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Free Monadic Three-valued Łukasiewicz Algebras

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Abstract

In this work the concept of *free monadic extension* of a three-valued Łukasiewicz algebra is defined and used to obtain the free monadic three-valued Łukasiewicz algebra with a finite set of free generators G up from the free three-valued Łukasiewicz algebra with the same set of free generators, following a method introduced by P. Halmos in [5]. This method also allows us to know the coordinates of the generators on each axis. As particular cases, free monadic boolean and three-valued Post algebras with a finite set of generators are determined, as well as the corresponding free monadic algebras over a given finite poset.

P. Halmos' technique has been used by R. Cignoli in the case of Q-distributive lattices [2] and by A. Petrovich in the case of monadic De Morgan algebras [15], of which the monadic three-valued Łukasiewicz algebras can be seen as a particular case.

1 Introduction

Definition 1.1 A *monadic three-valued Łukasiewicz algebra*, [11] is an algebra $(A, \wedge, \vee, \nabla, \sim, \exists, 1)$ of type $(2,2,1,1,1,0)$ such that $(A, \wedge, \vee, \nabla, \sim, 1)$ is a three-valued Łukasiewicz algebra, i.e. the following axioms are verified:

L0) $x \vee 1 = 1$ L1) $x \wedge (x \vee y) = x$
L2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$ L3) $\sim\sim x = x$
L4) $\sim (x \wedge y) = \sim x \vee \sim y$ L5) $\sim x \vee \nabla x = 1$
L6) $x \wedge \sim x = \sim x \wedge \nabla x$ L7) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$

and $\exists$ is a unary operator on A, called *existential quantifier* satisfying:

$\exists_0$) $\exists 0 = 0$ $\exists_1$) $x \leq \exists x$
$\exists_2$) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$ $\exists_3$) $\nabla \exists x = \exists \nabla x$
$\exists_4$) $\sim \nabla \sim \exists x = \exists \sim \nabla \sim x$.

A derived operator Δ is defined on A by $\Delta x = \sim \nabla \sim x$, and we have the following properties: (see [8],[11])

L8) $\nabla(x \vee y) = \nabla x \vee \nabla y$ L9) $\Delta(x \vee y) = \Delta x \vee \Delta y$
$\exists_5$) $\exists(x \vee y) = \exists x \vee \exists y$ $\exists_6$) $\exists \sim \exists x = \sim \exists x$.

Let $\mathbf{T}$ denote the three-valued Łukasiewicz algebra: $\mathbf{T} = \{0, c, 1\}$ where $0 < c < 1, \sim 0 = 1, \sim c = c, \sim 1 = 0, \nabla 0 = 0, \nabla c = \nabla 1 = 1$. With $\mathbf{B}$ we will indicate the subalgebra of $\mathbf{T}$ formed by 0 and 1.

Let $B(A)$ denote the set of boolean elements of a three-valued Łukasiewicz algebra A. It is easy to prove, [6] that: $B(A) = \{x \in A : \nabla x = x\} = \{x \in A : \Delta x = x\}$. Let $\mathcal{A}(B)$ be the set of atoms of a non-trivial finite boolean algebra B.

An element e of a three-valued Łukasiewicz algebra A is an *axis* of A if $\Delta e = 0$ and $\nabla x \leq \Delta x \vee \nabla e$, for all $x \in A$, [7]. We will call *constants* the elements of a monadic three-valued Łukasiewicz algebra A such that $\exists x = x$. The set of all the constants is a Łukasiewicz subalgebra of A and if A has an axis e, then $\exists e = e$, [11]. The *monadic subalgebra* generated by a subset X of A, $SM(X)$ is the intersection of all the monadic subalgebras of A containing X.

Definition 1.2 A mapping h from a three-valued Łukasiewicz algebra A to a three-valued Łukasiewicz algebra A' is called an *hemimorphism* if for all $p, q \in A$:

$H_1)$ $h(0) = 0$; $\qquad H_2)$ $h(p \vee q) = h(p) \vee h(q)$;

$H_3)$ $h(\nabla x) = \nabla h(x)$; $\qquad H_4)$ $h(\Delta x) = \Delta h(x)$.

If, furthermore

$$H_5) \quad h(1) = 1$$

is verified, then h is called an *1-hemimorphism*.

Note that this definition is different from the one given by L. Monteiro in [10].

Lemma 1.1 *Every existential quantifier is a 1-hemimorphism.*

In what follows, A will be a non-trivial finite three-valued Łukasiewicz algebra. It is well known and easy to check that $\pi(A)$, the set of join irreducible elements of A has the following properties: each $p \in \pi(A)$ is an atom of $B(A)$ or p precedes one and only one atom in $B(A)$ and conversely, every atom of $B(A)$ is an atom of A or is a join irreducible element of A preceded by only one atom in A. Therefore, $\pi(A) = \{a_i\}_{1 \leq i \leq j+k} \cup \{c_{j+i}\}_{1 \leq i \leq k}$, where a_i is an atom of $B(A), 1 \leq i \leq j+k$, and c_{j+i} is the atom of A preceding a_{j+i} for $1 \leq i \leq k$. It is clear that $\pi(B(A)) = B(A) \cap \pi(A) = \{a_i\}_{1 \leq i \leq j+k}$ and that if $p \in \pi(A)$, then $\Delta p \in \pi(A) \cup \{0\}$ and $\nabla p \in \pi(A)$. Then A is isomorphic to (and will be identified with) $\mathbf{B}^j \times \mathbf{T}^k$, with $j, k \in N \cup \{0\}$ and not simultaneously zero, [11]. Notice that the element $e = (e_t)_{1 \leq t \leq j+k}$ with $e_t = 0$ if $1 \leq t \leq j$ and $e_t = c$ if $j+1 \leq t \leq j+k$ is the axis of $\mathbf{B}^j \times \mathbf{T}^k$.

Lemma 1.2 *An hemimorphism h from $A = \mathbf{B}^j \times \mathbf{T}^k$ to the three-valued Łukasiewicz algebra $\mathbf{T}$ is uniquely determined by the values that h takes on the $j+k$ atoms of $B(A)$. This values must be in $B(\mathbf{T}) = \mathbf{B}$ and therefore, each application $g : \mathcal{A}(B(A)) \to \{0,1\} \subseteq \mathbf{T}$ can be extended to a unique hemimorphism from A to $\mathbf{T}$.*

Proof: Let h be an hemimorphism. If $a \in \mathcal{A}(B(A))$ then $\Delta h(a) = h(\Delta a) = h(a)$, so $h(a) \in B(\mathbf{T}) = \mathbf{B}$. It is easy to check that if b is an atom of A, then $h(b)$ is determined by $h(a)$, being a the only atom in $B(A)$ that b precedes. Thus, h is fixed for every element in $\pi(A)$. As hemimorphisms preserve joins and every nonzero element in A is join of the elements in $\pi(A)$ preceding it, h is uniquely determined for every element in A. □

Lemma 1.3 *Let h be an hemimorphism from A to $\mathbf{T}$; h is an homomorphism if and only if there is one and only one atom $a_i \in \mathcal{A}(B(A)), 1 \leq i \leq j+k$ such that $h(a_i) = 1$.*

Proof: Let h be an homomorphism. Then, if $h(a_i) = 0$ for every $i, 1 \leq i \leq j+k$, then $h(1) = 0$, a contradiction, so there is an index i such that $h(a_i) = 1$. If for two different elements $a, a' \in \mathcal{A}(B(A))$, $h(a) = h(a') = 1$, then $1 = h(a) \wedge h(a') = h(a \wedge a') = h(0) = 0$, so there must be one and only one $a_i \in \mathcal{A}(B(A))$ such that $h(a_i) = 1$.

In the other hand, it is easy to check that an hemimorphism h such that $h(a_i) = 1$ for just one $a_i \in \mathcal{A}(B(A))$, (and therefore $h(a_j) = 0$ for $j \neq i$) satisfies $h(x \wedge y) = h(x) \wedge h(y)$ for every $x, y \in A$. Then, by Theorem 3 in [10], h is an homomorphism. □

By the characterizations of the Lemmas 1.2 and 1.3, the next two Lemmas follow straightforwardly.

Lemma 1.4 *If h is an 1-hemimorphism from A to $\mathbf{T}$, then there is an homomorphism preceding it.*

Lemma 1.5 *Every 1-hemimorphism from A to $\mathbf{T}$ is supremum of the homomorphisms preceding it.*

Lemma 1.6 *Let $Y = Hom(A, \mathbf{T})$ be the set of all the homomorphisms from A to $\mathbf{T}$. A is isomorphic to the three-valued Łukasiewicz algebra $\mathbf{P} = \prod_{y \in Y} y(A)$.*

Proof: By Lemma 1.3, there is a bijection between the homomorphisms from A to $\mathbf{T}$ and the elements in $\mathcal{A}(B(A))$. It suffices then to observe that for each $a_i, 1 \leq i \leq j$, the image of A under the corresponding homomorphism is $\mathbf{B}$, and if $j+1 \leq i \leq j+k$, the image is $\mathbf{T}$. □

According to the previous Lemma, we may indicate an element in A by the value of each homomorphism in that element.

Let A and A' be three-valued Łukasiewicz algebras, $X = Hom(A, \mathbf{T})$ and $Y = Hom(A', \mathbf{T})$. Given a function $f : Y \to X$ such that:

$$(*) \quad f(Y_{\mathbf{B}}) \subseteq X_{\mathbf{B}},$$

where $X_{\mathbf{B}} = \{x \in X : x(A) = \mathbf{B}\}$ and $Y_{\mathbf{B}} = \{y \in Y : y(A') = \mathbf{B}\}$, we can define an homomorphism $f^* : A \to A'$ by: For all $y \in Y, p \in A$,

$$yf^*p = (fy)p.$$

As seen in Lemma 1.6, it suffices to show for f^*p the value of each homomorphism of Y on that point. To see that f is well defined, it is enough to show that for each $y \in Y_{\mathbf{B}}, yf^*p$ assumes a value in $\mathbf{B}$. Indeed, if $y \in Y_{\mathbf{B}}$, then by $(*)$, $fy \in X_{\mathbf{B}}$ and therefore $(fy)p \subset (fy)(A) \simeq \mathbf{B}$. It is easy to check that f^* is an homomorphism and that every homomorphism from A to A' may be obtained in this way. If f is injective, then f^* is an epimorphism, and if f is surjective, f^* is a monomorphism. These results generalize the ones obtained by M. Abad y A. V. Figallo for epimorphims between three-valued Łukasiewicz algebras, [1], and may be compared to those of R. Sikorski for boolean algebras, [16]. More information on this kind of dualities may be found in [3].

2 FREE MONADIC EXTENSIONS

A monadic three-valued Lukasiewicz algebra L is a *free monadic extension* of a three-valued Lukasiewicz algebra A if:

(i) A is a subalgebra of L,

(ii) L is the monadic subalgebra generated by A, i.e. $L = SM(A)$,

(iii) every homomorphism of three-valued Lukasiewicz algebras g from A to an arbitrary monadic three-valued Lukasiewicz algebra C has a (necessarily unique) extension to a monadic homomorphism f from L to C.

We now give a construction of the free monadic extension for the case in which the algebra A is finite ($A \cong \mathbf{B}^j \times \mathbf{T}^k$), following the method advanced by P. Halmos [5]. Let $Y = Hom(A, \mathbf{T})$ the set of homomorphisms from A to $\mathbf{T}$ and V the set of the 1-hemimorphisms from A to $\mathbf{T}$. It is easy to see that Y has $j + k$ elements and V has $2^{j+k} - 1$ elements.
Let $X = \{(y, v) : y \in Y, v \in V, y \leq v\}$. From Lemmas 1.2, 1.3 and 1.4, it follows that for each homomorphism in Y there are 2^{j+k-1} 1-hemimorphisms $v \in V$ such that $y \leq v$. Therefore, $X \subset Y \times V$ has $(j + k)(2^{j+k-1})$ elements.
Let L be the three-valued Lukasiewicz algebra:

$$L = \prod_{(y,v) \in X} v(A).$$

Let us now consider $X_{\mathbf{B}} = \{(y, v) \in X : v(A) \simeq \mathbf{B}\}$. To calculate the cardinality of $X_{\mathbf{B}}$, note that there are j homomorphisms in Y such that their image is isomorphic to $\mathbf{B}$, and each of them is dominated by 2^{j-1} 1-hemimorphisms with image $\mathbf{B}$. Therefore, $X_{\mathbf{B}}$ has $j \cdot 2^{j-1}$ elements and L is isomorphic to

$$\mathbf{B}^{j.2^{j-1}} \times \mathbf{T}^{(j+k).2^{j+k-1} - j.2^{j-1}}$$

Define for all $p \in L$,

$$(\exists p)(y, v) = \bigvee\{p(u, v) : u \in Y, u \leq v\}. \tag{1}$$

It is easy to check that $\exists$ is an existential quantifier over L, being the crucial step:
$\exists_2$) For all $p, q \in L$, and each $(y, v) \in X$,

$$\exists(p \wedge \exists q)(y, v) = \bigvee_{u \leq v} (p \wedge \exists q)(u, v) = \bigvee_{u \leq v} [p(u, v) \wedge (\exists q)(u, v)] =$$

$$[\bigvee_{u \leq v} p(u, v)] \wedge [\bigvee_{u \leq v} (\exists q)(u, v)] = (\exists p)(y, v) \wedge [\bigvee_{u \leq v} (\bigvee_{w \leq v} q(w, v))] =$$

$$= (\exists p)(y, v) \wedge (\bigvee_{w \leq v} q(w, v)) = (\exists p)(y, v) \wedge (\exists q)(y, v) = (\exists p \wedge \exists q)(y, v)$$

Therefore $\exists(p \wedge \exists q) = \exists p \wedge \exists q$ for all $p, q \in L$.
We will prove that the monadic three-valued Lukasiewicz algebra L is the free monadic extension of a subalgebra of L isomorphic to A.

The elements of $\mathcal{A}(B(L))$ are $f_{y,v}$, where

$$f_{y,v}(u,w) = \begin{cases} 1 & \text{si } (u,w) = (y,v), \\ 0 & \text{si } (u,w) \neq (y,v). \end{cases} \qquad (y,v),(u,w) \in X.$$

Consider now the representation of L by homomorphisms from L to $\mathbf{T}$. The elements of $H = Hom(L, \mathbf{T})$ are $h_{y,v}$ where $h_{y,v}$ is the homomorphism corresponding to the atom $f_{y,v}$ of $B(L)$. It is clear that $H_{\mathbf{B}} = \{h_{y,v} \in H : v(A) \cong \mathbf{B}\}$ and $H_{\mathbf{T}} = \{h_{y,v} \in H : v(A) \cong \mathbf{T}\}$. Let $c : H \to Y$ the function defined by $c(h_{y,v}) = y$. So c is surjective and $c(H_{\mathbf{B}}) \subseteq Y_{\mathbf{B}}$. Then, the homomorphism $h = c^* : A \to L$ is injective, and $h_{y,v}hp = (ch_{y,v})p = yp$, i.e.:

$$(hp)(y,v) = y(p). \tag{2}$$

$h(A)$ is a subalgebra of L isomorphic to A, and it is in this sense that (i) is verified. Our principal result can now be stated as follows.

Theorem 2.1 *The monadic three-valued Łukasiewicz algebra L with the quantifier $\exists$ defined in (1) is a free monadic extension of its three-valued Łukasiewicz subalgebra $h(A)$.*

From the definition of h and Lemma 1.5, we get:

$$(\exists hp)(y,v) = \bigvee_{u<v} (hp)(u,v) = \bigvee_{u<v} u(p) = v(p). \tag{3}$$

Let us now see that (ii) $SM(h(A)) = L$. By Lemma 0.3.10 in [11], p. 16, it suffices to show that $B(L) \subseteq SM(h(A))$ and $e \in SM(h(A))$, where e is the axis of L.
Let e' be the axis of A. Then for all $(y,v) \in X$:

$$he'(y,v) = y(e') = \begin{cases} c & \text{if } y(A) \cong \mathbf{T}, \\ 0 & \text{if } y(A) \cong \mathbf{B}. \end{cases}$$

Therefore $he' = e$ and $e \in h(A) \subseteq SM(h(A))$.
We shall prove now that

$$f_{y,v} = ha_y \wedge (\bigwedge_{va_z=1} \exists ha_z) \wedge (\bigwedge_{va_z=0} \sim \exists ha_z), \tag{4}$$

where a_y is the only element of $\pi(A)$ such that $y(a_y) = 1$. As $ha_y, ha_z \in h(A)$ and therefore $ha_y, \exists ha_z, \sim \exists ha_z \in SM(h(A))$, we will have that $\mathcal{A}(B(L)) \subseteq SM(h(A))$. As an immediate consequence, $B(L) \subseteq SM(h(A))$.
Let

$$\begin{aligned} q(u,w) &= [ha_y \wedge (\bigwedge_{va_z=1} \exists ha_z) \wedge (\bigwedge_{va_z=0} \sim \exists ha_z)](u,w) \\ &= ua_y \wedge (\bigwedge_{va_z=1} wa_z) \wedge (\bigwedge_{va_z=0} \sim wa_z) \quad \text{(by (2) and (3))}. \end{aligned}$$

In particular, $q(y,v) = ya_y \wedge (\bigwedge_{va_z=1} va_z) \wedge (\bigwedge_{va_z=0} \sim va_z) = 1$.

If $(u,w) \neq (y,v)$ then $u \neq y$ or $w \neq v$. If $u \neq y$, then $ua_y = 0$ and $q(u,w) = 0$. If $w \neq v$, then by Lemma 1.2 there exists j such that $wa_j \neq va_j$. If $va_j = 0$ then $wa_j = 1$ and $\sim wa_j = 0$, therefore $\bigwedge_{va_z=0} \sim wa_z = 0$ and $q(u,w) = 0$.

We have just proved that $q = f_{y,v}$. Let now C be an arbitrary monadic three-valued Lukasiewicz algebra, and g an homomorphism of Lukasiewicz algebras from $h(A)$ to C. If we prove that a monadic homomorphism from L to C extending g exists, the demonstration of the fact that L is the free monadic extension of $h(A)$ will be complete.

Let us consider the subalgebra $S = SM(g(h(A)))$ of C. By Theorem III.3.4 in [11], p.76, S is finite and therefore we can use its representation by the set of homomorphisms $Z = Hom(S, \mathbf{T})$, i.e., $S \cong \prod_{z\in Z} z(S)$.

Let $a : Z \longrightarrow Y$ be defined by $(az)p = zghp$; and $b : Z \longrightarrow V$ defined by $(bz)p = z\exists ghp$ (where the existential quantifier corresponds to the algebra C). az is an homomorphism from A to $\mathbf{T}$ because it is a composition of homomorphisms. In a similar way, bz is an hemimorphism from A to $\mathbf{T}$. (Note that $\exists$ is an hemimorphism from C to C). Let $r(z) = h_{(az,bz)}$. As for all $p \in A, ghp \leq \exists ghp$, and then $zghp \leq z\exists ghp$, i.e., $(az)p \leq (bz)p$, then $(az, bz) \in X$. So r is a map from Z to H. It is clear that if $z \in Z_{\mathbf{B}}, (bz)p = z\exists ghp \in \mathbf{B}$, so we can say that $f = r^*$ is an homomorphism from L to C. Furthermore, for all $q \in L, z \in Z$, $z(fq) = q(az, bz)$.

To show that (iii) is verified, it remains to prove that f restricted to $h(A)$ is equal to g, and f is a monadic homomorphism.

If $q \in h(A)$ then $q = h(p)$ for some $p \in A$ and

$$zfq = q(az,bz) = h(p)(az,bz) \overset{(2)}{=} (az)(p) \overset{def.}{=} zghp = zgq, \text{ for all } z \in Z,$$

i.e., $fq = gq$.

f is a monadic homomorphism, i.e. for all $q \in L, z \in Z, z\exists fq = zf\exists q$.

If $q = h(p)$ for some $p \in L$ then $zf\exists q = zf\exists hp = (\exists hp)(az,bz) = (bz)p = z\exists ghp = z\exists fhp = z\exists fq$. In particular, this yields $zf\exists e = z\exists fe$.

If $q \in \mathcal{A}(B(L))$ then $q = ha_y \wedge (\bigwedge_{va_w=1} \exists ha_w) \wedge (\bigwedge_{va_w=0} \sim \exists ha_w)$ for some $(y,v) \in X$.

Then

$$\begin{aligned}
zf\exists q &= (\exists q)(az,bz) = \\
&= \exists[ha_y \wedge (\bigwedge_{va_w=1} \exists ha_w) \wedge (\bigwedge_{va_w=0} \sim \exists ha_w)](az,bz) = \\
&= [\exists ha_y \wedge (\bigwedge_{va_w=1} \exists ha_w) \wedge (\bigwedge_{va_w=0} \sim \exists ha_w)](az,bz) = \\
&= (\exists ha_y)(az,bz) \wedge [\bigwedge_{va_w=1} (\exists ha_w)(az,bz)] \wedge [\bigwedge_{va_w=0} \sim (\exists ha_w)(az,bz)] = \\
&= (bz)a_y \wedge [\bigwedge_{va_w=1} (bz)a_w] \wedge [\bigwedge_{va_w=0} \sim (bz)a_w] = \\
&= z\exists gha_y \wedge [\bigwedge_{va_w=1} z\exists gha_w] \wedge [\bigwedge_{va_w=0} \sim z\exists gha_w)] = \\
&= z\exists[fha_y \wedge (\bigwedge_{va_w=1} \exists fha_w) \wedge (\bigwedge_{va_w=0} \sim \exists fha_w)] =
\end{aligned}$$

$$= z\exists[fha_y \wedge (\bigwedge_{va_w=1} f\exists haw) \wedge (\bigwedge_{va_w=0} \sim f\exists ha_w)] =$$

$$= z\exists f[ha_y \wedge (\bigwedge_{va_w=1} \exists ha_w) \wedge (\bigwedge_{va_w=0} \sim \exists ha_w)] = z\exists fq.$$

As L is finite, $B(L)$ is finite, so every element different from 0 in $B(L)$ is supremum of elements in $\mathcal{A}(B(L))$. Since $\exists$ is an hemimorphism, we can conclude that for all $p \in B(L), f\exists p = \exists fp$.
Let now p be an arbitrary element in L. As e is the axis of L, $p = (\Delta p \vee e) \wedge \nabla p$. Since we also have $\exists e = e$ and fe is the axis of C,

$$f\exists p = f\exists(\Delta p \vee (\nabla p \wedge e)) \overset{\exists_5)}{=} f(\exists\Delta p \vee \exists(\nabla p \wedge e)) = f\exists\Delta p \vee f(\exists\nabla p \wedge \exists e).$$

Since $\Delta p, \nabla p \in B(L)$, this is equal to:

$$\exists f\Delta p \vee (f\exists\nabla p \wedge f\exists e) = \exists f\Delta p \vee (\exists f\nabla p \wedge \exists fe) = \exists f\Delta p \vee \exists(f\nabla p \wedge \exists fe) =$$

$$= \exists(f\Delta p \vee (f\nabla p \wedge fe)) = \exists f(\Delta p \vee (\nabla p \wedge e)) = \exists fp,$$

which concludes the proof of Theorem 2.1.
If $f : Y \times V \rightarrow \mathbf{T}$ is such that $f(y,v) = f(y,v')$, for every y, v, v', then we say that the function f is *independent* from V. In a similar way, if $f(y,v) = f(y',v)$ for all y, y', v, f is independent from Y.
Looking at (2), it is clear that the functions in $h(A)$ are independent from V. Furthermore, $h(A)$ consists exactly of those functions in L that are independent from V. The constants in L are independent from Y. Indeed, by (1),

$$\exists p(y,v) = \bigvee_{u \leq v} p(u,v) = \exists p(y',v).$$

As a particular case we may obtain the free monadic extension of the boolean algebra $\mathbf{B}^j$, which is $\mathbf{B}^{j2^{j-1}}$ just as indicated in [5](see also [17]). In a similar way, the free monadic extension of the three-valued Post algebra $\mathbf{T}^k$ is $\mathbf{T}^{k2^{k-1}}$ (see [12]).

3 The free monadic three-valued Lukasiewicz algebra with n free generators

The preceding results can be applied to the free three-valued Lukasiewicz algebra generated by an arbitrary finite set G. Any map from G to a monadic three-valued Lukasiewicz algebra C has a (necessarily unique) extension to an homomorphism of three-valued Lukasiewicz algebras g that maps A to C. The homomorphism of three-valued Lukasiewicz algebras g has a (necessarily unique) monadic extension f that maps L to C. We conclude from this that the free monadic extension of a free three-valued Lukasiewicz algebra is a free monadic three-valued Lukasiewicz algebra.
It is well known that the free three valued Lukasiewicz algebra with n free generators is $\mathbf{B}^{2^n} \times \mathbf{T}^{3^n-2^n}$. Then, according to the results in §2, the free monadic three-valued Lukasiewicz algebra with n generators is isomorphic to:

$$\mathbf{B}^{[2^{(2^n+n-1)}]} \times \mathbf{T}^{[3^n.2^{(3^n-1)}-2^{(2^n+n-1)}]}.$$

This result was obtained by L. Monteiro in [11], using a different method.
As an example, when $n = 1$ and $G = \{g\}$, the free three-valued Łukasiewicz algebra generated by G, $L(1)$, is $\mathbf{B}^2 \times \mathbf{T}$, with $g = (0, 1, c)$, [14]. The corresponding free monadic extension is $\mathbf{B}^4 \times \mathbf{T}^8$. If we denote with y_i the homomorphism from $L(1)$ to $\mathbf{T}$ such that $y_i(a_i) = 1, i = 1, 2, 3$, then the 1-hemimorphisms from $L(1)$ to $\mathbf{T}$ are $y_1, y_2, y_3, v_1 = y_1 \vee y_2, v_2 = y_1 \vee y_3, v_3 = y_2 \vee y_3, v_4 = y_1 \vee y_2 \vee y_3$. The elements of X as well as the value that $h(g)$ takes in those elements, are indicated in the following table:

x	(y_1, y_1)	(y_1, v_1)	(y_2, y_2)	(y_2, v_1)	(y_1, v_2)	(y_1, v_4)
$h(g)(x)$	0	0	1	1	0	0

x	(y_2, v_3)	(y_2, v_4)	(y_3, y_3)	(y_3, v_2)	(y_3, v_3)	(y_3, v_4)
$h(g)(x)$	1	1	c	c	c	c

Starting with the free boolean algebra with n generators and following the same procedure, we get the free monadic boolean algebra with n generators ([5], [17],[13] and the bibliography indicated there) and in a similar way, from the free three-valued Post algebra we get the free monadic three-valued Post algebra, $\mathbf{T}^{3^n 2^{3^n - 1}}$, just as it is indicated in [12]. Applying this method to the free boolean, three-valued Łukasiewicz or Post algebras over a finite ordered set ([9], [4]), the corresponding free monadic algebras over those ordered sets are obtained.

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