Revista de la
Unión Matemática Argentina
Volumen 41, 2, 1998.

ON THE MACÍAS-SEGOVIA METRIZATION OF QUASI–METRIC SPACES

Hugo Aimar, Bibiana Iaffei and Liliana Nitti

Abstract: We give a direct proof of a theorem of Macías and Segovia ([M-S] on the metrization (X, ρ) of quasi–metric spaces (X, d), without an explicit use of the uniform structure on $X \times X$. Then we show how our construction can be extended to some generalized quasi-metrics.

A distance or metric on a set X is a real non-negative, symmetric function vanishing on the diagonal of $X \times X$ for which the triangle inequality

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z) \tag{1}$$

holds true for every x, y and z in X and is faithful

$$\rho(x,y) = 0 \quad \textit{implies} \quad x = y. \tag{2}$$

When only (1) *is satisfied usually ρ is called a pseudo–metric. Sometimes the function ρ satisfies the strictly stronger triangle inequality*

$$\rho(x,z) \leq \max\{\rho(x,y), \rho(y,z)\}. \tag{1.a}$$

In 1970, R. Coifman and M. de Guzmán ([CG]) introduce the weaker notion of quasi–distance in an attempt to include functions like $d(x,y) = |x-y|^n$ on $\mathbb{R}^n$ for which some central questions of harmonic analysis remain true (see also [CW]). A quasi–distance d on X share with a metric all its properties except perhaps the triangle inequality which now can take the weaker form

---

Supported by: CONICET and UNL (CAI+D Program).
Keywords and phrases: Spaces Homogeneous Type.
AMS Subjet Classification: 28C10

(1.b) *there exists* $K \geq 1$ *such that*

$$d(x,z) \leq K\,[d(x,y) + d(y,z)]$$

for every x, y *and* z *in* X.
Let us observe that (1.b) is now equivalent to the following
(1.c) there exists K *such that*

$$d(x,z) \leq K \max\{d(x,y), d(y,z)\},$$

with an eventually different constant K.

A set X with a quasi–distance d is called a quasi–metric space. The deepest result concerning a quasi-distance d on a set X was obtained by R. Macías and C. Segovia ([MS]) by realizing that d produces a uniform structure on $X \times X$ with a countable basis generating a metrizable topology on X. Frink's theorem on metrization of uniformities with countable basis (see [K]), provides also a quantitative relation that allows to construct a metric ρ and a real number α larger than one such that ρ^α is equivalent to d, in the sense that ρ^α/d is bounded above and below. In the terminology introduced by Macías and Segovia, every quasi-distance d is equivalent to a quasi-distance d' of order β: *there is a constant* C *such that for every* x, y *and* z *satisfying* $d'(x,y) < r$ *and* $d'(x,z) < r$ *we have*

$$|d'(x,y) - d'(x,z)| \leq Cr^{1-\beta} d'(y,z)^\beta.$$

The last inequality constitutes also the source of non–trivial Lipschitz functions on quasi–metric spaces.

In this note we give a more explicit construction of the metric induced by a quasi–distance d, making use of the key point of Frink's argument but without the use of uniformities. Let us point out that the so defined metric ρ coincides with the given d when this is already a metric.

Theorem I: *Let* X *be a set and let* d *be a quasi–distance on* X. *Then there exists* $0 < \beta \leq 1$, *depending only on* K, *such that*

$$\rho(x,y) = \inf\left\{\sum_{i=1}^{n} d^\beta(x_i, x_{i+1}) : x_1 = x, x_2, \ldots, x_{n+1} = y \in X, n \in I\!N\right\}$$

is a metric on X *with* $\rho^{1/\beta}$ *equivalent to the given* d

The next lemma shows that the function ρ given in the statement of Theorem I is a pseudo–metric, for every positive β.

Lemma 1: *Let* X *be a set and* $g : X \times X \to I\!R^+_o$ *be a symmetric function vanishing on the diagonal. Then the function*

$$\rho(x,y) = \inf \sum_{i=1}^{n} g(x_i, x_{i+1})$$

is a pseudo-metric bounded above by $g(x,y)$ if the infimum is taken over all finite chains $x_1 = x, x_2, ..., x_{n+1} = y$ in X joining x with y.

Proof:

We only need to check the triangle inequality. Let x, y and z be three given points in X and let $\epsilon > 0$. There exist two chains of points $x = x_1, x_2, ..., x_n = y$ and $y = x_n, x_{n+1}, ..., x_m = z$ such that

$$\sum_{i=1}^{n-1} g(x_i, x_{i+1}) < \rho(x,y) + \epsilon \quad \text{and} \quad \sum_{i=n}^{m-1} g(x_i, x_{i+1}) < \rho(y,z) + \epsilon,$$

adding the above inequalities we see that

$$\rho(x,z) \leq \sum_{i=1}^{m-1} g(x_i, x_{i+1}) < \rho(x,y) + \rho(y,z) + 2\,\epsilon$$

for every positive ϵ. ∎

Proof of Theorem I:

Let us first observe that if d is a metric, by taking $\beta = 1$ and $g = d$ in Lemma 1, we have that $\rho = d$ since the inequality $d \leq \rho$ follows easily from the triangle inequality for d. In the general case we start by applying Lemma 1 to the function $g(x,y) = d^\beta(x,y)$, with $\beta \leq \frac{1}{log_2 3K^2}$. So that

$$\rho(x,y) = \inf \sum_{i=1}^{n} d^\beta(x_i, x_{i+1}),$$

is a pseudo-metric bounded by $d^\beta(x,y)$, where the infimum is taken over all finite chains in X joining x with y. Of course in order to complete the proof of the theorem we only need to show that

$$d^\beta \leq 2\rho. \tag{3}$$

In fact, (3) proves the remainder inequality for the equivalence of d and $\rho^{1/\beta}$, moreover if x and y are two points in X with $\rho(x,y) = 0$ we necessarily have $d(x,y) = 0$ and $x = y$, so ρ becomes a distance. Let us prove (3) which actually is the main point of the theorem. We have to prove that for any finite sequence $x_1, ..., x_n$ of points, or chain of length n, in X we have that

$$d^\beta(x_1, x_n) \leq 2 \sum_{i=1}^{n-1} d^\beta(x_i, x_{i+1}). \tag{4}$$

Let us prove (4) by induction on the length n of the chain. For $n = 2$, (4) is obvious. Assume (4) holds for every chain of length less than or equal to n, and take $x_1, x_2, ..., x_n, x_{n+1}$ a chain of length $(n+1)$ of points in X. Let $0 < \delta = \sum_{i=1}^{n} d^\beta(x_i, x_{i+1})$. Three cases are possible

$$(i) \qquad d^\beta(x_1, x_2) > \delta/2,$$

$$(ii) \qquad \sum_{i=1}^{n-1} d^\beta(x_i, x_{i+1}) \leq \delta/2,$$

(iii) there exists $k = 1, 2, ..., n-2$ such that

$$\sum_{i=1}^{k} d^\beta(x_i, x_{i+1}) \leq \delta/2 \text{ and } \sum_{i=1}^{k+1} d^\beta(x_i, x_{i+1}) > \delta/2.$$

(i) Since $\delta = \sum_1^n$ and $d^\beta(x_1, x_2) > \delta/2$ we have that $\sum_2^n \leq \delta/2$. Now $x_2, ..., x_{n+1}$ has length n and we can apply (4) to obtain

$$d^\beta(x_2, x_{n+1}) \leq 2 \sum_{i=2}^{n} d^\beta(x_i, x_{i+1}) \leq 2.\delta/2 = \delta\ ,$$

and, since $d^\beta(x_1, x_2) \leq \delta$, by the triangle inequality we have

$$\begin{aligned} d^\beta(x_1, x_{n+1}) &\leq (K(d(x_1, x_2) + d(x_2, x_{n+1})))^\beta \\ &\leq (K(\delta^{1/\beta} + \delta^{1/\beta}))^\beta \\ &\leq (2K)^\beta \delta \\ &\leq (3K^2)^\beta \delta. \end{aligned}$$

Finally, the choice of β gives $d^\beta(x_1, x_{n+1}) \leq 2\delta$.
(ii) The chain $x_1, ..., x_n$ has length n so that from (4) $d^\beta(x_1, x_n) \leq 2.\delta/2 = \delta$, since $d^\beta(x_n, x_{n+1}) \leq \delta$ the same kind of estimate in (i) gives us $d^\beta(x_1, x_{n+1}) \leq 2\delta$.

(iii) Since $\sum_1^{k+1} > \delta/2$ and $\sum_1^n = \delta$, then $\sum_{k+2}^n \leq \delta/2$. Also $\sum_1^k \leq \delta/2$ and $d^\beta(x_{k+1}, x_{k+2}) \leq \delta$. By applying (4) to $x_1, ..., x_{k+1}$ and to $x_{k+2}, ..., x_{n+1}$ we get the three estimates

$$\begin{aligned} d^\beta(x_1, x_{k+1})) &\leq \delta \\ d^\beta(x_{k+1}, x_{k+2}) &\leq \delta \\ \text{and } d^\beta(x_{k+2}, x_{n+1}) &\leq \delta. \end{aligned}$$

Now, the triangle inequality gives

$$\begin{aligned} d^\beta(x_1, x_{n+1})) &\leq (K(d(x_1, x_{k+1}) + d(x_{k+1}, x_{n+1})))^\beta \\ &\leq (K^2(d(x_1, x_{k+1}) + d(x_{k+1}, x_{k+2}) + d(x_{k+2}, x_{n+1})))^\beta \\ &\leq (3K^2)^\beta \delta \\ &\leq 2\delta. \blacksquare \end{aligned}$$

Let us now introduce a generalization of quasi-metric spaces. If $|x - y|$ is the usual euclidean distance from x to y in $I\!R^n$, the function $d(x,y) = |x-y|^n$ brings a normalized structure for which the measure of the ball of radious r equals a constant times r. But d is no longer a distance, nevertheless it is a quasi–distance

$$d(x,z) \leq 2^n \max\{d(x,y), d(y,z)\} \leq 2^n (d(x,y) + d(y,z)).$$

If the $n^{\underline{th}}$ power of $|x-y|$ is substituted by a function φ continuous and increasing with $\varphi(0) = 0$ in order to produce $d(x,y) = \varphi(|x-y|)$, we are led to a d satisfying a generalized triangle inequality of the type

$$d(x,z) \leq \eta\,(\max\{d(x,y), d(y,z)\}),$$

for some increasing function $\eta(t) \geq t$. This remark suggests the next definition as an extension of the notion of quasi–distance. Let $\eta : I\!R_o^+ \to I\!R_o^+$ be a continuous, increasing and convex function with $\eta(0) = 0$. An η–metric on the set X is a non–negative symmetric function d vanishing on the diagonal of $X \times X$ satisfying

(5) $d(x,y) = 0$ implies $x = y$,

(6) $d(x,z) \leq \eta\,(\max\{d(x,y), d(y,z)\})$ for every x, y and z in X.

By taking $y = z$ in (6) we see that $\eta(t) \geq t$. From now on, without loosing generality, we shall assume that $\eta(t) > 2t$ since (6) is obviously satisfied by $\overline{\eta} = 3\eta$. Let us finally remark that quasi–distances are η–metrics with $\eta(t) = Kt$.

The metrization of η–metric spaces is given by the next theorem

Theorem II: *Let X be a set and let d be an η–metric on X. Then*
(II.1) there is an increasing, continuous and concave function ψ on $I\!R^+ \cup \{0\}$ such that

$$\rho(x,y) = \inf\left\{\sum_{i=1}^{n} \psi(d(x_i, x_{i+1})) : x_1 = x, x_2, ..., x_{n+1} = y \in X, n \in I\!N\right\}$$

is a metric on X with $\psi^{-1}(\rho(x,y))$ equivalent to the given d in the following sense

$$\psi^{-1}(\rho) \leq d \leq \psi^{-1}(2\rho),$$

(II.2) the function $d' = \psi^{-1}(\rho)$ satisfies the following property of "order ψ"

$$|d'(x,z) - d'(x,y)| \leq \psi(d'(y,z)) / \frac{d}{dr}\psi(r),$$

where r is any positive number larger than $\max\{d'(x,z), d'(x,y)\}$ for which the derivative $\frac{d}{dr}\psi(r)$ exists.

The function ψ is given in the next lemma

Lemma 2: *Let η be a continuous, non decreasing and convex function defined on $\mathbb{R}_o^+$ satisfying $\eta(t) > 2t$ for every positive t and $\eta(0) = 0$, then the inequality*

$$\psi \circ \eta \circ \eta \leq 2\psi$$

has at least one solution ψ increasing, continuous and concave with $\psi(1) = 1$ and $\psi(0) = 0$.

The proof of Theorem II follows the same pattern of that of Theorem I, using here the ψ provided by Lemma 2 instead of the power β used there.

Proof of Theorem II:

We start by applying Lemma 1 to the function $g(x, y) = \psi(d(x, y))$ where ψ is given by Lemma 2. So that

$$\rho(x, y) = \inf \sum_{i=1}^{n} \psi(d(x_i, x_{i+1})),$$

is a pseudo–metric bounded by $\psi(d(x, y))$, where the infimum is taken over all finite chains in X joining x with y. Since ψ is increasing we have that $\psi^{-1}(\rho) \leq d$. Notice now that in order to finish the proof of (II.1) we only need to show that

$$(7) \qquad \psi(d) \leq 2\rho.$$

Let us prove (7). We have to prove that for any finite sequence $x_1, ..., x_n$ of points, or chain of length n, in X we have that

$$(8) \qquad \psi(d(x_1, x_n)) \leq 2 \sum_{i=1}^{n-1} \psi(d(x_i, x_{i+1})).$$

As in the proof of Theorem I, let us prove (8) by induction on the length n of the chain. Assume (8) holds and take $x_1, x_2, ..., x_n, x_{n+1}$ a chain of length $(n+1)$ of points in X. Let $0 < \delta = \sum_{i=1}^{n} \psi(d(x_i, x_{i+1}))$. Again, we only need to consider the three following cases

$$(i) \qquad \psi(d(x_1, x_2)) > \delta/2,$$

$$(ii) \qquad \sum_{i=1}^{n-1} \psi(d(x_i, x_{i+1})) \leq \delta/2,$$

(iii) there exists $k = 1, 2, ..., n-2$ such that

$$\sum_{i=1}^{k} \psi(d(x_i, x_{i+1})) \leq \delta/2 \text{ and } \sum_{i=1}^{k+1} \psi(d(x_i, x_{i+1})) > \delta/2.$$

For each case we argue as in the proof of Theorem I, so that we shall only write the estimates for $\psi(d(x_1, x_{n+1}))$ by applying the triangle inequality (6) and the definition of ψ given by Lemma 2.

(i)

$$\begin{aligned}\psi(d(x_1, x_{n+1})) &\le \psi(\eta(\max\{d(x_1,x_2), d(x_2,x_{n+1})\}))\\ &\le \psi(\eta(\psi^{-1}(\delta)))\\ &\le \psi((\eta\circ\eta)(\psi^{-1}(\delta)))\\ &\le 2\psi(\psi^{-1}(\delta)) = 2\,\delta.\end{aligned}$$

(iii)

$$\begin{aligned}\psi(d(x_1, x_{n+1})) &\le \psi(\eta(\max\{d(x_1,x_{k+1}), \eta(\max\{d(x_{k+1},x_{k+2}), d(x_{k+2},x_{n+1})\}\})))\\ &\le \psi(\eta\circ\eta(\psi^{-1}(\delta)))\\ &\le 2\,\delta.\end{aligned}$$

Let us now prove (II.2).Since ψ is concave, ψ^{-1} is a convex function, so that, for any choice of $0 < t_1 < t_2 < t_3 < t_3 + h$, we have that

$$\frac{\psi^{-1}(t_2) - \psi^{-1}(t_1)}{t_2 - t_1} \le \frac{\psi^{-1}(t_3 + h) - \psi^{-1}(t_3)}{h}.$$

Take now $t_1 = \min\{\rho(x,z), \rho(x,y)\}$, $t_2 = \max\{\rho(x,z), \rho(x,y)\}$ and $t_3 = \psi(r) > t_2$ such that derivative of ψ at r exists. Let us now take $h \to 0$. Then

$$\frac{|d'(x,z) - d'(x,y)|}{|\rho(x,z) - \rho(x,y)|} \le \frac{1}{\frac{d}{dr}\psi(r)}.\blacksquare$$

Proof of Lemma 2:

Let us denote by $\tilde{\eta}$ the composition $\eta\circ\eta$. We have to solve the inequality $\psi(\tilde{\eta}(t)) \le 2\psi(t)$ with $\psi(1) = 1$. Since $\tilde{\eta}(t) > 4t$ we have that $\tilde{\eta}^{(k)}(1)$ is an increasing sequence for $k \in \mathbb{Z}$, with $\tilde{\eta}^{(k)} = \tilde{\eta}\circ\tilde{\eta}\circ\cdots\circ\tilde{\eta}$, k times, $\tilde{\eta}^{(-k)} = \tilde{\eta}^{(-1)}\circ\tilde{\eta}^{(-1)}\circ\cdots\circ\tilde{\eta}^{(-1)}$ and $\tilde{\eta}^{(0)}$ the identity. Moreover

$$\lim_{k\to\infty} \tilde{\eta}^{(k)}(1) = +\infty \quad\text{and}\quad \lim_{k\to-\infty} \tilde{\eta}^{(k)}(1) = 0.$$

Define ψ on the sequence $\tilde{\eta}^{(k)}(1)$ by

$$\psi(\tilde{\eta}^{(k)}(1)) = 2^k; \qquad k \in \mathbb{Z},$$

observe that $\psi(1) = 1$. Of course, for $t_k = \tilde{\eta}^{(k)}(1)$ we have equality:

$$\psi(\tilde{\eta}(t_k)) = \psi(\tilde{\eta}^{(k+1)}(1)) = 2^{k+1}$$

$$= 2\psi(\tilde{\eta}^{(k)}(1)) = 2\psi(t_k).$$

Let us show that ψ defined on $I\!R^+$ by piecewise linear interpolation of the points $(\tilde{\eta}^{(k)}(1), 2^k); k \in \mathbb{Z}$, satisfies the required properties. Of course ψ is increasing since so is on t_k. Let us call m_k the slope of the $k^{\underline{th}}$ segment of the graph of ψ. In other words

$$m_k = \frac{2^{k+1} - 2^k}{\tilde{\eta}^{k+1}(1) - \tilde{\eta}^k(1)} = \frac{2^k}{\tilde{\eta}^{k+1}(1) - \tilde{\eta}^k(1)}.$$

The concavity of ψ is equivalent to the inequality $m_{k+1} \leq m_k$ and this, in turn, follows from $\tilde{\eta}(t) \geq 4t$. In fact, $m_{k+1} \leq m_k$ if and only if $2(\tilde{\eta}^{(k+1)}(1) - \tilde{\eta}^{(k)}(1)) \leq \tilde{\eta}^{(k+2)}(1) - \tilde{\eta}^{(k+1)}(1)$ which is implied by $3\tilde{\eta}^{(k+1)}(1) < 4\tilde{\eta}^{(k+1)}(1) \leq \tilde{\eta}(\tilde{\eta}^{(k+1)}(1)) = \tilde{\eta}^{(k+2)}(1)$. It remains only to show that ψ is a solution for the inequality $(\psi \circ \tilde{\eta})(t) \leq 2\psi(t)$. For $t = t_k$ we have equality, we may then assume

$$\tilde{\eta}^{(k)}(1) < t < \tilde{\eta}^{(k+1)}(1),$$

so that

$$\tilde{\eta}^{(k+1)}(1) < \tilde{\eta}(t) < \tilde{\eta}^{(k+2)}(1).$$

The convexity of $\tilde{\eta}$ implies the inequality

$$\frac{\tilde{\eta}(t) - \tilde{\eta}^{(k+1)}(1)}{t - \tilde{\eta}^{(k)}(1)} \leq \frac{\tilde{\eta}^{(k+2)}(1) - \tilde{\eta}^{(k+1)}(1)}{\tilde{\eta}^{(k+1)}(1) - \tilde{\eta}^{(k)}(1)}. \tag{9}$$

From the definition of ψ we have

$$\frac{\psi(t) - 2^k}{t - \tilde{\eta}^{(k)}(1)} = m_k \tag{10}$$

and

$$\frac{\psi(\tilde{\eta}(t)) - 2^{k+1}}{\tilde{\eta}(t) - \tilde{\eta}^{(k+1)}(1)} = m_{k+1}. \tag{11}$$

Using (11), (9) and finally (10) we get

$$\psi(\tilde{\eta}(t)) = \frac{\tilde{\eta}(t) - \tilde{\eta}^{(k+1)}(1)}{\tilde{\eta}^{(k+2)}(1) - \tilde{\eta}^{(k+1)}(1)} 2^{k+1} + 2^{k+1}$$

$$\leq (t - \tilde{\eta}^{(k)}(1)) \frac{\tilde{\eta}^{(k+2)}(1) - \tilde{\eta}^{(k+1)}(1)}{\tilde{\eta}^{(k+2)}(1) - \tilde{\eta}^{(k+1)}(1)} \frac{2^{k+1}}{\tilde{\eta}^{(k+1)}(1) - \tilde{\eta}^{(k)}(1)} + 2^{k+1}$$

$$= 2(\psi(t) - 2^k) + 2^{k+1} = 2\psi(t) \blacksquare$$

REFERENCES

[CG] R. Coifman and M. de Guzmán, *Singular integrals and multipliers on homogeneous spaces.* Rev. Union Mat. Argentina 35 (1970), 137–144.

[CW] R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains spaces homogènes, ètude de certaines intégrales singulières*, Lecture Notes in Math., Vol 242, Springer–Verlag, Berlin, 1971.

[K] J.L. Kelley, *General Topology*, van Nostrand 1955.

[MS] R.A. Macías and C. Segovia. *Lipschitz functions on spaces of homogeneous type*, Advances in Mathematics, Vol. 33 (1979), 257–270.

Recibido en diciembre de 1997