physicists call a "mathematical" fractal set. We have $dim_H(K) = log(2)/log(3)$, a number strictly between 0 and 1, where $dim_H(K)$ is the Hausdorff dimension of the set K. However, there are other types of fractal sets Ω arising from the study of physical phenomena. Let us consider the forced pendulum, with internal frequency ω. When plotting the winding number W as a function g of ω, we have that, for a certain critical value of the parameters involved, $W = g(\omega)$ is a Cantor-like staircase. It means that $g(\omega)$ is constant in the so called intervals of resonance I_k (k a natural number) of the variable ω, each I_k producing a step of the staircase. The complement of the union of the interior of the intervals I_k is a

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fractal set Ω, and $dim_H(\Omega)$ is again a number strictly between 0 and 1 [1]. This Ω, given by a natural process or a physical phenomenon, is very different from K. It does not have the regularity of self similarity shown in the process of formation of K.

Cvitanovic, Jensen, Kadanoff, and Procaccia [2] dicovered a property of the staircase $W = g(\omega)$: Let $\frac{P}{Q}$ and $\frac{P'}{Q'}$ be the values of W for a pair of intervals of resonance I and I', such that all intervals of resonance in the gap between I and I' are smaller in size than both I and I'. Then, there is an interval of resonance I'' in this gap such that the corresponding constant value of W is $\frac{P''}{Q''} = \frac{P+P'}{Q+Q'}$. This interval I'' is the widest of all intervals of resonance in the gap between I and I'. This is a purely empirical finding.

Now, given two positive rationals $\frac{P}{Q}$ and $\frac{P'}{Q'}$, we have that $\frac{P}{Q} < \frac{P+P'}{Q+Q'} < \frac{P'}{Q'}$. This way of interpolating a rational number $\frac{P+P'}{Q+Q'}$ strictly between two others is known as the Farey-Brocot (F-B) interpolation. By (F-B) interpolating rational numbers between $\frac{P}{Q} = \frac{0}{1}$ and $\frac{P'}{Q'} = \frac{1}{1}$ we obtain finer and finer (F-B) partitions of the unit segment [0,1]:

$$(F-B)_0 = \{\frac{0}{1}, \frac{1}{1}\};\ (F-B)_1 = \{\frac{0}{1}, \frac{0+1}{1+1}, \frac{1}{1}\};$$

$$(F-B)_2 = \{\frac{0}{1}, \frac{0+1}{1+2}, \frac{1}{2}, \frac{1+1}{2+1}, \frac{1}{1}\},\ ...etc$$

Notice that $(F-B)_n$ divides [0,1] into 2^n segments. By assigning equal measure to each of these we induce a measure P in [0,1] called the (F-B) measure.

Now, let us go back to our Cantor staircase $W = g(\omega)$, $W \in [0,1]$. The probability measure P in [0,1] induces via $W = g(\omega)$, another probability measure μ on Ω. This μ measure on the ω axis is called the Farey tree partition of Ω.

Our "non-mathematical" fractal set Ω is now a measure space (Ω, μ).

Cvitanovic et al. [2] and Halsey et al. [3] have different examples of physical phenomena exhibiting Cantor staircases with such (F-B) arrangements. Bruinsma and Bak [4] studied the magnetic structure of ferromagnetic quasicrystals by plotting the ratio of up spins against the strength of the magnetic field applied to the quasicrystalline structure, when only 2 values of each spin are allowed, i.e. + and -, or up and down. Again the result is a Cantor staircase with the (F-B) arrangement [5]. We can find this arrangement in some of the staircases shown in [1] including the one associated with the chemical reaction of Belusov-Zabotinsky.

Procaccia, Jensen and others [3] devised a way of decomposing a "natural" —as opposed to "mathematical" — fractal (Ω, μ) into more self-similar subsets $\Omega_\alpha \subset \Omega$, $\alpha \in [\alpha_{min}, \alpha_{max}]$, an interval on the real line. If we denote by $f(\alpha)$ the $\dim_H(\Omega_\alpha)$, then the curve $(\alpha, f(\alpha))$ is considered an important characteristic of

the physical phenomenon yielding the fractal (Ω, μ). Such a curve is called the multifractal spectrum of (Ω, μ).
Key properties of multifractal spectra of such measure spaces (Ω, μ) are due to the Farey tree partition μ —a partition inherited from the (F-B) probability measure P of the unit segment. We found a significant discrepancy between the theory of Multifractal Spectra $(\alpha, f(\alpha))$ and concrete examples: some statements of the general theory are not true (see below) for the spectrum $(\alpha, f(\alpha))$ of the unit segment with the (F-B) measure P.

NOTE. Many ideas underlying this paper come from previous papers: we studied geometrical properties of the (F-B) partition in [6]; in [7], [8], and [9] we began to study the multifractal spectrum of the (F-B) partition P, whereas in [10], [4], and [11] we studied connections between the $(\alpha, f(\alpha))$ spectrum of the (F-B) measure of $[0,1]$ and Number Theory.

1.2. A NOTE FOR THOSE INTERESTED IN NUMBER THEORY. AN EXTENSION OF JARNIK CLASSES.

The (F-B) partition is naturally associated with the decomposition of an irrational number ξ in continued fractions, $\xi = [n_1, n_2, ..., n_N, ...]$, as will be detailed below. Jarnik is concerned with the set E_m of irrational numbers $\xi = [n_1, n_2 ... n_N ...]$ such that $n_N \leq m \ \forall N$. He proves [12] that the $\dim_H(E_m)$ grows very much like $1 - \frac{constant}{m}$. Notice that Theorem 4 below is an extension of this result (so far as we know, the first one since 1928): we obtain an analogous result when the Jarnik condition "$n_N \leq m$" is replaced by the more general one $\frac{\sum_{i=1}^{N} n_i}{N} \leq m \ \forall N$.

2. NOTATION.

Any real number $\xi \in (0,1)$ can be expressed as a continued fraction:

$$\xi = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \ddots}}} = [n_1, n_2, n_3, ...], \quad \text{with } n_i \in I\!N.$$

The sequence is finite if and only if ξ is rational.
If ξ is irrational, and we consider the N^{th} rational approximant to ξ

$$\frac{p_N}{q_N} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ddots \cfrac{1}{n_N}}} = [n_1, n_2, ..., n_N],$$

then q_N is the so-called "cumulant", a polynomial in the variables $n_1, ..., n_N$.

Let $\Omega \subset I\!R$ be a fractal set constructed in steps by an iterative process. Let C_k be the covering (by intervals I_k) of Ω given by the canonical partition of Ω in the k^{th} step of its construction —e.g. the set of 2^k intervals I_k of length $\frac{1}{3^k}$ that cover the Cantor ternary in the k^{th} step of its construction.
We consider Ω endowed with a probability measure P.
Following Procaccia, Jensen and others [3], we recall that the α-index of Procaccia relates lengths of intervals I_k in C_k to the corresponding probabilities $P(I_k \cap \Omega)$ thus:

$$P(I_k \cap \Omega) = |I_k|^{\alpha(I_k)}$$

or

$$\alpha(I_k) = \frac{log(P(I_k \cap \Omega))}{log(|I_k|)}$$

where $| \ |$ denotes the usual measure in the real line.
Now let Ω be the unit segment and let P be its (F-B) partition. Let $\xi \in (0,1)$. Let k be a natural number. In the k^{th} step of (F-B), there is a unique $I_k = I_k(\xi)$ to which ξ belongs. The k^{th} approximation of the α-index of ξ —abbreviated as $\alpha^k(\xi)$— is by definition,

$$\alpha^k(\xi) = \frac{log(P(I_k(\xi) \cap \Omega))}{log(|I_k(\xi)|)} = \frac{log(1/2^k)}{log(|I_k(\xi)|)}$$

and

$$\alpha(\xi) = \lim_{k \to \infty} \alpha^k(\xi)$$

when this limit exists.

3. THE THEOREMS.

In the next sections we prove

Theorem 1. *Let*

$$\xi = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \frac{1}{n_3 + \ldots}}} = [n_1, n_2, n_3 \ldots],$$

each n_i a natural number.

We have: the sequence $\{\alpha^k(\xi)\}$ *is bounded if and only if the sequence* $\{\frac{\sum_{i=1}^k n_i}{k}\}$ *is bounded. Moreover, if* $A_M = \{\xi/\ \alpha^k(\xi) \leq M\ \forall k\}$, *M an arbitrarily large constant, then there exists* $K = K(M) > 0$ *such that* $\frac{\sum_{i=1}^k n_i}{k} \leq K\ \forall k$, *for any* $\xi \in A_M$.

Theorem 2. *The set of all* ξ *such that the sequence* $\{\alpha^k(\xi)\}$ *is bounded has zero Lebesgue measure.*

Theorem 3. *The set of all* ξ *such that the sequence* $\{\alpha^k(\xi)\}$ *is bounded has Hausdorff dimension unity.*

Theorem 4. *Let* $m > 1$ *be an arbitrarily large constant. Then*

$$\dim_H(\{\xi = [n_1, n_2, ..., n_N ...]\ /\ \frac{\sum_{i=1}^N n_i}{N} \leq m\ \forall\ N\ \})\ <\ 1.$$

From Theorems 1 to 4 we can infer

Corollary.

$$\dim_H(\{\xi/\ \alpha_k(\xi) \leq M\ \forall\ k\})\ <\ 1.$$

From this we can infer:

4. CONCLUSION.

Multifractality theory states that the multifractal spectrum $(\alpha, f(\alpha))$ of a fractal Ω reaches a maximum

$$\max{}_{\alpha\in[\alpha_{min},\ \alpha_{max}]}\ f(\alpha)\ =\ \dim_H(\Omega).$$

Nevertheless, from our corollary we infer that, for the $(\alpha, f(\alpha))$ corresponding to the Farey tree in the unit segment we have

$$\max{}_{\alpha\in[\alpha_{min},\ \alpha_{max}]}\ f(\alpha)\ <\ 1 = \dim_H([0,1]).$$

There is, then, a significant discrepancy between multifractal theory and this example —the Farey partition. Since this example is, as noticed above, both common and important in Physics and other disciplines, the physical interpretation of this discrepancy should be explored.

It remains to prove Theorems 1 to 4 now.

5. PROOF OF THEOREM 1.

Following the definition of $\alpha^k(\xi)$, we have to estimate the length of $I_k(\xi)$.

Claim 1. *Let $\xi = [n_1, ..., n_i, ...]$. Let $k \in I\!N$, and let us write $k = \sum_{i=1}^{N} n_i + h$; $N = N(k)$; $0 \leq h < n_{N+1}$. Then we have:*

$$|I_k(\xi)| = \frac{1}{q_N(hq_N + q_{N-1})},$$

where q_N is the cumulant of the N^{th} step of the development of ξ as a continued fraction.

Proof of Claim 1.

$$\xi \in \left(0, \frac{1}{n_1}\right) \quad \text{since} \quad \cfrac{1}{n_2 + \cfrac{1}{n_3 + \ddots}} > 0,$$

and therefore

$$\xi = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots}}} < \frac{1}{n_1}:$$

Step 1: $\frac{0}{1} < \xi < \frac{1}{1}$; $\xi \in [0,1] = I_1(\xi)$; $|I_1(\xi)| = 1$,

Step 2: $\frac{0}{1} < \xi < \frac{1}{2} < \frac{1}{1}$; $\xi \in [0, \frac{1}{2}] = I_2(\xi), |I_2(\xi)| = \frac{1}{2}$,

$\vdots$

Step n_1: $\frac{0}{1} < \xi < \frac{1}{n_1} < \frac{1}{2}$; $\xi \in [0, \frac{1}{n_1}] = I_{n_1}(\xi)$; $|I_{n_1}(\xi)| = \frac{1}{n_1}$.

On the other hand, $\xi = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{\ddots}}} > \frac{1}{n_1+1}$ $\left(\text{since } \frac{1}{n_2 + \frac{1}{\ddots}} < 1\right)$, and therefore

$\xi \in \left(\frac{1}{n_1+1}, \frac{1}{n_1}\right)$:

Step n_1+1: $\frac{0}{1} < \frac{1}{n_1+1} < \xi < \frac{1}{n_1} < \frac{1}{2}$; $\xi \in \left[\frac{1}{n_1+1}, \frac{1}{n_1}\right] = I_{n_1+1}(\xi)$;

$$|I_{n_1+1}(\xi)| = \frac{1}{n_1(n_1+1)}.$$

Let us suppose that $2 < n_2$. Then, $\frac{2}{2n_1+1} = \frac{1}{n_1+\frac{1}{2}} < \xi$, and $\xi \in \left[\frac{2}{2n_1+1}, \frac{1}{n_1}\right]$:

Step n_1+2: $\frac{0}{1} < \frac{1}{n_1+1} < \frac{2}{2n_1+1} < \xi < \frac{1}{n_1}$; $\xi \in \left[\frac{2}{2n_1+1}, \frac{1}{n_1}\right] = I_{n_1+2}(\xi)$;

$$|I_{n_1+2}(\xi)| = \frac{1}{n_1(2n_1+1)}.$$

$\vdots$

Step n_1+h: $\frac{0}{1} < \frac{1}{n_1+1} < \frac{2}{2n_1+1} < \frac{h}{hn_1+1} < \xi < \frac{1}{n_1}$; $|I_{n_1+h}(\xi)| = \frac{1}{n_1(hn_1+1)}$;

$\vdots$

Step n_1+n_2: $\frac{0}{1} < \frac{n_2}{n_2n_1+1} < \xi < \frac{1}{n_1}$; $|I_{n_1+n_2}(\xi)| = \frac{1}{n_1(n_2n_1+1)}$.

Now, let us recall that $q_0 = 1$, $q_1 = n_1$, and $q_2 = n_1n_2+1$, where q_i is the i^{th} cumulant polynomial, so $|I_{n_1+h}(\xi)| = \frac{1}{q_1(hq_1+q_0)}$, and $|I_{n_1+n_2}(\xi)| = \frac{1}{q_1q_2}$. Iterating this process a few more steps, and knowing that $q_{N+1} = n_{N+1}q_N + q_{N-1}$, we have

$$|I_{n_1+n_2+h}(\xi)| = \frac{1}{q_2(hq_2+q_1)},$$

and

$$|I_{n_1+n_2+n_3}(\xi)| = \frac{1}{q_2(n_3q_2+q_1)} = \frac{1}{q_2q_3}, \ldots$$

and, in general, we obtain the result of Claim 1.

The proof of Theorem 1 is divided into two cases:
A) $h=0$, B) $h \neq 0$, where h is as in Claim 1.

From Claim 1 we know that, with $k = \sum_{i=1}^{N} n_i + h$, $0 \le h < n_{N+1}$, we have

$$\alpha^k(\xi) = \frac{\ln 1/2^k}{\ln|I_k(\xi)|} = \frac{\ln 2\left(\sum_{i=1}^{N} n_i + h\right)}{\ln\{q_N(hq_N + q_{N-1})\}}.$$

A) Let $h = 0$. In this case the corresponding formula for α is:

$$\alpha^{k(N)}(\xi) = \alpha^N(\xi) = \frac{\ln 2\sum_{i=1}^N n_i}{\ln\left(q_N q_{N-1}\right)},$$

and, since $q_N^2 \geq q_N q_{N-1} \geq q_N$, we obtain:

$$\ln 2\frac{\sum_{i=1}^N n_i}{2\ln q_N} \leq \frac{\ln 2\sum_{i=1}^N n_i}{\ln(q_N q_{N-1})} \leq \frac{\ln 2\sum_{i=1}^N n_i}{\ln q_N},$$

so, for the purpose of studying the finiteness of $\alpha^N(\xi)$, we will deal with $\alpha^N(\xi) = \frac{\sum_{i=1}^N n_i}{\ln q_N}$ for short.

We will find upper and lower estimates for q_N:

Claim 2. *Let $\xi = [n_1, n_2, ...n_i...]$ as before, and let q_N be its N^{th} cumulant polynomial.*
Then $c_1\phi^N \leq q_N \leq c_2\prod_{i=1}^N n_i\ \phi^N$, where $\phi = \frac{1+\sqrt{5}}{2}$ and c_1 and c_2 are absolute constants.

Proof of Claim 2. q_N is a polynomial in the variables $n_1, n_2...n_i...n_N$, each variable n_i appears in each monomial with degree zero or unity, the coefficient of each monomial is unity, and the monomial with the biggest degree is $\prod_{i=1}^N n_i$.
Therefore, in order to estimate the minimal value $\overline{q}_N$ of q_N, we have to give to each variable its minimal value $n_i = 1$. Then, $\overline{q}_N$ is the total number of monomials in q_N.
We want to calculate this number:

$$\frac{\overline{p_N}}{\overline{q_N}} = \frac{1}{1+\frac{1}{1+\ddots_1}} = [\underbrace{1, 1...1}_{N\ \text{times}}],$$

from which $\overline{q_N} = F_N$, the N^{th} Fibonacci number. Fibonacci numbers follow the rule $F_k = F_{k-1} + F_{k-2}$, $k \in I\!N$, $F_0 = 1$, $F_1 = 1$; and they are well approximated below and above by $c_1\phi^k$ and $c_2\phi^k$ respectively for high values of k, where $\phi = \frac{1+\sqrt{5}}{2}$ and c_1 and c_2 are absolute constants.
This finishes the first inequality of Claim 1.

In order to tackle the second inequality we need to observe that the largest monomial in q_N is $\prod_{i=1}^N n_i$, and we have F_N monomials altogether; therefore,

$$q_N \leq F_N \prod_{i=1}^N n_i \leq c_2\phi^N \prod_{i=1}^N n_i,$$

which finishes the proof of Claim 2.

Let us continue with the proof of Theorem 1.
Let us recall that $\alpha^N(\xi) = \frac{\sum_{i=1}^N n_i}{\ln q_N}$.
Using Claim 2 we have

$$\frac{\sum_{i=1}^N n_i}{\ln\left(c_2\phi^N \prod_{i=1}^N n_i\right)} \le \alpha^N(\xi) \le \frac{\sum_{i=1}^N n_i}{\ln \phi^N c_1},$$

from which we obtain:

$$\frac{1}{\frac{N\ln\phi}{\sum\limits_{i=1}^N n_i} + \frac{\sum\limits_{i=1}^N \ln n_i}{\sum\limits_{i=1}^N n_i} + \frac{\ln c_2}{\sum\limits_{i=1}^N n_i}} \le \alpha^N(\xi) \le \frac{\sum\limits_{i=1}^N n_i}{N\left(\ln\phi + \frac{\ln c_1}{N}\right)} \le C\frac{\sum\limits_{i=1}^N n_i}{N}, \tag{1}$$

where C is an absolute constant. From here we obtain that the boundedness of $\left\{\frac{\sum_{i=1}^N n_i}{N}\right\}_{N\in I\!N}$ is a sufficient condition for $\{\alpha^N(\xi)\}_{N\in I\!N}$ to be bounded. It remains to show that it is also a necessary one.
We need

Claim 3. *Let $\{n_i\}_{i\in I\!N}$ be a sequence with $n_i \in I\!N$. If $\frac{\sum_{i=1}^N n_i}{N} \underset{N\to\infty}{\longrightarrow} \infty$, then $\frac{\sum_{i=1}^N \ln n_i}{\sum_{i=1}^N n_i} \underset{N\to\infty}{\longrightarrow} 0$.*

Proof of Claim 3. For each $N \in I\!N$, we will group n_i, $1 \le i \le N$, into two sets: those which do not exceed the average $\frac{\sum_{i=1}^N n_i}{N} = P_N$, and those which do. We write

$$\frac{\sum_{i=1}^N \ln n_i}{\sum_{i=1}^N n_i} = \frac{\sum_{n_i\le P_N} \ln n_i}{\sum_{i=1}^N n_i} + \frac{\sum_{n_i> P_N} \ln n_i}{\sum_{i=1}^N n_i} = A(N) + B(N),$$

and we will prove that $A(N)$ and $B(N)$ tend to zero when $N \to \infty$.
Let us deal first with

$$A(N) = \frac{\sum_{n_i\le P_N} \ln n_i}{\sum_{i=1}^N n_i} \le \frac{\ln P_N \sum_{n_i\in P_N} 1}{\sum_{i=1}^N n_i} \le \frac{\ln P_N \cdot N}{\sum_{i=1}^N n_i} = \frac{\ln P_N}{P_N} \underset{N\to\infty}{\longrightarrow} 0$$

because $P_N \to \infty$ with N.
Next,

$$B(N) = \frac{\sum_{n_i>P_N} \ln n_i}{\sum_{i=1}^N n_i} = \frac{\sum_{n_i>P_N}^N \frac{\ln n_i}{n_i} \cdot n_i}{\sum_{i=1}^N n_i}.$$

Let us estimate the numerator. Let us consider $\left\{\frac{\ln n_i}{n_i}\right\}_{n_i>P_N}$ There are values $n_{\underline{i}}$ and $n_{\bar{i}}$ in $\{n_i\}_{n_i>P_N}$, such that

$$\frac{\ln n_{\underline{i}}}{n_{\underline{i}}} \leq \frac{\ln n_i}{n_i} \leq \frac{\ln n_{\bar{i}}}{n_{\bar{i}}} .$$

Therefore,

$$\frac{\ln n_{\underline{i}}}{n_{\underline{i}}} \sum_{n_i>P_N} n_i \leq \sum_{n_i>P_N} \frac{\ln n_i}{n_i} n_i \leq \frac{\ln n_{\bar{i}}}{n_{\bar{i}}} \sum_{n_i>P_N} n_i$$

and

$$\frac{\ln n_{\underline{i}}}{n_{\underline{i}}} \leq \frac{\sum_{n_i>P_N} \ln n_i}{\sum_{n_i>P_N} n_i} \leq \frac{\ln n_{\bar{i}}}{n_{\bar{i}}}.$$

Now, $\frac{\ln x}{x}$ is monotonically decreasing in $x \in [e,\infty)$. In our case $x > P_N$ which tends to ∞ with N, so we do work in $[e,\infty)$.
Then, there exists λ_N such that

$$\frac{\ln \lambda_N}{\lambda_N} = \frac{\sum_{n_i>P_N} \ln n_i}{\sum_{n_i>P_N} n_i}, \qquad P_N < n_{\underline{i}} \leq \lambda_N \leq n_{\bar{i}} .$$

Since $P_N \to \infty$ with N, so does λ_N; therefore,

$$B(N) = \frac{\sum_{n_i>P_N} \ln n_i}{\sum_{i=1}^{N} n_i} \leq \frac{\sum_{n_i>P_N} \ln n_i}{\sum_{n_i>P_N} n_i} = \frac{\ln \lambda_N}{\lambda_N} \longrightarrow 0 \text{ as } N \to \infty .$$

Let us finish the proof of case A). We want to show that $\left\{\frac{\sum_{i=1}^{N} n_i}{N}\right\}_{N\in I\!N}$ bounded implies $\{\alpha^N(\xi)\}_{N\in I\!N}$ bounded.
Let us suppose that $\left\{\frac{\sum_{i=1}^{N} n_i}{N}\right\}_{N\in I\!N}$ is not bounded. Then there exists a subsequence $\{N_j\} \subset I\!N$ such that

$$P_{N_j} = \frac{\sum_{i=1}^{N_j} n_i}{N_j} \to \infty \quad \text{as} \quad j \to \infty.$$

Then Claim 3 can be adapted so as to imply that

$$\frac{\sum_{i=1}^{N_j} \ln n_i}{\sum_{i=1}^{N_j} n_i} \underset{j\to\infty}{\longrightarrow} 0.$$

For these values N_j the first inequality of Eq.(1) is

$$\frac{1}{\frac{\ln \phi}{P_{N_j}} + \frac{\sum_{i=1}^{N_j} \ln n_i}{\sum_{i=1}^{N_j} n_i} + \frac{\ln c_2}{\sum_{i=1}^{N_j} n_i}} \leq \alpha^{N_j}(\xi)$$

which implies $\alpha^{N_j}(\xi) \underset{j\to\infty}{\longrightarrow} \infty$, which in turn implies that $\{\alpha^N(\xi)\}_{N\in I\!N}$ is unbounded.

Case B) $h \neq 0$. Let us recall that, with $k = \sum_{i=1}^N n_i + h$, $1 \leq h < n_{N+1}$, we had

$$\alpha^k(\xi) = \frac{\sum_{i=1}^N n_i + h}{\ln q_N(hq_N + q_{N-1})},$$

forgetting about $\ln 2$.
Let us suppose $\left\{\frac{\sum_{i=1}^N n_i}{N}\right\}_{N\in I\!N}$ bounded, $\frac{\sum_{i=1}^N n_i}{N} \leq C$, $\forall\ N \in I\!N$. We want to show that $\{\alpha^k(\xi)\}_{k\in I\!N}$ is bounded.
We know that $q_{N-1} + hq_N \geq q_N$; therefore

$$q_N(q_{N-1} + hq_N) \geq q_N^2 \geq F_N^2 \geq c\phi^{2N}.$$

Therefore,

$$\begin{aligned}\alpha^k(\xi) &= \frac{\sum_{i=1}^N n_i + h}{\ln q_N(hq_N + q_{N-1})} \leq \frac{\sum_{i=1}^{N+1} n_i}{\ln c + 2N\ln\phi} \\ &= \frac{\sum_{i=1}^{N+1} n_i}{N+1} \cdot \frac{N+1}{\ln c + 2N\ln\phi} \leq \frac{C}{\ln\phi} \qquad \forall\ k \in I\!N\end{aligned}$$

This proves that $\left\{\frac{\sum_{i=1}^N n_i}{N}\right\}_{N\in I\!N}$ bounded is sufficient for $\{\alpha^k(\xi)\}_{k\in I\!N}$ bounded.
The necessity is proven in case A) q.e.d.

In Theorem 1, we left rational numbers out. The sequence $[n_1, \ldots n_i \ldots]$ associated with ξ is an infinite one:

Proposition. *Let $\xi = [n_1, \ldots, n_k]$ be a rational number. Then $\alpha(\xi) = \infty$.*

We leave the proof as an exercise.

6. PROOF OF THEOREM 2.

It is a consequence of Theorem 1 and a classical theorem by Borel and Bernstein [13, p.167]: "If $\phi(i)$ is any increasing function of i for which $\sum_i 1/\phi(i)$ is divergent, then the set of ξ for which $n_i \leq \phi(i)$ for all sufficiently large i, is null".
Let $F = \{\xi = [n_1, n_2, \ldots n_i \ldots]\ /\ \exists\ c = c(\xi)$ with $\frac{\sum_{i=1}^N n_i}{N} \leq c(\xi)\ \ \forall\ N \in I\!N\}$.
Let F_m denote $\{\xi = [n_1, \ldots n_i \ldots]\ /\ \frac{\sum_{i=1}^N n_i}{N} \leq m\ \forall\ N \in I\!N\}$.
Clearly $F = \bigcup_{m\in I\!N} F_m$. We will show that $|F_m| = 0\ \forall\ m \in I\!N$.
Let $\xi \in F_m$. Then we have:

1) $\frac{n_1}{1} \leq m$, then $n_1 \leq m = 1(m-1)+1$
2) $\frac{n_1+n_2}{2} \leq m$, then $n_2 \leq 2m - n_1 \leq 2m-1 = 2(m-1)+1$.

$\vdots$

N) $\frac{n_1+n_2+\ldots+n_N}{N} \leq m$, then $n_N \leq mN - n_1 - n_2 - \ldots - n_{N-1} \leq mN - (N-1) = N(m-1)+1$.

$\vdots$

Let us write $\phi_m(i) = i(m-1)+1$. $\phi_m(i)$ is increasing, and $\sum_i 1/\phi_m(i)$ is divergent. Since $n_i \leq \phi_m(i)\ \forall\, i \in I\!N$ we have $|F_m| = 0$ by the Borel-Bernstein theorem.

7. PROOF OF THEOREM 3.

We will use a classical theorem of Jarnik [12]:

"Let $E_m = \{\xi = [n_1, ..., n_i, ...] \ / \ n_i \leq m\ \forall\, i \in I\!N\}$.

For $m > 8$, we have

$$1 - \frac{1}{m \log 2} \leq \dim_H E_m \leq 1 - \frac{1}{8m \log m}\text{''}.$$

Let F and F_m be as in Section 6. Obviously $E_m \subset F_m$ and $F_m \subset F\ \forall\, m \in I\!N$. Therefore

$$\dim_H(F) \geq \dim_H(F_m) \geq \dim_H(E_m) \geq 1 - \frac{1}{m \log 2},$$

by Jarnik's theorem. Letting $m \to \infty$ finishes the proof.

8. A SKETCH OF THE PROOF OF THEOREM 4.

From Good [14] we can deduce: Let $\sigma \in (0,1)$. If there exist constants $C = C(\sigma)$ and $n_0 \in I\!N$ such that

$$\sum_{\substack{[n_1,\ldots,n_N] \\ \frac{\sum_{i=1}^r n_i}{r} \leq m\ \forall r \leq N}} \frac{1}{q_N(n_1,\ldots,n_N)^{2\sigma}} \leq C \quad \forall N \geq n_0 \tag{2}$$

Then we have $\dim_H(F_m) \leq \sigma$, where F_m is as in the proof of Theorem 2.

OBSERVATION. Let us consider $\left(E_\infty - \bigcup_{k \in I\!N} E_k\right) \cap F_m = A$. Each element $[n_1 \ldots n_N \ldots] \in A$ has a subsequence $n_{i_j} \to \infty$. For clarity, let us consider that the condition $P_N \leq m \forall N$ is achieved by compensating each n_{i_j} going to infinity with a string of "1's". As $n_{i_j} \to \infty$ so does the length of such string, contributing nothing to the dimension of A. A moment of reflection shows that the n_i responsible for $\dim_H A$ are those bounded (by some constant), which suggests that, in order to study $\dim_H F_m$, it is enough to consider $\dim_H(F_m \cap E_k)\ \forall\, k \in I\!N$. Moreover, the

elements in A have cumulants q_N far larger than those in $F_m \cap E_k$, being thus associated with much smaller intervals in the canonical coverings of $[0,1]$.
We leave out the details of the rigorous proof of this observation, for, on the one hand, they involve long and tedious combinatorial algebra... on the other hand, the underlying idea is simple.

We have to prove that $\dim_H(F_m \cap E_k)$ is bounded away from 1.

Let k be as in E_k, and N be as in $[n_1 ... n_N ...]$. Let us fix both of them, $q_N(n_1, ..., n_N)$ the cumulant associated to $[n_1, ..., n_N]$. Let $[F_m \cap E_k]_N = C_N$ be the set of $[n_1 ... n_N]$ such that $[n_1, ... n_N, n_{N+1} ...] \in F_m \cap E_k$.
Let us partition C_N in disjoint classes $C_N^{\ell_1 ... \ell_k}$: Let $\ell_1 ... \ell_k$ be in $\mathbb{N}$, $0 \le \ell_i \le N$, $\ell_1 + ... + \ell_k = N$. We will say that $[n_1 ... n_N] \in C_N^{\ell_1 ... \ell_k}$ if ℓ_1 elements n_i are equal to 1, ℓ_2 elements n_i are equal to 2,... etc.
The condition $P_N \le m$ implies $\frac{\ell_1 + 2\ell_2 + ... + k\ell_k}{N} \le m$. Now, eq. (2) reads

$$\sum_{\ell_1 ... \ell_k} \sum_{[n_1 ... n_N] \in C_N^{\ell_1 ... \ell_k}} \frac{1}{q_N(n_1 ... n_N)^{2\sigma}} \le C \tag{3}$$

Next, we need two Claims.

Claim 1. $q_N(n_1 ... n_N) \ge \left[\frac{1}{2}\left(\sqrt{\mu^2+4}+\mu\right)\right]^N$, $\mu = 2^{\frac{\ell_2}{N}} ... k^{\frac{\ell_k}{N}} = 2^{\lambda_2} ... k^{\lambda_k}$, *for any* $[n_1, ..., n_N] \in C_N^{\ell_1 ... \ell_k}$

Claim 2.

$$\text{Cardinal}\left(C_N^{\ell_1 ... \ell_k}\right) \cong \left(\frac{1}{\lambda_1^{\lambda_1} \lambda_2^{\lambda_2} ... \lambda_k^{\lambda_k}}\right)^N .$$

With Claims 1 and 2, the left hand side of (3) can be bounded by

$$\sum_{\ell_1 ... \ell_k} \frac{\left(\frac{1}{\lambda_1^{\lambda_1} \lambda_2^{\lambda_2} ... \lambda_k^{\lambda_k}}\right)^N}{\left\{\left[\frac{1}{2}\left(\sqrt{\mu^2+4}+\mu\right)\right]^N\right\}^{2\sigma_k}} = \sum_{\ell_1 ... \ell_k} \left[\frac{\phi(\lambda_1, ..., \lambda_k)}{F(\lambda_1, ..., \lambda_k)^{2\sigma_k}}\right]^N ,$$

for short. The last expression will be bounded uniformly in the variable N when the expressions between [ ] are smaller than unity. The smallest σ_k verifying this condition is the maximum of the function $\sigma_k(\lambda_1, ..., \lambda_k) = \frac{\ln \phi}{2 \ln F}(\lambda_1 ... \lambda_k)$, with the conditions $\lambda_1 + ... + \lambda_k = 1$ and $\lambda_1 + 2\lambda_2 + ... + k\lambda_k \le m$.
An adaptation of the Lagrange method yields a system of equations which can be reduced, after tedious algebra, to a single trascendental one. Numerical methods yield values of σ_k bounded away from unity as $k \to \infty$.

It remains to sketch the proofs of Claims 1 and 2.

Sketch of the Proof of Claim 1. Let us consider

$$[a_1...a_N] = [1,2,3,1,2,3,...1,2,3]; \quad [b_1,...b_N] = [1,1,2,2,3,3,...1,1,2,2,3,3]$$

and

$$[c_1,...c_N] = [1,1,1,2,2,2,3,3,3,...1,1,1,2,2,2,3,3,3] .$$

In all three cases we have $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$, $\lambda_i = 0 \ \forall \ i \geq 4$. We have $q_N(a_1,...a_N) < q_N(b_1,...b_N) < q_N(c_1,...c_N)$; $[a_1...a_n]$ being the most equidistributed arrangement for these values of $\{\lambda_i\}_{1\leq i\leq k}$. The fact is quite general: the smallest q_N, for a certain set of values of λ_i, corresponds to the most equidistributed possible arrangement. The proof of this fact involves elementary but rather long calculations, and we leave it as an exercise to the reader. From now on, $\lambda_1...\lambda_k$ are fixed, and we work with equidistributed arrangements.
Next, we can write q_N as

$$(n_1...n_N)\left\{1 + \sum_i (n_i n_{i+1})^{-1} + \sum_{i_1+1<i_2} (n_{i_1} n_{i_1+1} n_{i_2} n_{i_2+1})^{-1} + ... \right.$$
$$\left. + \sum_{\substack{i_p+1<i_{p+1} \\ 1\leq p\leq r}} (n_{i_1} n_{i_1+1} ... n_{i_r} n_{i_r+1})^{-1} + ... \right\},$$

and let us consider the rth term involved:

$$\sum_{\substack{i_p+1<i_{p+1} \\ 1\leq p\leq r}} \frac{n_1...n_N}{n_{i_1} n_{i_1+1} n_{i_2} ...} . \tag{4}$$

The number of terms in (4) is $\binom{N-r}{r}$ (see [14]). Because of the equidistribution noted above, we have that the proportion of "ones", "twos", "threes"... etc. in each monomial in (4) is, precisely, $\lambda_1...\lambda_k$ —this fact is guaranteed only when the length of such monomials is large, i.e. $N-2r$ has to be large. Then (4) is well approximated by

$$\binom{N-r}{r} 1^{\lambda_1(N-2r)} 2^{\lambda_2(N-2r)} ... k^{\lambda_k(N-2r)} , \tag{5}$$

and a sufficiently accurate expression of q_N is given by the maximum (with r as a variable) of expression (5).
By using Stirling's formula, (5) becomes well approximated by

$$\frac{(N-r)^{N-r}}{r^r (N-2r)^{N-2r}} 2^{\lambda_2(N-2r)} ... k^{\lambda_k(N-2r)} ,$$

and by replacing the discrete variable r by a continuous one $x = \frac{r}{N}$ we have the maximum value of the last expression to be $\left[\frac{\sqrt{\mu^2+4}+\mu}{2}\right]^N$, where $\mu = 2^{\lambda_2}3^{\lambda_3}...k^{\lambda_k}$.

Sketch of the Proof of Claim 2. First let us observe that, if $[n_1...n_N]$ has $\ell_1...\ell_k$ as before, $\frac{\ell_1+2\ell_2+...+k\ell_k}{N} \leq m$, then some cyclic permutation —or rotation— of the n_i is in $C_N^{\ell_1...\ell_k}$.
Given N and $\ell_1, ..., \ell_k$, there are

$$\binom{N}{\ell_1}\binom{N-\ell_1}{\ell_2}\binom{N-\ell_1-\ell_2}{\ell_3}...\binom{\ell_k}{\ell_k}$$

N-strings $[n_1...n_N]$ with such ℓ_i, whether they belong to $C_N^{\ell_1...\ell_k}$ or not. Since there are N of such cyclic permutations, we have

$$\frac{1}{N}\binom{N}{\ell_1}...\binom{\ell_k}{\ell_k} \leq \text{Card}\left(C_N^{\ell_1...\ell_k}\right) \leq \binom{N}{\ell_1}...\binom{\ell_k}{\ell_k},$$

and from this inequality and Stirling's formula we obtain

$$\text{Card}\left(C_N^{\ell_1...\ell_k}\right) \cong \left(\frac{1}{\lambda_1^{\lambda_1}...\lambda_k^{\lambda_k}}\right)^N.$$

NOTE.

We want to stress that the lower bound obtained for q_N in Claim 1 is representative of $C_N^{\ell_1...\ell_k}$, for $\text{Card}\left(C_N^{\ell_1...\ell_k}\right)$ is the Cardinal of the equidistributed N-strings $[n_1...n_N]$ (we will not sketch the proof of this fact in this paper).

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