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RING OF DIFFERENTIAL OPERATORS AND A RELATED COMPLETELY INTEGRABLE SYSTEM

LUIS A. PIOVAN

ABSTRACT. We represent the affine ring of an elliptic curve as a ring of matrix differential operators. As an application, we embed the phase variables of the rigid body motion on $SO(3)$ (Euler Top) into commuting differential operators with matrix coefficients. Thus, showing that this algebraic completely integrable system is a piece of an infinite dimensional (four-component) **KP** hierarchy.

1. Introduction.

In the usual Geometric Realization of Conformal Field Theory on Riemann Surfaces [KNTY], the basic "Krichever" data consist of quintuples $((\mathcal{C},(\alpha,\beta)),p,\mathcal{L},z,t)$, where $(\mathcal{C},(\alpha,\beta))$ is a Riemann surface together with a choice of a canonical homology basis, $p\in\mathcal{C}$ a point at infinity, $\mathcal{L}$ a line bundle on $\mathcal{C}$, z a local parameter about p and t a trivialization of $\mathcal{L}$ at p. To this data, one relates points in the Universal Grassmann Manifold of Sato **UGM** by $t(H^0(\mathcal{C},\mathcal{L}(*p)))\in$ **UGM** [Mul 2]. By dividing the projectivization of the quintuples above by the action of $Sp(2g,\mathbb{Z})$ if $g(\mathcal{C})>1$ (the action of $Sp(2g,\mathbb{Z})\times Aut(\mathcal{C})$ if $g(\mathcal{C})\leq 1$), we get the so called moduli space of framed and gauged Riemann surfaces and an embedding of this space into **UGM**. Moreover, the deformation of these data along the Jacobian directions is determined by the action of the **KP** flows on the points $t(H^0(\mathcal{C},\mathcal{L}(*p)))\in$ **UGM**. Also, there is a bijection between the triples $(\mathcal{C},p,\mathcal{L})$ with certain conditions on $\mathcal{L}$ and the affine rings $\mathcal{O}(\mathcal{C}-p)$ [Mu 1].

Quite a similar data can be associated to smooth elliptic curves with a divisor $\mathcal{D}$ instead of a point p at infinity. Consider for instance the data $(E,\mathcal{D},\mathcal{F}=[\tau_x^{-1}\mathcal{D}-\mathcal{D}],z,t)$, where E is an elliptic curve, $\mathcal{D}=\Sigma p_i$ a divisor on E, $\mathcal{F}$ a line bundle, $z=\{z_i\}$ local equations about the points p_i of $\mathcal{D}$ and $t=\{t_i\}$ trivializations of $\mathcal{F}$ about the points of $\mathcal{D}$. Then, one associates to it the point $\Pi_i t_i(H^0(E,\mathcal{F}(*\mathcal{D})))\in$ **UGM** under suitable identifications.

As generally believed [Sa] et al. integrable systems, finite and infinite, can be viewed as pieces of infinite dimensional dynamical systems like **KP** or multicomponent **KP** hierarchies [Ad-B].

The main step is to define a map from the dynamical phase space of the integrable system into an appropriate moduli space whose points are characterized by some sort of Krichever data modulo relations like the quintuples above. One can bypass this by directly defining the map from the phase space into **UGM** with the help of a basis for $H^0(E,\mathcal{F}(*\mathcal{D}))$ (the Baker-Akhiezer sections) , a representation

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of the phase variables with some matrix differential operators and an identification of a holomorphic flow on the elliptic curves with a multicomponent **KP** flow.

One of the results of this paper is that the data $(\mathcal{C}, \mathcal{D}, \mathcal{L})$ with $h^0(\mathcal{L}) = h^1(\mathcal{L}) = 0$ and $\mathcal{D}$ a particularly chosen divisor, determines an embedding of the affine ring $\mathcal{O}(\mathcal{C} - \mathcal{D})$ into commutative ring of differential operators. This yields a generalization for elliptic curves of Krichever prescription for the dictionary $(\mathcal{C}, p, \mathcal{L}) \to \mathcal{O}(\mathcal{C} - p)$ [Mu 1].

We apply part of the above program to the rigid body motion on $SO(3)$ (Euler top). The Euler top is a system that describes the rigid body motion around a fixed center of gravity. In the angular moment coordinates, it reduces to the equations

$$
\left\{
\begin{aligned}
\dot{v}_1 &= (\lambda_2 - \lambda_3) v_2 v_3 \\
\dot{v}_2 &= (\lambda_3 - \lambda_1) v_3 v_1 \\
\dot{v}_3 &= (\lambda_1 - \lambda_2) v_1 v_2
\end{aligned}
\right.
\tag{1}
$$

It has two independent integrals

$$
\begin{aligned}
Q_1 &= v_1^2 + v_2^2 + v_3^2 \\
Q_2 &= \lambda_1 v_1^2 + \lambda_2 v_2^2 + \lambda_3 v_3^2
\end{aligned}
\tag{2}
$$

which commute with respect to the Poisson bracket. Q_1 being the trivial invariant and Q_2 the nontrivial Hamiltonian.

Although the real geometry of integrable systems is described, to some degree, by the Arnold-Liouville theorem [Ar], their complex geometry is more subtle. The nature of the solutions to integrable systems depends heavily on the complex geometry. If we require the solutions to be expressible in terms of theta functions related to abelian varieties , then, we call such systems algebraic completely integrable (a.c.i.). Many of these systems were known classically in Mechanics and studied in detail by several people. To mention a few, Adler and Van Moerbeke [A-VM 1,2], Dubrovin [Du], Moser [Mo], Mumford [Mu 1,3].

In the picture introduced by Adler and Van Moerbeke for (a.c.i) systems, the real phase space $\mathbb{R}^{2n+k}$ is complexified, and the integrals are polynomials. The complexified invariant manifolds $\tilde{M}_c = \{v = (v_1, \cdots, v_{2n+k}) \in \mathbb{C}^{2n+k},\ F_i(v) = c_i,\quad i = 1, \ldots, n+k\}$ are affine varieties in $\mathbb{C}^{2n+k}$. They are affine pieces of abelian varieties A_c in such a way that the coordinates v_i become nontrivial abelian functions on A_c. Thus $v_i \in L(\mathcal{D})$ = functions on A_c that blow up at a divisor $\mathcal{D}$ of A_c, and $\tilde{M}_c = A_c \backslash \{$ the reduced divisor $\mathcal{D}\}$. Moreover, the nontrivial holomorphic vector fields $X_{F_1}, \ldots, X_{F_n}$ have a linear motion on A_c.

For instance, in the Euler top case, one obtains (by setting Q_1 and Q_2 to constants) the affine part of an elliptic curve in $\mathbb{P}^3 = \mathbb{P}(L(\mathcal{D}))$ with $\mathcal{D}$ = divisor at infinity = 4 points. X_{Q_2} yields linear motion on the affine elliptic curve $E_c = \{v \in \mathbb{C}^3, Q_1(v) = c_1, Q_2(v) = c_2\}$ and X_{Q_1} vanishes on E_c.

The paper is organized as follows. In section **2** we construct a kind of Baker-Akhiezer functions which are suitable to represent the Euler Top phase variables in terms of matrix differential operators. It is possible to identify the Hamiltonian flow with a Multicomponent **KP** flow under a suitable embedding. The Lemmas and Propositions in this section describe this identification. In section **3** we give a construction of a commutative ring of differential operators associated to the data $(E, \mathcal{D}, \mathcal{F})$, where E is an elliptic curve, $\mathcal{D}$ a divisor on E and $\mathcal{F} = [\tau_x^{-1}\mathcal{D} - \mathcal{D}]$ a line bundle such that $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$. We prove a theorem for the embedding of the affine ring of elliptic curves into a ring of differential operators. In particular, this will hold for elliptic curves in $\mathbb{P}^3$; which are related to the Euler Top.

There are two appendices: Appendix **1** deals with some basics about Multicomponent **KP** hierarcy. In Appendix **2** we construct Weierstrass $\wp$-functions on an abelian variety A with the help of the defining equations for a divisor $\mathcal{D}$ on A. The construction is quite similar to hyperelliptic $\wp$-functions [Mu 3]. These functions and their related meromorphic differentials of second kind are used to define Baker-Akhiezer functions. It is hoped that some results obtained for the elliptic curves can be extended to abelian varieties.

2. Baker functions defined on an elliptic curve.

In this section we present several examples. There are different attempts to defining a Baker-Akhiezer function for the divisor $\mathcal{D} = p_1 + p_2 + p_3 + p_4 =$ sum of points. The relevant examples 6.1 and 6.6 allow us to identify the Euler Top flow with a Multicomponent **KP** flow.

First, we consider the usual method for constructing Baker-Akhiezer functions.

Given the divisor $\mathcal{E}$, one considers the ϑ-function Θ associated to it ([We], [Ig]), i.e. Θ vanishes once on $\mathcal{E}$. Let $\mathcal{A}_{x_\alpha} : A \to H^0(A, \Omega^1)^*/H_1(A, \mathbb{Z})$ be a set of Albanese's maps, $\mathcal{A}_{x_\alpha}(x) = \left(\int_{x_\alpha}^{x} \omega\right)$, for some conveniently chosen $x_\alpha \in A$. Here, the integrals are along a path γ joining x_α and x. For elliptic curves these maps are isomorphisms and any two of them differ by a translation on E.

There is a holomorphic differentials ω, and basis of homology cycles $\{a, b\}$, such that the period matrix has the form $\left(\int_a \omega, \int_b \omega\right) = (1, \tau)$. According to Igusa [Ig] any ϑ-function Θ can be written as a linear combination of ϑ-series of the form

$$\Theta_m(\tau, z) = \sum_{p \in \mathbb{Z}} \mathrm{e}\left(\tfrac{1}{2}(p + m')\tau(p + m') + (p + m')(z + m'')\right) \tag{3}$$

where $m = (m'm'')$ and m', m'' in $\mathbb{R}$ and $\mathrm{e}(x) = \exp(2\pi i x)$. Such a ϑ-series satisfies

$$\Theta_m(\tau, z + n'\tau + n'') = \Theta_m(\tau, z)\ \mathrm{e}(-\tfrac{1}{2}n'\tau n' - n'z)\ \mathrm{e}(m'n'' - n'm'') \tag{4}$$

for any element $n'\tau + n''$, $(n', n'' \in \mathbb{Z})$, belonging to the lattice of the elliptic curve.

Moreover, if δ is the integer defining the polarization type of $\mathcal{E}$, then there exist real numbers $m', m'' \in \mathbb{R}$ such that

$$\Theta(z) = \sum_{r \bmod \mathbb{Z}} \text{constant} \cdot \Theta_{(r + m'\delta^{-1}, m'')}(\tau, z), \tag{5}$$

where r runs over a complete set of representatives of $\left(\frac{1}{\delta}\mathbb{Z}\right)/\mathbb{Z}$.

Following [Du], [Sh] and [Ma–Ka] we define the Baker-Akhiezer function associated to the divisors $\mathcal{D}$ and $\mathcal{E}$ as follows.

$\psi(u, t, x_\alpha, x)$, $u \in \mathbb{C}$, $\Theta(u) \neq 0$, $t \in \mathbb{C}^\infty$, $x \in A - \mathcal{D} = \mathcal{U}_0$,
z_α a local parameter around $x_\alpha \in \mathcal{D}$ defined on the chart U_α such that the $\mathcal{U}_\alpha$'s are disjoint.

$$\psi(u, t, x_\alpha, x) = \mathrm{e}\left(\sum t_i \int_{x_\beta}^{x} \omega_\alpha^i\right) \frac{\Theta\left(\mathbb{B}^t t + u - \mathcal{A}_{x_\alpha}(x)\right)}{\Theta\left(u - \mathcal{A}_{x_\alpha}(x)\right)}, \tag{6}$$

where ω_α^i's are normalized 2^{nd} kind differentials and $\mathbb{B}$ their matrix of b-period: $\mathbb{B} = \left(\int_b \omega_\alpha^j\right)$. The ω_α^i's have local expansions around $\mathcal{D} \cap U_\alpha$,
$\omega_\alpha^i \sim (-1)^i c_i \dfrac{dz_\alpha}{z_\alpha^{i+1}} + O(z_\alpha^{-i})\, dz_\alpha$.

As we increase w by the period $n'\tau + n''$ we get the change

$$\begin{aligned}\Theta(w + n'\tau + n'') &= \sum_{r \bmod \mathbb{Z}} c_r \Theta_{(r+m'\delta^{-1}, m'')}(\tau, w + n'\tau + n'') \\ &= \left[\sum_{r \bmod \mathbb{Z}} c_r \Theta_{(r+m'\delta^{-1}, m'')}(\tau, w)\, \mathrm{e}\left((r + m'\delta^{-1})n''\right)\right] \cdot \\ &\quad \cdot \mathrm{e}\left(-\tfrac{1}{2}n'\tau n' - n'\,(m'' + n'') - n'w\right).\end{aligned}$$

Thus $\dfrac{\Theta(w + w_0)}{\Theta(w)}$ is changed by $\mathrm{e}(-n'w_0)\dfrac{\Theta[n''](w + w_0)}{\Theta[n''](w)}$, where $\Theta[n''](w)$ is a theta function vanishing on a divisor linearly equivalent to $\mathcal{E}$. Since we want the same θ-function we have to ask $\ell(\mathcal{E}) = 1$ and therefore $\delta = 1$.

Now, changing $\sum t_i \int_{x_\beta}^{x} \omega_\alpha^i$ by the homology cycle $n'b + n''a$ produces the extra factor $\mathrm{e}\left(\sum_i t_i n'\, {}^t\left(\int_b \omega_\alpha^i\right)\right)$ in ψ, which cancels with the contribution of the term $\mathrm{e}\left(-n'\left(\sum_i t_i\, {}^t\left(\int_b \omega_\alpha^i\right)\right)\right) = \mathrm{e}(-n'\mathbb{B}^t t)$ due to the quotient of theta functions.

This shows that the function (6) extends to a well defined meromorphic function on the open set $\mathcal{U}_0$ that blows up once at $\mathcal{E}$ where $\mathcal{E} = \{x \in A : \Theta\left(u - \mathcal{A}_{x_\alpha}(x)\right) = 0\}$ and has essential singularities at the points of $\mathcal{D}$.

Let t be a uniformizing parameter and $z_i = O(t)$ the local parameter at the piece p_i of the divisor $\mathcal{D} = p_1 + p_2 + p_3 + p_4$. Ω_i^n the normalized differential of 2^{nd} kind with a single pole of order $n + 1$ at p_i and holomorphic everywhere else.

Consider the map $\varphi : E \to \mathrm{Pic}^0(E)$ defined by $\varphi(x) = [\tau_x\mathcal{D} - \mathcal{D}]$ (the canonical map). This has a finite kernel (the translation group $H(\mathcal{D})$). Let $\mathcal{E}$ be a divisor in $\mathrm{Pic}^0(E)$ such that $D = \varphi^{-1}\mathcal{E}$. Then $\theta(\varphi(p))$ is a theta function for the divisor D.

A Baker function can be obtained as

$$\psi_{i,\nu}^n(x) = \exp\left(\nu\, t_n \int_{p_0}^{x} \Omega_i^n\right) \frac{\theta_1\left(\int_{p_0}^{x} \omega + t_n U_i^n + \xi\right)}{\theta_2\left(\int_{p_0}^{x} \omega + \xi\right)}, \tag{7}$$

where ω is a nonzero holomorphic differential, and θ_i theta functions associated to translates of $\mathcal{D}$. As we go around a b-cycle of E we pick a b-period of Ω_i^n. So the exponential gets increased by the factor $\exp(t_n \underbrace{\int_{b_n} \Omega_i^n}_{U_i^n})$, which will cancel out with factors of θ_1 and θ_2.

Lemma 2.1. *The expression (7) is a Baker function at p_i associated to the divisor $\mathcal{D}$. It has the expansions*

$$\psi_{i,\nu}^n(x) = \begin{cases} e^{\nu\, t_n/z_i^n}\,(1+O(z_i)) & \text{around } p_i \\ e^{\nu\, t_n \alpha_{ij}}\,(1+O(z_j)) & \text{around } p_j,\ j \neq i. \end{cases} \tag{8}$$

Proof. Assume θ_1, θ_2 are θ functions of order ν with characteristics $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, i.e. satisfy a relation of the type

$$\theta_\nu \begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z+2\pi i N + BM) = \\ \exp\left\{-\frac{\nu}{2}\langle BM, M\rangle - \nu\langle M, z\rangle + 2\pi i(\langle \alpha, N\rangle - \langle \beta, M\rangle)\right\}\theta_\nu \begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z).$$

If θ_1 and θ_2 are of the same type and order then all the factors cancel except the factor $\exp\{-\nu\langle M, t_n U_i^n\rangle\} = \exp\left\{-\nu\, t_n \int_{Mb_i} \Omega_i^n\right\}$.

So if we add the factor ν in the exponential of the Baker function we obtain the desired cancelling, i.e. (7) is a well defined meromorphic function outside p_i, with zeroes at $\theta_1\left(\int_{p_0}^x \omega + t_n U_i^n + \xi\right) = 0$ and poles at $\theta_2\left(\int_{p_0}^x \omega + \xi\right) = 0$

As candidates for θ_1 and θ_2 one can pick the functions $\theta\begin{bmatrix}(\alpha+\gamma)/\nu\\ \beta\end{bmatrix}(\nu z|\nu B)$.

Around p_i we have

$$\Omega_i^n = -\frac{n.dz_i}{z_i^{n+1}} + O(1), \tag{9}$$

so

$$\int_{p_0}^x \Omega_i^n = \frac{1}{z_i^n(x)} + O(1) \qquad \text{and} \qquad e^{\nu\, t_n \int_{p_0}^x \Omega_i^n} = e^{\nu\, t_n/z_i^n}(1+O(t_n, z_i)). \tag{10}$$

One can pick as z_i the time parameter t of a holomorphic vector field in E. One also has the expressions (8) around p_i.

Let τ_{ij} be the translation that sends $p_i \to p_j$, i.e., addition by $p_j - p_i$, and let $\Omega_j = \tau_{ij}^*\Omega_i$ be the pull back of Ω_i. Then Ω_j blows up at p_j.

Now we have the formula $\int_a \Omega_j = \int_{\tau_{ij}a} \Omega_i = .\int_a \Omega_i$ (since $a + \tau_{ij}$ is homologous to a) for a period a of Ω_i (i.e., the periods are the same). (Notice that one can choose a so that all the translates $\tau_{ij}a$ of a do not meet the poles of Ω_i or its translate Ω_j.)

If we let $c_{ij} = \int_{p_0}^{p_0+p_i-p_j} \Omega_j$ then we have $\int_{p_0}^{x} -\Omega_i - \int_{p_0}^{x+p_i-p_j} \Omega_j = -c_{ij}$. In other words, one can interpret the cycle c_{ij} as the difference between the infinite integrals $\int_{p_0}^{x+p_i-p_j} \Omega_j - \int_{p_0}^{x} \Omega_i$ as $x \to p_i$. One has $c_{ij} + c_{jk} = c_{ik}$.

The correlation function f_{ij} defined by $df_{ij} = \Omega_i - \Omega_j$ is defined on the universal cover of E. Up to a constant we can pick $f_{ij} = \int^x \Omega_i - \int^x \Omega_j$ which is a function that blows up at p_i and p_j.

Now, we can write $\int^x \Omega_i = \int^x \Omega_j + f_{ij}$, and let $\alpha_{ij} = \int^{p_j} \Omega_i$, $i \neq j$. Thus, we have

$$\begin{aligned} \psi_i^n &= e^{\nu\, t_n \alpha_{ij}}(1 + O(z_j)) \qquad \text{about } p_j \\ &= O(1) \quad \square \end{aligned} \tag{11}$$

Lemma 2.2. *We have the estimates*

$$\frac{d}{dt_n}\psi_{i,\nu}^n(x) = \begin{cases} \left(\dfrac{\nu}{z_i^n(x)} + O(z_i)\right)\psi_{i,\nu}^n(x) & \text{if } x \text{ is around } p_i, \\ O(1) & \text{if } x \text{ is around } p_j,\ j \neq i. \end{cases} \tag{12}$$

Proof.

$$\begin{aligned} \frac{d}{dt_n}\left(e^{\nu\, t_n/z_i^n}(1 + O(z_i))\right) &= \frac{\nu}{z_i^n} e^{\nu\, t_n/z_i^n}(1 + O(z_i)) + e^{\nu\, t_n/z_i^n} O_1(z_i) \\ &= \left(\frac{\nu}{z_i^n} + O_2(z_i)\right)\psi_{i,\nu}^n. \end{aligned}$$

$$\frac{d}{dt_n} e^{\nu\, t_n \alpha_{ij}}(1 + O(z_j)) = e^{\nu\, t_n \alpha_{ij}}\left(\nu \alpha_{ij} + O_1(z_j)\right). \quad \square$$

Proposition 2.3. *There is a unique function, up to an element in $H(\mathcal{E}_0)$, having essential singularity at the point p_i, zeroes at $\mathcal{E}_0$ and blowing up at $\mathcal{E}_\infty$.*

Proof. If ψ and $\tilde{\psi}$ are two Baker functions then $\tilde{\psi}/\psi$ is meromorphic on the elliptic curve because the essential singularities cancel. The poles at $\mathcal{E}_\infty$ also cancel. Thus, the divisor of $\tilde{\psi}/\psi$ comes from the zeroes of $\tilde{\psi}$ and ψ, namely $\tilde{\mathcal{E}}_0$ and $\mathcal{E}_0$. So, we have $\tilde{\mathcal{E}}_0$ linearly equivalent to $\mathcal{E}_0$ for all $|t_n| \ll 1$. Since the group of divisors linearly equivalent to $\mathcal{E}_0$ is finite (the translation group $H(\mathcal{E}_0)$) we have that such a Baker function is unique up to an element in the Translation group of $\mathcal{E}_0$. $\square$

Note 2.4. It follows from Proposition 7.3 the following lemma:

Lemma 2.5. *On an elliptic curve, a Baker function with expansion $\psi = O(z)e^{t_1/z}$ and no other zero or pole has to be zero.*

Note 2.6. For elliptic curves it will be shown in Theorem 3.1 that there is an embedding $\mathcal{R} = \Gamma(A - \mathcal{D}, \mathcal{O}_A)$ into a commutative ring of differential operators with matrix coefficients.

Example 2.7. In order to illustrate Note 2.6 we draw Table I with the expansions of $\psi_1, \ldots, \psi_4$ and $D\psi_1, \ldots, D\psi_4$ around the points $p_1, \ldots, p_4$, where the p_i's are the points of the divisor of the Euler Top. Let $\{v_1, v_2, v_3\}$ be the generators of the affine ring associated to the Euler top system which satisfies equations (2). The invariant manifolds of this system have divisor at infinity $\mathcal{D} = \Sigma p(\delta_1, \delta_2) = p(1,1) + p(1,-1) + p(-1,1) + p(-1,-1) = p_1 + p_2 + p_3 + p_4$, and the expansion of the functions $\{v_1, v_2, v_3\}$ about $\mathcal{D}$, in terms of the time evolution parameter t associated with the Euler top flow, are

$$(13) \qquad \begin{cases} V_1 = \sqrt{\alpha\beta}\, v_1 = \delta_1 \left(\dfrac{1}{t} - (u+v)t + \cdots \right) & \delta_1^2 = \delta_2^2 = 1, \\ V_2 = \sqrt{\alpha\gamma}\, v_2 = \delta_2 \left(\dfrac{1}{t} + ut + \cdots \right) & \alpha = \lambda_1 - \lambda_2, \quad \beta = \lambda_3 - \lambda_1, \\ V_3 = \sqrt{\gamma\beta}\, v_3 = \delta_1\delta_2 \left(\dfrac{1}{t} + vt + \cdots \right) & \gamma = \lambda_2 - \lambda_3. \end{cases}$$

Table I

	$p(1,1)$	$p(1,-1)$	$p(-1,1)$	$p(-1,-1)$
ψ_1	$e^{\nu t_1/z_1}(1+O(z_1))$	$e^{\nu t_1 \alpha_{12}}(1+O(z_2))$	$e^{\nu t_1 \alpha_{13}}(1+O(z_3))$	$e^{\nu t_1 \alpha_{14}}(1+O(z_4))$
ψ_2	$e^{\nu t_1 \alpha_{21}}(1+O(z_1))$	$e^{\nu t_1/z_2}(1+O(z_2))$	$e^{\nu t_1 \alpha_{23}}(1+O(z_3))$	$e^{\nu t_1 \alpha_{24}}(1+O(z_4))$
ψ_3	$e^{\nu t_1 \alpha_{31}}(1+O(z_1))$	$e^{\nu t_1 \alpha_{32}}(1+O(z_2))$	$e^{\nu t_1/z_3}(1+O(z_3))$	$e^{\nu t_1 \alpha_{34}}(1+O(z_4))$
ψ_4	$e^{\nu t_1 \alpha_{41}}(1+O(z_1))$	$e^{\nu t_1 \alpha_{42}}(1+O(z_2))$	$e^{\nu t_1 \alpha_{43}}(1+O(z_3))$	$e^{\nu t_1/z_4}(1+O(z_4))$
$D\psi_1$	$\left(\frac{\nu}{z_1} + O(z_1)\right)\psi_1$	$O(1)$	$O(1)$	$O(1)$
$D\psi_2$	$O(1)$	$\left(\frac{\nu}{z_2} + O(z_2)\right)\psi_2$		
$D\psi_3$			$\left(\frac{\nu}{z_3} + O(z_3)\right)\psi_3$	
$D\psi_4$				$\left(\frac{\nu}{z_4} + O(z_4)\right)\psi_4$
$V_1\psi_1$	$\frac{1}{t}\psi_1 + \cdots$	$\frac{1}{t}\psi_1$	$-\frac{1}{t}\psi_1$	$-\frac{1}{t}\psi_1$

Notice that $\psi_j(x + p_i - p_j) = \exp(\nu\, t_n c_{ij})\, \psi_i(x)$ (with $\int_{p_0}^{p_j} \Omega_i = \int_{p_o}^{p_i} \Omega_j - c_{ij}$), once one chooses convenient θ functions to construct the remaining Baker functions from a given one. This is because we have

$$(14) \qquad \int_{p_o}^{x} \Omega_i = \int_{p_0}^{x+p_i-p_j} \Omega_j - c_{ij}$$

and

$$(15) \qquad \int_{p_0}^{x} \omega = \int_{p_0}^{x+p_i-p_j} \omega + \int_{x+p_i-p_j}^{x} \omega = \int_{p_0}^{x+p_i-p_j} \omega + \int_{p_i}^{p_j} \omega,$$

since ω are translation invariant 1-forms on an elliptic curve.

Example 2.8. Consider now a 2^{nd} kind normalized differential form Ω that blows up at the $\frac{1}{2}$-periods p_1, p_2, p_3 and p_4 to order two, thus having local expansion $-\frac{dz_i}{z_i^2}$, where z_i is the local parameter at the point p_i. Let τ_{ij} be the translation by the vectors $p_j - p_i$. Assume that these translations are all $\frac{1}{2}$-periods.

We assume that the differential Ω is invariant under the group of translations $\tau_{ij}(x) = x + p_j - p_i$. This is a subgroup of the group of translations associated to the divisor $\mathcal{D} = p_1 + p_2 + p_3 + p_4$. We have the following relation:

$$(16) \quad \int_{p_0}^{x_i} \Omega = \int_{p_0}^{x_i} \tau_{ij}^{*}\Omega = \int_{p_0+p_j-p_i}^{x_i+p_j-p_i} \Omega = \int_{p_0}^{x_j} \Omega - \int_{p_0}^{p_0+p_j-p_i} \Omega = \int_{p_0}^{x_j} \Omega - c_{ij},$$

where $x_j = x_i + p_j - p_i$ and x_i is close to p_i (and x_j is close to p_j).

One can pick p_0 so that $\int_{p_0}^{x_1} \Omega = \frac{1}{z_1(x_1)} + O(z_1(x_1)) = \frac{1}{z_j(x_j)} - c'_{ij} + O(z_j(x_j))$ with $x_j = x_1 + p_j - p_1$ and for certain coefficients c'_{ij} satisfying the cocycle condition $c'_{ij} + c'_{jk} = c'_{ik}$, $c'_{ij} = -c'_{ji}$.

Now on the long range curve γ_i we have

$$\int_{\gamma_i(x)} \Omega = \int_{p_0}^{x_i} \Omega + \int_{x_i}^{x} \Omega = \int_{\gamma_j(x)} \Omega + \text{periods} = \int_{p_0}^{x_j} \Omega + \int_{x_j}^{x} \Omega + \text{periods of } \Omega.$$

Namely

$$(17) \quad c_{ij} = \int_{p_0}^{x_j} - \int_{p_0}^{x_i} \Omega = \int_{x_i}^{x} \Omega - \int_{x_j}^{x} \Omega + \text{ periods of } \Omega \qquad (x_i \text{ close to } p_i).$$

On the other hand, for a holomorphic normalized *translation invariant* differential ω we have

$$(18) \quad \int_{p_0}^{x_j} \omega = \int_{p_0}^{x_i+p_j-p_i} \omega = \int_{p_0}^{x_i} \omega + \int_{x_i}^{x_i+p_j-p_i} \omega = \int_{p_0}^{x_i} \omega + \int_{p_i}^{p_j} \omega,$$

where we assume $\int_{p_i}^{p_j} \omega$ is a $\frac{1}{2}$-period. Also, modulo a period

$$\int_{x_i}^{x} \omega \equiv \int_{x_j}^{x} \omega + \int_{p_i}^{p_j} \omega. \tag{19}$$

Given the ϑ-functions ϑ_1, ϑ_2 related to any of the points p_i's, and of the same order, we define the following Baker functions

$$\psi_i(x) = \exp\left(\nu t \int_{x_i}^{x} \Omega\right) \frac{\vartheta_1\left(\int_{x_i}^{x} \omega + t \int_b \Omega + \xi\right)}{\vartheta_2\left(\int_{x_i}^{x} \omega + \xi\right)}, \tag{20}$$

where the points x_i are in a chart U_i about p_i.

One can relate the behaviour of ψ_i as x approaches p_j. We have

$$\begin{aligned}
\psi_i(x) &= \exp(\nu t c_{ij}) \exp\left(\nu t \int_{x_j}^{x} \Omega\right) \frac{\vartheta_1\left(\int_{x_j}^{x} \omega + \int_{p_i}^{p_j} \omega + t \int_b \Omega + \xi\right)}{\vartheta_2\left(\int_{x_j}^{x} \omega + \int_{p_i}^{p_j} \omega + \xi\right)} \cdot \\
&\qquad \frac{\vartheta_1(\ldots)}{\vartheta_2(\ldots)} \cdot \frac{\vartheta_2(\ldots)}{\vartheta_1(\ldots)} \\
&= \exp(\nu t c_{ij})\, \psi_j(x) \left\{ \frac{\tau^* \vartheta_1\left(\int_{x_j}^{x} \omega + t \int_b \Omega + \xi\right)}{\vartheta_1\left(\int_{x_j}^{x} \omega + t \int_b \Omega + \xi\right)} \cdot \frac{\vartheta_2\left(\int_{x_j}^{x} \omega + \xi\right)}{\tau^* \vartheta_2\left(\int_{x_j}^{x} \omega + \xi\right)} \right\} \\
&= \exp(\nu t c_{ij})\ \psi_j(x)\, \psi_{ij}(x)
\end{aligned}$$

where τ^* represents translation by the $\frac{1}{2}$-period $\int_{p_i}^{p_j} \omega$.

Now, we would like to estimate the term within braces as $x \longrightarrow p_j$ and $t \longrightarrow 0$. We take $\vartheta_1 = \vartheta_{00}$ and $\vartheta_2 = \vartheta_{11}$, the elliptic θ-functions with $\frac{1}{2}$-integer characteristics. If ϑ represents the Riemann θ-function associated to the elliptic curve of lattice $\mathbb{Z}\{1, \tau\}$, then we have the usual relations:

$$\begin{aligned}
\vartheta_{00}(z,\tau) &= \vartheta(z,\tau); \quad \vartheta_{01}(z,\tau) = \vartheta(z + \tfrac{1}{2}, \tau), \\
\vartheta_{10}(z,\tau) &= \exp(\pi i \tau/4 + \pi i z)\, \vartheta(z + \tfrac{1}{2}\tau, \tau), \\
\vartheta_{11}(z,\tau) &= \exp(\pi i \tau/4 + \pi i (z + \tfrac{1}{2}))\, \vartheta(z + \tfrac{1}{2}(1+\tau), \tau) \\
\vartheta(z + \alpha\tau + \beta, \tau) &= \exp(-\pi i \alpha^2 \tau - 2\pi i \alpha z)\, \vartheta(z,\tau).
\end{aligned}$$

and the relations on page 19 [Mu 2].

Now, let $p_{ij} = \int_{p_i}^{p_j} \omega$, so that $p_{12} = \frac{1}{2}$, $p_{13} = \frac{1}{2}\tau$, $p_{14} = \frac{1}{2}(1+\tau)$. By our choice and use of tables we obtain

$$\psi_{12} = -\frac{\vartheta_{01}(U)}{\vartheta_{00}(U)} \cdot \frac{\vartheta_{11}(V)}{\vartheta_{10}(V)} = \frac{\vartheta(U+\frac{1}{2})}{\vartheta(U)} \cdot \frac{\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(V+1+\frac{1}{2}\tau)} \cdot \frac{\exp(\pi i(V+\frac{1}{2}))}{\exp(\pi i(V))},$$
$$\psi_{13} = -i\frac{\exp(-\pi i\tau/4 - \pi iU)}{\exp(-\pi i\tau/4 - \pi iV)} \cdot \frac{\vartheta_{10}(U)}{\vartheta_{00}(U)} \cdot \frac{\vartheta_{11}(V)}{\vartheta_{01}(V)}, \qquad V = \int_{x_i}^{x} \omega + \xi,$$
$$\psi_{14} = i\frac{\exp(-\pi i\tau/4 - \pi iU)}{\exp(-\pi i\tau/4 - \pi iV)} \cdot \frac{\vartheta_{11}(U)}{\vartheta_{00}(U)} \cdot \frac{\vartheta_{11}(V)}{\vartheta_{00}(V)}, \qquad U = \int_{x_i}^{x} \omega + t\int_b \Omega + \xi\ .$$

One uses the period relations

$$\vartheta_{01}(z+\alpha\tau+\beta) = \exp(-\pi i\alpha - \pi i\alpha^2\tau - 2\pi i\alpha z)\,\vartheta_{01}(z),$$
$$\vartheta_{00}(z+\alpha\tau+\beta) = \exp(\pi i\beta - \pi i\alpha^2\tau - 2\pi i\alpha z)\,\vartheta_{10}(z),$$
$$\vartheta_{11}(z+\alpha\tau+\beta) = \exp(\pi i(\beta-\alpha) - \pi i\alpha^2\tau - 2\pi i\alpha z)\,\vartheta_{11}(z),.$$

to find

$$\psi_{21} = \frac{\vartheta_{01}(U)}{\vartheta_{00}(U)} \cdot \frac{(+\vartheta_{11}(V))}{\{-(-\vartheta_{10}(V))\}} = -\psi_{12} = \psi_{43},$$
$$\psi_{23} = i\exp(-\pi i(U-V))\,\frac{\{-\vartheta_{11}(U)\}\,\vartheta_{11}(V)}{\vartheta_{00}(U)\,\{\vartheta_{00}(V)\}} = -\psi_{14},$$
$$\psi_{31} = -i\exp(-\pi i(U-V))\,\frac{\{\exp(-\pi i\tau + 2\pi iU)\vartheta_{10}(U)\}\,\vartheta_{11}(V)}{\vartheta_{00}(U)\,\{\exp(+\pi i - \pi i\tau + 2\pi iV)\,\vartheta_{01}(V)\}}$$
$$= -\exp(2\pi i(U-V))\,\psi_{13},$$
$$\psi_{32} = i\exp(-\pi i(U-V))\,\frac{\{\exp(\pi i - \pi i\tau + 2\pi iU)\,\vartheta_{11}(U)\}\,\vartheta_{11}(V)}{\vartheta_{00}(U)\,\{\exp(-\pi i\tau + 2\pi iV)\,\vartheta_{00}(V)\}}$$
$$= -\exp(2\pi i(U-V))\,\psi_{14},$$
$$\psi_{41} = +i\exp(-\pi i(U-V))\,\frac{\{\exp(-\pi i\tau + 2\pi iU)\,\vartheta_{11}(U)\}\,\vartheta_{11}(V)}{\vartheta_{00}(U)\,\{\exp(-\pi i\tau + 2\pi iV)\,\vartheta_{00}(U)\}}$$
$$= \exp(2\pi i(U-V))\,\psi_{14},$$

$$\psi_{24} = \psi_{13},$$

$$\psi_{34} = \psi_{12},$$

$$\psi_{42} = \psi_{31},$$

$$\psi_{ii} = 1.$$

Thus a suitable change of basis matrix (or of the coefficients ψ_{ij}) is

$$M = \begin{pmatrix} 1 & \psi_{12} & \psi_{13} & \psi_{14} \\ -\psi_{12} & 1 & -\psi_{14} & \psi_{13} \\ -e^{2\pi i(U-V)}\psi_{13} & -e^{2\pi i(U-V)}\psi_{14} & 1 & \psi_{12} \\ e^{2\pi i(U-V)}\psi_{14} & -e^{2\pi i(U-V)}\psi_{13} & -\psi_{12} & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ -e^{2\pi i(U-V)}B & A \end{pmatrix}$$

We can obtain other expressions for ψ_{12}, ψ_{13} and ψ_{14}:

$$\psi_{12} = -i\frac{\vartheta(U+\frac{1}{2})}{\vartheta(U)} \cdot \frac{\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(V+\frac{1}{2}\tau)} = -\exp(\pi i\tfrac{1}{2})\cdots$$

$$\psi_{13} = -i\exp(-\pi i(U-V))\cdot$$
$$\cdot\frac{\exp(\pi i\tau/4+\pi iU)\,\vartheta(U+\frac{1}{2}\tau)\,\exp(\pi i\tau/4+\pi i(V+\frac{1}{2}))\,\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(U)\,\vartheta(V+\frac{1}{2})}$$
$$= -\exp(\pi i\tfrac{1}{2}(1+\tau))\,\exp(2\pi iV)\cdot\frac{\vartheta(U+\frac{1}{2}\tau)\,\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(U)\,\vartheta(V+\frac{1}{2})}$$

$$\psi_{14} = i\exp(-\pi i(U-V))\cdot$$
$$\cdot\frac{\exp(\pi i\tau/4+\pi i(U+\frac{1}{2})+\pi i\tau/4+\pi i(V+\frac{1}{2}))}{\vartheta(U)}$$
$$\cdot\frac{\vartheta(U+\frac{1}{2}(1+\tau))\,\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(V)}$$
$$= -\exp(\pi i\tau/2)\,\exp(2\pi iV)\cdot\frac{\vartheta(U+\frac{1}{2}(1+\tau))\,\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(U)\,\vartheta(V)}$$

Lemma 2.9. $\det M \neq 0$.

Proof. If $t=0$ then $U=V$, $\psi_{12} = -\frac{a_1a_3}{a_0a_2}$, $\psi_{13} = -i\frac{a_2a_3}{a_0a_1}$, $\psi_{14} = i\frac{a_3^2}{a_0^2}$ and

$$\det M = \left(1+\left(\frac{a_1a_3}{a_0a_2}\right)^2\right)^2\left(1-\left(\frac{a_3a_2}{a_0a_1}\right)^2\right)^2 \neq 0$$

for appropriate values of a_1, a_2, a_3, a_0. □

In a similar fashion as we did in the previous example we can construct a table of the expansions for the functions ψ_i around the points p_i.

We have $\psi_{ij}(x) = \alpha_{ij}(\xi) + O(z_j)$ and the expansions in Table II:

Table II

	ψ_1	permutations	$D\psi_1$	permutations	$V_1\psi_1$
$p(1,1)$	$e^{t_1/z_1}(1+O(z_1))$	$\dots$	$\left(\frac{1}{z_1}+O(z_1)\right)\psi_1$	$\dots$	$\frac{1}{t}\psi_1$
$p(1,-1)$	$e^{t_1c_{12}}\psi_2(x)(\alpha_{12}+\cdots)$	$\dots$	$\dots$	$\dots$	$\frac{1}{t}\psi_1$
$p(-1,1)$	$e^{t_1c_{13}}\psi_3(x)(\alpha_{13}+\cdots)$	$\dots$	$\dots$	$\dots$	$-\frac{1}{t}\psi_1$
$p(-1,-1)$	$e^{t_1c_{14}}\psi_4(x)(\alpha_{14}+\cdots)$	$\dots$	$\dots$	$\dots$	$-\frac{1}{t}\psi_1$

With the expansions we have for $D\psi_1$, $D\psi_2$, $D\psi_3$, $D\psi_4$ in table II we get an expression $V_1.\psi_i = \Sigma\lambda_{ij}D\psi_j + C(1)$, since the matrix (α_{ij}) is nonsingular by the lemma. (In here we identify the time evolution parameter with the local parameters z_i about p_i and with the deformation parameter t_1).
Therefore obtaining a matrix differential operator in $M_4[[t]][D]$. Also, we obtain a commutative ring of differential operators in $M_4[\mathbb{C}[[t]]][D]$, as follows from the representation to be obtained for the v_i's.

We want to study in more detail the relations arisen from the action of the translation group $G \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, whose elements we indicate by $\tau_{ij} = p_j - p_i$. The action on functions is defined by $\tau_{ij}^* f(x) = f(x+\tau_{ij})$ and one can show the formula

$$\psi_{p_k}(x+\tau_{ij}) = e^{tc_{ij}}\psi_{\tau_{ij}(p_k)}(x+\tau_{ij})\,\psi_{ij}^{\tau_{ij}(p_k)}(x+\tau_{ij}), \tag{21}$$

where $\psi_{ij}^{\tau_{ij}(p_k)}$ is defined as follows:

$$\psi_{ij}^{\tau_{ij}(p_k)}(x+\tau_{ij}) = \frac{\vartheta_1\left(\int_{\tau_{ij}(p_k)}^{x+\tau_{ij}}\omega + t\int_b\Omega + \xi + \int_{p_i}^{p_j}\omega\right)}{\vartheta_2\left(\int_{\tau_{ij}(p_k)}^{x+\tau_{ij}}\omega + \xi + \int_{p_i}^{p_j}\omega\right)} \cdot \frac{\vartheta_2\left(\int_{\tau_{ij}(p_k)}^{x+\tau_{ij}}\omega + \xi\right)}{\vartheta_1\left(\int_{\tau_{ij}(p_k)}^{x+\tau_{ij}}\omega + t\int_b\Omega + \xi\right)}.$$

The above formula translates into the multiplicative cocycle formula

$$\psi_{\tau_{ij}(p_k)}(y) = \phi_{ij}^{\tau_{ij}(p_k)}(y)\,\psi_{p_k}(y). \tag{$*$}$$

Indeed, identifying the elements of G with the translation points $\{p_i\}$ and with the translations $\tau_{ij} = p_j - p_i$ once an origin $p_0 \in \{p_i\}$ is chosen , we have the elements $\{\psi_\sigma\}$, $\psi_\sigma \in \Gamma(E \times \{|t| < \epsilon\}, \mathcal{F}^*(*\mathcal{D})) = \hat{S}$, which is a ring that contains

$\mathcal{S} = \Gamma(E, \mathcal{O}(*\mathcal{D}))$ and $\phi_{ij}^{\tau(\sigma)} = \tau \cdot \psi_\sigma = 1 + O(t, z)$ which are also elements in $\hat{\mathcal{S}}$. Thus, equation $(*)$ is the cocycle relation $\tau \cdot \psi_\sigma = \psi_{\tau\sigma}/\psi_\sigma$.

Now, if we differentiate with respect to t we obtain

$$(22) \qquad D\psi_{\tau_{ij}(p_k)}(y) = \left\{ D\phi_{ij}^{\tau_{ij}(p_k)}(y) + \phi_{ij}^{\tau_{ij}(p_k)}(y)\, D\log\psi_{p_k}(y) \right\} \psi_{p_k}(y).$$

Let the cocycle relation $(*)$ be written $\phi_{\sigma,\tau} = \tau \cdot \psi_\sigma = \psi_{\tau\sigma}/\psi_\sigma$.

Now, one has the expansions around the points $\nu \in \{p_i\}$

$$\psi_\sigma(\text{about } \nu) = e^{t/z}(\alpha_{\sigma,\nu}(t) + \beta_{\sigma,\nu}(t)z + \cdots).$$

One obviously has $D\psi_\sigma(\text{about } \nu) = \left[\frac{1}{z} + O(1)\right] \psi_\sigma(\text{about } \nu)$ (assuming $\alpha_{\sigma,\nu}(0) \neq 0$). Around the points ν the expansions of the coordinate V_i (about ν) $= \frac{\alpha_\nu}{z} + O(1) = \frac{\alpha_\nu}{z} + \beta_\nu + O(z)$, where α_ν is a constant. Thus

$$V_i \text{ (about } \nu) \cdot \psi_\sigma \text{ (about } \nu) = \Sigma\lambda_{\sigma,\rho} D\psi_\rho \text{ (about } \nu) + O(1)e^{\frac{t}{z}}.$$

Since the poles in z have to be peeled off, this leads to the equation

$$(23) \qquad \alpha_\nu\, \alpha_{\sigma,\nu} = \sum_\rho \lambda_{\sigma,\rho}\, \alpha_{\rho,\nu}(t).$$

This means that $(\lambda_{\sigma\rho})(\alpha_{\rho,\nu}(t)) = (\alpha_{\sigma,\nu}(t))\,\mathrm{diag}(\alpha_\nu)$; namely

Lemma 2.10. *$(\lambda_{\sigma\rho})$ is diagonalizable and nonsingular if* $\det \mathrm{diag}(\alpha_\nu) \neq 0$.

In an analogous way we obtain a relation for the coefficients $\lambda_{\sigma\rho}$ of the 0^{th} order part: we have the equations

$$(24) \qquad \alpha_\nu\beta_{\sigma,\nu} + \beta_\nu\alpha_{\sigma,\nu} = \sum_\rho \lambda_{\sigma,\rho}(\beta_{\rho,\nu} + \alpha'_{\rho,\nu}) + \sum_\rho \mu_{\sigma,\rho}\alpha_{\rho,\nu}.$$

Namely

$$(\mu_{\sigma,\rho})(\alpha_{\rho,\nu}(t)) = (\beta_{\sigma,\nu}(t))\,\mathrm{diag}(\alpha_\nu) + (\alpha_{\sigma,\nu}(t)\,\mathrm{diag}(\beta_\nu) - (\lambda_{\sigma\rho})[(\beta_{\rho,\nu}(t)) + (\alpha'_{\rho\nu})].$$

Let $\mu = (\mu_{\sigma,\rho})$, $\alpha = (\alpha_{\rho,\nu})$, $\beta = (\beta_{\sigma,\nu})$, $r = \mathrm{diag}(\alpha_\nu)$, $s = \mathrm{diag}(\beta_\nu)$, $\lambda = (\lambda_{\sigma,\rho})$; then we can write the operator as follows: $\lambda D + \mu$, but $\alpha^{-1}(\lambda D + \mu)\alpha = \alpha^{-1}\lambda\alpha D + \alpha^{-1}\lambda\alpha' + \alpha^{-1}\mu\alpha = rD + s + [\alpha^{-1}\beta, r]$. Thus, by an appropriate conjugation the operator is almost with constant coefficients.

Actually, by looking at the expansions of V_i we obtain $s = 0$ so that the representation of V_i as differential operator is $r_i D + [a, r_i]$, $a = \alpha^{-1}\beta$.

Proposition 2.11. *There is a unique pseudodifferential operator Ψ_i associated with $\mathbb{D}_i = r_i\partial + [a, r_i]$, and a unique $W = 1 + \sum_{i=1}^{\infty} s_{-i}\partial^{-i}$ pseudodifferential operator such that $\mathbb{D}_i = W^{-1}\Psi_i W$ for any i. Any such W differ by a diagonal matrix.*

If $\Psi_i = r_i\partial + \sum_{k=1}^{\infty} a_{-k}\partial^{-k}$, then the equality $W\mathbb{D}_i = \Psi_i W$ yields the following equations:

$$(25)\qquad \begin{aligned} &[s_{-1} + a, r_i] = 0 \\ &[s_{-2}, r_i] = a_{-1} + r_i s'_{-1} - s_{-1}[a, r_i] \\ &[s_{-(n+1)}, r_i] - a_{-n} = r_i s'_{-n} + \sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}(-1)^k\binom{n-j-1}{k} a_{-(n-k-j)} s_{-j}^{(k)} \\ &\qquad - \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k} s_{-(n-k)}[a, r_i]^{(k)}. \end{aligned}$$

One can choose $s_{-1} = -a$, and the remaining s_{-k} such that $[s_{-k}, r_i] = 0$ for any $i = 1, 2, 3$. Since $r_1 = \mathrm{diag}(1, 1, -1, -1)$, $r_2 = \mathrm{diag}(1, -1, 1, -1)$, $r_3 = \mathrm{diag}(1, -1, -1, 1)$, being commutative with the group of matrices generated by $\langle r_1, r_2\rangle$ means that s_{-k} is diagonal, $k > 1$. Then, the values of the a_{-k} are uniquely determined. If we perturbe the coefficients of W by diagonal matrices we obtain another solution to this representation.

Proposition 2.12. *Given the operator $r\partial + [a, r]$, with r constant diagonal matrix, $r^2 = 1$, then, there exists a pseudo-differential operator $K = 1 + \Sigma w_{-i}\partial^{-i}$ such that $r\partial + [a, r] = K(r\partial)K^{-1}$. Any such a solution K differs from a given one by a constant matrix pseudodifferential operator commuting with r.*

Proof. Let $L(x) = [x, r]$; this is a linear derivation and satisfies $rL(x) + L(x)r = 0$. We want to find a solution K to the equation

$$(26)\qquad (r\partial + L(a))(1 + \Sigma w_{-i}\partial^{-i}) = (1 + \Sigma w_{-i}\partial^{-i})(r\partial).$$

This gives a system that implies the differential equations in w_{-i}

$$(**)\qquad \begin{cases} L(a) = L(w_{-1}) \\ L(w_{-(i+1)}) = rw'_{-i} + L(a)w_{-i} = P(w_{-i}). \end{cases}$$

Notice that we have the following identities:

(27) $$LP(x) = PL(x) - 2rL(a)x \quad \text{and} \quad P(rx) = rP(x) - 2rL(a)x.$$

Also

(28) $$L^2(x) = [L(x), r] = 2(x - rxr) = -2rL(x) = L(x)(2r).$$

Now any 4×4 matrix x can be written as $x = -\frac{1}{2}rL(x) + d$, with $L(d) = 0$. Indeed, this follows from the above properties of the operator L. Let us decompose $w_{-i} =: -\frac{1}{2}rL(w_{-i}) + d_{-i}$. On one hand we have

$$-2rL(w_{-(i+1)}) = L^2(w_{-(i+1)}) = LP(w_{-i}) = PL(w_{-i}) - 2rL(a)w_{-i}.$$

Namely,

(29) $$L(w_{-(i+1)}) = -\tfrac{1}{2}rPL(w_{-i}) + L(a)w_{-i}.$$

Replacing, we obtain

$$\begin{aligned} L(w_{-(i+1)}) &= -\frac{1}{2}L(w'_{-i}) - \tfrac{1}{2}rL(a)L(w_{-i}) - \tfrac{1}{2}L(a)rL(w_{-i}) + L(a)d_{-i} \\ &= -\tfrac{1}{2}L(w_{-i}) + L(a)d_{-i} = L(-\tfrac{1}{2}w'_{-i} + ad_{-i}). \end{aligned}$$

This implies that $w_{-(i+1)} = -\frac{1}{2}w'_{-i} + ad_{-i} + d_{-(i+1)}$, where $d_{-(i+1)}$ belongs to the kernel of L.

In order to solve (**), we will represent the solution $w_{-(i+1)}$ as the sum of a term in Image of L + a term in Ker L. Thus, we can write the following recursion formula for w_{-i}:

(30) $$w_{-(i+1)} = \tfrac{1}{4}rL(w_{-i})' - \tfrac{1}{2}rL(a)d_{-i} + d_{-(i+1)},$$

where the d_{-i} are to be determined so as to satisfy the system (**) since we have

$$\begin{aligned} L(w_{-(i+1)}) &= L(\tfrac{1}{4}rL(w'_{-i}) - \tfrac{1}{2}rL(a)d_{-i} + d_{-(i+1)}) = -\tfrac{1}{2}L(w_{-i})' + L(a)d_{-i} \\ &= rw'_{-i} - rd'_{-i} + L(a)\left(w_{-i} + \tfrac{1}{2}rL(w_{-i})\right) \\ &= P(w_{-i}) - rd'_{-i} + \tfrac{1}{2}L(a)rL(w_{-i}). \end{aligned}$$

Assuming that $L(w_{-i})$ is known, it follows that $d'_{-i} = -\frac{1}{2}L(a)L(w_{-i})$. This element belongs to $\operatorname{Ker} L$ since $L(L(x)L(y)) = -2(rL(x) + L(x)r)L(y) = 0$, and gives, up to a constant matrix commuting with r, the solution we want. □

The first terms are

$$w_{-1} = -\tfrac{1}{2}rL(a) + d_{-1} \text{ where } d_{-1} = -\tfrac{1}{2}\int L(a)^2$$

$$w_{-2} = \tfrac{1}{4}rL(a)' - \tfrac{1}{2}rL(a)d_{-1} + d_{-2} \text{ where } d'_{-2} = -\tfrac{1}{2}L(a)\left(\tfrac{1}{4}rL(a)' - \tfrac{1}{2}rL(a)d_{-1}\right)$$

We now determine the differential operator part of the pseudo-differential operator $K(r\partial^2)K^{-1}$. If $K = 1+\Sigma w_{-i}\partial^{-i}$, $K^{-1} = 1-w_{-1}\partial^{-1}+(w_{-1}^2-w_{-2})\partial^{-2}+\cdots$.

$$\begin{aligned} K(r\partial^2)K^{-1} &= (1 + w_{-1}\partial^{-1} + w_{-2}\partial^{-2} + \cdots) \\ &\quad (r\partial^2 - rw_{-1}\partial - 2rw'_{-1} + r(w_{-1}^2 - w_{-2}) + \cdots) \\ &= r\partial^2 + L(w_{-1})\partial + L(w_{-2}) - 2rw'_{-1} + rw_{-1}^2 - w_{-1}rw_{-1} + \cdots \end{aligned}$$

The independent term can be written as:

$$\begin{aligned} rw'_{-1} + L(a)w_{-1} - 2r\left(-\tfrac{1}{2}rL(a)' - \tfrac{1}{2}L(a)^2\right) + (rw_{-1} - w_{-1}r)w_{-1} &= \\ = -rw'_{-1} = \tfrac{1}{2}L(a)' + \tfrac{1}{2}rL(a)^2. \end{aligned}$$

Thus

$$(*_i) \qquad (K(r_i\partial^2)K^{-1})_+ = r_i\partial^2 + L_i(a)\partial + \tfrac{1}{2}L_i(a)' + \tfrac{1}{2}r_iL_i(a)^2.$$

Example 2.13. Assume now that the coordinates V_1, V_2, V_3 (having the expansions shown in Example 2.7) satisfy the Euler top equations

$$(31) \qquad \begin{cases} \dfrac{dV_1}{dt} = -V_2V_3 \\ \dfrac{dV_2}{dt} = -V_1V_3 \\ \dfrac{dV_3}{dt} = -V_1V_2 \end{cases} \qquad \text{with relations} \qquad \begin{aligned} \frac{V_1^2}{\alpha_2\alpha_3} + \frac{V_2^2}{\alpha_3\alpha_1} + \frac{V_3^2}{\alpha_1\alpha_2} &= 1 \\ \frac{\lambda_1V_1^2}{\alpha_2\alpha_3} + \frac{\lambda_2V_2^2}{\alpha_3\alpha_1} + \frac{\lambda_3V_3^2}{\alpha_1\alpha_2} &= h. \end{aligned}$$

Here $V_1 = \frac{\epsilon_1}{t} - \epsilon_1(u+v)t + \cdots$, $V_2 = \frac{\epsilon_2}{t} + \epsilon_2ut + \cdots$, $V_3 = \frac{\epsilon_1\epsilon_2}{t} + \epsilon_1\epsilon_2vt$, $\epsilon_1^2 = \epsilon_2^2 = 1$, $u = \frac{1}{6}((\lambda_3 - h)\alpha_3 + (h - \lambda_1)\alpha_1)$, $v = \frac{1}{6}((h - \lambda_2)\alpha_2 + (\lambda_1 - h)\alpha_1)$, $w = -(u+v) = \frac{1}{6}((h - \lambda_3) + (\lambda_2 - h)\alpha_2)$.

We have seen that the differential operator associated with V_i is $D_i = r_i\partial + L_i(a)$, $L_i(a) = [a, r_i]$. Thus, $D_i^2 = \partial^2 + r_iL_i(a') + L_i(a)^2$ and $D_iD_j = D_jD_i = r_k\partial^2 + L_k(a)\partial + r_iL_j(a') + L_i(a)L_j(a)$ (cycle $i \to j \to k$). We wish to compare the operator $(*_k)$ with the operator associated to the function $-\frac{dV_k}{dt}$, i.e., D_iD_j.

Since the operators D_i satisfy the equations

$$(32) \qquad \sum_{i=1}^{3} \alpha_i D_i^2 = \alpha_1\alpha_2\alpha_3, \qquad \sum_{i=1}^{3} \lambda_i\alpha_i D_i^2 = \alpha_1\alpha_2\alpha_3 h,$$

we obtain the relation $D_i^2 - D_j^2 = \alpha_k(h - \lambda_k)$, $i \to j \to k \to i$.

If $s_i = r_iL_i(a') + L_i(a)^2$, then we can also write $s_i - s_j = \alpha_k(h - \lambda_k)$. Let $T = r_iL_j(a') + L_i(a)L_j(a) = r_jL_i(a') + L_j(a)L_i(a)$, then it follows

$$(33) \qquad [L_i(a), L_j(a)] = r_jL_i(a') - r_iL_j(a') = r_jL_k(a')r_j \qquad (i \to j \to k \to i).$$

Also

$$(33') \quad r_kT = s_j - L_j(a)^2 + r_kL_i(a)L_j(a) = s_i - L_i(a)^2 + r_kL_j(a)L_i(a) \quad (r_k = r_ir_j),$$

which yields, using the relations

$$(33'') \qquad L_k(a) = r_iL_j(a) + L_i(a)r_j = r_jL_i(a) + L_j(a)r_i \quad (i \to j \to k)$$

$$(34) \qquad \begin{aligned} L_k(a)(r_jL_i(a) - r_iL_j(a)) &\overset{\text{by (33)}}{=} L_k(a)r_jL_k(a)r_j \\ &= s_i - s_j = \alpha_k(h - \lambda_k) \quad (i \to j \to k \to i) \end{aligned}$$

Let us compute the differences between the independent terms of the operators $(K_k(r_k\partial^2)K_k^{-1})_+$ and D_iD_j. This is $2S = 2T - r_ks_k$

$$\begin{aligned} 2r_kS &= r_kT + r_kT - s_k \\ &\overset{\text{by (33')}}{=} s_i - L_i(a)^2 + r_kL_j(a)L_i(a) + r_jL_j(a') + r_kL_i(a)L_j(a) - s_k \\ &\overset{\text{by (33)}}{=} s_i - s_k - L_i(a)^2 + r_kL_j(a)L_i(a) + \\ &\qquad + r_k(L_k(a)L_i(a) - L_i(a)L_k(a))r_i + r_kL_i(a)L_j(a) \\ &\overset{\text{by (33'')}}{=} s_i - s_k - L_k(a)r_jL_i(a) + r_kL_k(a)L_i(a)r_i + r_kL_i(a)r_kL_i(a) \\ &= s_i - s_k + r_k(L_i(a)r_kL_i(a)r_k)r_k \\ &= s_i - s_k + r_k(\alpha_i(h - \lambda_i)r_k \\ &= \alpha_i(h - \lambda_i) - \alpha_j(h - \lambda_j). \end{aligned}$$

Thus

$$\left(K_k(r_k\partial^2)K_k^{-1}\right)_+ = D_iD_j + \tfrac{1}{2}\left\{\alpha_j(h-\lambda_j) - \alpha_i(h-\lambda_i)\right\} r_k = D_iD_j + c_k r_k,$$

and we can write

$$\left(K_k(r_k\partial^2 - c_k r_k)K_k^{-1}\right)_+ = D_iD_j,$$

with $r_k\partial^2 - c_k r_k = G_k(r_k\partial^2)G_k^{-1}$, G_k being a scalar differential operator and therefore commuting with r_k.

3. Ring of differential operators.

We consider here the construction of a map $(A, \mathcal{D}, \mathcal{F}) \to R$ into a commutative ring of differential operators R for the data related to a smooth elliptic curve A, an ample divisor $\mathcal{D}$ on A, and a line bundle $\mathcal{F}$ on A such that $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$. The triple $(A, \mathcal{D}, \mathcal{F})$ will be called Krichever data as similar to the Krichever data in [Mu 1]. One can keep in mind the example of an elliptic curve in $\mathbb{P}^3$ with $\mathcal{D}$ the divisor cut out by an odd section (i.e. four points at infinity which form a group of translates isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$). The bundle $\mathcal{F}$ will be of the form $\mathcal{F} = [\tau_x^{-1}\mathcal{D} - \mathcal{D}]$ for some $x \in A$, i.e. $\mathcal{F} \in \mathrm{Pic}^0(A)$ will be the image of a direction vector $D \in Lie(A) = \mathbb{C}$.

We want to construct a line bundle $\mathcal{F}^*$ on $A \times \mathbb{C}^\infty$ ($\mathbb{C}^\infty := \underrightarrow{lim}\, \mathbb{C}^n$) in the following way. Take the covering formed by $(U - \mathcal{D}) \times \mathbb{C}^\infty = \mathcal{U}_0$ and neighborhoods $U_\alpha \times \mathbb{C}^\infty = \mathcal{U}_\alpha$ around the points $\{z_\alpha\} \times \mathbb{C}^\infty$ of $\mathcal{D} \times \mathbb{C}^\infty$, and let $\mathcal{F}^*$ be defined by $\mathcal{F} \otimes \mathcal{O}_{\mathbb{C}^\infty}$ on each of these open sets and given by the transition functions $g_{0,\alpha}(u, x_\alpha, t\!t) = \exp\left(\sum_{i\geq 1} t_i \text{polar part}\left(\int_{x_\alpha}^{u} \Omega_i\right)\right)$ at the overlappings $\mathcal{U}_\alpha \cap \mathcal{U}_0$. Here $t\!t = (t_1, t_2, \ldots, t_n, \ldots) \in \mathbb{C}^\infty$ and Ω_i are differential forms of 2^{nd} kind on A whose expansions around points of $\mathcal{D}$ is $(-1)^i \dfrac{dz}{z^{i+1}} c_i(x) + O(z^{-(i-1)})\, dz$ with $x \in \mathcal{D}$. An arsenal of such forms is gotten by taking the differential of derivatives of $log\, \vartheta$ (as will be shown in Appendix **2**), where ϑ is the theta function vanishing on $\mathcal{D}$. Of course, one makes sure that the $g_{0,\alpha}(u, x_\alpha, t\!t)$ are compatible transition functions. For instance, this is done by requiring the existence of a covering of $\mathcal{D}$ by contractible charts (small disks) U_α around the points x_α of $\mathcal{D}$.

For any line bundle $\mathcal{G}$ on A, we can also define similarly the line bundle $\mathcal{G}^*$ on $A \times \mathbb{C}^\infty$. This bundle will have transition functions $\tilde{g}(u, tt) = g_{\alpha,\beta}(u) g_{0,\gamma}(u, tt)$ where $g_{\alpha,\beta}$ is a set of transition functions for $\mathcal{G}$.

Notice that (around $\mathcal{D}$) $\int_{x_\alpha}^{x} \Omega_1 = \frac{a(x_\alpha)}{z} + c(x_\alpha) + O(z)$ where z is the local parameter about $x_\alpha \in \mathcal{D}$ such that $\frac{\partial}{\partial z}$ is a holomorphic vector field on A. We want to define a differential operator $\nabla : \mathcal{F}^* \longrightarrow \mathcal{F}^*(*\mathcal{D})$ such that

$$\nabla(s) = \frac{a(x)\, s}{z} + \text{ section of } \mathcal{F}^*$$

for a section s of $\mathcal{F}^*$.

Take $\nabla := \frac{\partial}{\partial t_1} = \frac{\partial}{\partial z}$, then $\nabla \tilde{g}_{\alpha\beta} = \frac{a(x)}{z}\tilde{g}_{\alpha\beta} + (c(x) + O(z))\tilde{g}_{\alpha\beta}$. Now, for a holomorphic section s of $\mathcal{F}^*$ we have $\nabla(s) = \nabla s_\alpha = \nabla \tilde{g}_{\alpha\beta} s_\beta = \frac{a(x)}{z} s_\alpha + \tilde{g}_{\alpha\beta}(\overbrace{\nabla s_\beta + c(x) + O(z)}^{t_\beta})$ on $U_{\alpha\beta}$, since there are holomorphic functions s_α such that $s_\alpha = \tilde{g}_{\alpha\beta} s_\beta$ on $U_{\alpha\beta}$.

On $U_{\alpha\beta\gamma}$ we have the relation $\frac{a(x)}{z} s + \tilde{g}_{\alpha\beta} t_\beta = \frac{a(x)}{z} s + \tilde{g}_{\alpha\gamma} t_\gamma$. Thus, $t_\beta = \tilde{g}_{\beta\gamma} t_\gamma$ over $U_{\alpha\beta\gamma}$, i.e. t is a section of $\mathcal{F}^*$.

We consider the situation where A is an elliptic curve and $\mathcal{D}$ = sum of different points $= \Sigma p_i$. Let $U_0 = A - \mathcal{D}$ be the affine piece and $V_\mathcal{D} = \cup U_{p_i}$ where $V_\mathcal{D}$ is a disjoint union of small disks around the points p_i and also take $\nabla := \frac{\partial}{\partial t_1}$.

For any n we have the exact sequence over $A \times \mathbb{C}^\infty$

$$0 \longrightarrow \mathcal{F}^*(n\mathcal{D}) \xrightarrow{\nabla} \mathcal{F}^*((n+1)\mathcal{D}) \longrightarrow \mathcal{F}^*((n+1)\mathcal{D})/\mathcal{F}^*(n\mathcal{D}) \longrightarrow 0 \tag{35}$$

($\mathcal{F}^*(n\mathcal{D}) = \mathcal{F}^* \otimes \mathcal{O}(n\mathcal{D})$).

This induces the exact sequence of cohomology groups

$$0 \longrightarrow \Gamma(\mathcal{F}^*(n\mathcal{D})) \longrightarrow \Gamma(\mathcal{F}^*((n+1)\mathcal{D})) \longrightarrow \Gamma(\mathcal{F}^*((n+1)\mathcal{D})/\mathcal{F}^*(n\mathcal{D})) \longrightarrow 0 \tag{36}$$

This follows because $H^i(\mathcal{F}^*) = 0$, $i = 0, 1$, and $H^1(\mathcal{F}^*(n\mathcal{D})) = 0$ for any $n \geq 1$.

Indeed, the hypothesis $H^i(\mathcal{F}) = 0$, $i = 0, 1$, implies $H^i(A \times \mathbb{C}^\infty, \mathcal{F}^*) = H^i(\pi_1^{-1}\mathfrak{U}, \mathcal{F}^*) = 0$, $i = 0, 1$, where $\mathfrak{U}$ is an affine cover of A for which $\mathcal{F}^*$ is isomorphic to $\mathcal{F}\otimes\mathcal{O}_{\mathbb{C}^\infty}$ on any open of $\pi_1^{-1}\mathfrak{U}$, and $\pi_1 : A\times\mathbb{C}^\infty \to A$ is the projection. Now, the sheaf $\mathcal{F}^*((n+1)\mathcal{D})/\mathcal{F}^*(n\mathcal{D})$ is supported at $\mathcal{D}\times\mathbb{C}^\infty$, thus $H^1(\mathcal{F}^*((n+1)\mathcal{D})/\mathcal{F}^*(n\mathcal{D}))=0$. Then, by using induction on the exact sequence

(37)

$$\cdots\to H^1(\mathcal{F}^*(n\mathcal{D}))\to H^1(\mathcal{F}^*((n+1)\mathcal{D}))\to H^1(\mathcal{F}^*((n+1)\mathcal{D})/\mathcal{F}^*(n\mathcal{D}))\to\cdots$$

follows that $H^1(\mathcal{F}^*(n\mathcal{D}))=0$, all n.

On the other hand,

$\Gamma(\mathcal{F}^*((n+1)\mathcal{D})/\mathcal{F}^*(n\mathcal{D}))\simeq\oplus_i\Gamma(U_{p_i}\times\mathbb{C}^\infty,\mathcal{F}^*((n+1)\mathcal{D})/\mathcal{F}^*(n\mathcal{D}))\simeq\oplus_i\mathbb{C}[[\mathit{t\!t}]]$ $\simeq\mathbb{C}[[\mathit{t\!t}]]^{\deg\mathcal{D}}$.

Given the $\mathbb{C}[[\mathit{t\!t}]]$-linearly independent sections $s_1,\ldots,s_k$ belonging to the space $\Gamma(\mathcal{F}^*(n\mathcal{D}))\backslash\Gamma(\mathcal{F}^*((n-1)\mathcal{D}))$, then, the sections $\nabla s_1,\ldots,\nabla s_k$ are in $\Gamma(\mathcal{F}^*((n+1)\mathcal{D}))\backslash\Gamma(\mathcal{F}^*(n\mathcal{D}))$ and they are $\mathbb{C}[[\mathit{t\!t}]]$-linearly independent.

Indeed, if $\Sigma_i\lambda_i\nabla s_i=0$ (module $\Gamma(\mathcal{F}^*(n\mathcal{D}))$), then $\nabla(\Sigma_i\lambda_i s_i)=\Sigma_i(\nabla\lambda_i)s_i+\Sigma_i\lambda_i\nabla s_i=0$ (module $\Gamma(\mathcal{F}^*(n\mathcal{D}))$) implies $\Sigma_i\lambda_i s_i\in\Gamma(\mathcal{F}^*((n-1)\mathcal{D}))$. Namely, $\Sigma_i\lambda_i s_i=0$ (module $\Gamma(\mathcal{F}^*((n-1)\mathcal{D}))$) , and from this follows $\lambda_i=0$.

Now, since the rank of $\Gamma(\mathcal{F}^*(n\mathcal{D}))$ is $n.deg(\mathcal{D})$, we have that if $s_1,\ldots,s_k$ $(k=\deg\mathcal{D})$ is a $\mathbb{C}[[\mathit{t\!t}]]$-basis of $\Gamma(\mathcal{F}^*(\mathcal{D}))$, then $\{\nabla^r s_1,\ldots,\nabla^r s_k;\quad r=0,1,\ldots,n\}$ is a $\mathbb{C}[[\mathit{t\!t}]]$-basis of $\Gamma(A\times\mathbb{C}^\infty,\mathcal{F}^*((n+1)\mathcal{D})$.

Now, we wish to show the representability of the affine ring $R=\Gamma(A-\mathcal{D},\mathcal{O}_A)$ as a ring of differential operators. Let $\mathcal{D}=\Sigma p_i$, there is an embedding $R=\Gamma(A-\mathcal{D},\mathcal{O}_A)\hookrightarrow\oplus_n\Gamma(A,\mathcal{O}(\mathcal{D})^{\otimes n})=$ homogeneous coordinate ring.

Also, we have an induced mapping

(38) $$\Gamma(A,\mathcal{O}(\mathcal{D})^n)\otimes\Gamma(A\times\mathbb{C}^\infty,\underline{\mathcal{F}}^*(k\mathcal{D}))\to\Gamma(A\times\mathbb{C}^\infty,\underline{\mathcal{F}}^*((n+k)\mathcal{D})),$$

and, if $\alpha\in R$, $\alpha=\sum\dfrac{\alpha_n(x_1)}{(z-z(x_1))^n}+$ lower terms $\in\Gamma(A,\mathcal{O}(\mathcal{D})^n)$.

Thus

(39) $$\alpha.s_i=\sum a^j_{ir}(\mathit{t\!t})\nabla^r s_j={}^t\!\left(\sum a^j_{ir}(\mathit{t\!t})\nabla^r\right)\begin{pmatrix}s_1\\ \vdots\\ s_k\end{pmatrix},$$

i.e.

(40) $$\alpha\begin{pmatrix}s_1\\ \vdots\\ s_k\end{pmatrix}=\sum_{r=0}^{n}\begin{pmatrix}a^1_{1r}(\mathit{t\!t}) & \ldots & a^k_{1r}(\mathit{t\!t})\\ \vdots & & \vdots\\ a^1_{kr}(\mathit{t\!t}) & \ldots & a^k_{kr}(\mathit{t\!t})\end{pmatrix}\nabla^r\begin{pmatrix}s_1\\ \vdots\\ s_k\end{pmatrix}.$$

We define an immersion ring map $\Phi : R \hookrightarrow M_k(\mathbb{C}[[t\!t]])[\nabla]$ by

$$\Phi(\alpha) = \sum_{r=0}^{n} \left(a^j_{ir}(t\!t)\right) \nabla^r.$$

Let $\mathcal{F}$ be associated to ε. For an elliptic curve and a divisor ε on it, $h^0(\varepsilon) = h^1(\varepsilon) = 0$ if $\deg \varepsilon = 0$. Conversely, if $\varepsilon \in$ Jacobian of $A = \{\varepsilon,\ \deg \varepsilon = 0\}$, then $\varepsilon \sim_\ell p - p_0$ and therefore $h^0(\varepsilon) = h^1(\varepsilon) = 0$ unless $\varepsilon \sim_\ell 0$ (e.g. Prop. 4.1.2, [Ha]).

Thus, the above proves the

Theorem 3.1. *Let $\mathcal{D} = \Sigma p_i$ be a divisor on an elliptic curve A with $A - \mathcal{D}$ affine and $\mathcal{F}$ a line bundle on A such that $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$. Suppose $\mathcal{D}$ gives rise to a set of compatible transition functions for the bundle $\mathcal{F}^*$. Then, there is an injection of the affine ring $R = \Gamma(A - \mathcal{D}, \mathcal{O}_A)$ into the ring $M_k(\mathbb{C}[[t\!t]])[\nabla]$, and the space $\Gamma(\mathcal{F}^*(\mathcal{D})/\mathcal{F}^*)$ has a finite $\mathbb{C}[[t\!t]]$-basis of k elements, ($k = \deg\mathcal{D}$).*

Example 3.2. If A is an elliptic curve in $\mathbb{P}^3$ and $\mathcal{D} = \sum_{i=1}^{4} p_i$ (typically the section cut out by an odd theta function). Then $\Gamma(\underline{\mathcal{F}}^*((n+1)\mathcal{D}))$ has generators $\{s_1, s_2, s_3, s_4, \ldots, \nabla^n s_1, \nabla^n s_2, \nabla^n s_3, \nabla^n s_4\}$, where $\{s_i\}$ is a $\mathbb{C}[[t\!t]]$-basis of $\Gamma(\underline{\mathcal{F}}^*(\mathcal{D}))$.

Thus, there is an embedding of $R = \Gamma(A - \mathcal{D}, \mathcal{O}_A)$ into $M_4(\mathbb{C}[[t\!t]])[\nabla]$.

Appendix 1. Multicomponent KP hierarchy.

Let us introduce some notation to consider the multicomponent KP equations. See [Ad-B]. We will consider wave functions of the form

$$w(t\!t) = \left(I + \sum_{i>0} w_i z^i\right) \phi(t\!t)$$

where w_i are $k \times k$ matrices depending on $t\!t$ and $\phi(t\!t)$ is the exponential diagonal matrix

$$\phi(t\!t) = \exp\left(\sum_{i>0} \begin{pmatrix} t_i^1 & & & \\ & t_i^2 & & \\ & & \ddots & \\ & & & t_i^k \end{pmatrix} z^{-i}\right)$$

and $tt = (t_1^1, t_1^2, \ldots, t_1^k, \ldots)$ is the vector of time variables t_i^j.

We have

$$\partial_{t_i^j}\phi(tt) = \frac{1}{z^i}\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}\phi(tt),$$

and if $\partial = \sum_{j=1}^k \partial_{t_1^j}$, $\partial\phi(tt) = \frac{1}{z}\phi(tt)$.

Given the matrix pseudodifferential operator $W = I + \sum_{i=1}^\infty w_i\partial^{-i}$ we have

$$W\phi(tt) = \left(I + \sum_{i>0} w_i z^i\right)\phi(tt) = w(tt).$$

The multicomponent KP equations can be written as the set of Lax equations

$$\partial_{t_i^j} Q = [Q, [R_j^i]_+] \tag{41}$$

where $Q = W^{-1}(A\partial)W$, $A =$ constant diagonal matrix with nonzero entries and $R_j^i = W^{-1}E_{jj}\partial^i W$, $E_{jj} = \mathrm{diag}(0,\ldots,0,\overset{j}{1},0,\ldots,0)$, and $[\ ]_+$ indicates the differential operator part of R_j^i.

The set of equations (41) is also equivalent to the equations in the wave operator W

$$\partial_{t_i^j} W = -W[R_j^i]_- \tag{42}$$

where $[\ ]_-$ is the formal pseudodifferential operator part.

Proposition A.1.1. *Given the wave function $w^*(tt) = W^{-1}\phi(tt)$, there is a matrix differential operator P_i^j, such that*

$$\frac{\partial}{\partial t_i^j} w^*(tt) = P_i^j w^*(tt).$$

Proof:

$\frac{\partial}{\partial t_i^j} w^*(tt) = O(z)\phi(tt) + (I + \sum_{i>0} w_i^* z^i)E_{jj}z^{-i}\phi(tt)$. On the other hand $\partial^i w^*(tt)$ $= O(z)\phi(tt) + (I + \sum_{i>0} w_i z^i)\frac{1}{z^i}\phi(tt)$. Therefore $\frac{\partial}{\partial t_i^j} - \partial^i E_{jj}$ is a differential operator that acting on $w^*(tt)$ has order $O(\frac{1}{z^{i-1}})$ and we continue by induction.

Then we can write

$$\frac{\partial}{\partial t_i^j}(W^{-1}\phi(t)) = \frac{\partial W^{-1}}{\partial t_i^j}\phi(t) + W^{-1}\frac{\partial}{\partial t_i^j}\phi(t) = \frac{\partial W^{-1}}{\partial t_i^j}\phi(t) + W^{-1}E_{jj}\frac{1}{z^i}\phi(t)$$
$$= \frac{\partial W^{-1}}{\partial t_i^j}\phi(t) + W^{-1}E_{jj}\partial^i\phi(t) = P_i^j W^{-1}\phi(t),$$

with

$$(43) \qquad \frac{\partial W^{-1}}{\partial t_i^j} + W^{-1}(E_{jj}\partial^i) = P_i^j W^{-1}$$

Using the relation $W^{-1}\partial_{t_i^j}W + R_j^i = [R_j^i]_+$ from (42) we find in particular $P_i^j = [W^{-1}(E_{jj}\partial^i)W]_+ = [R_j^i]_+$.

Now, if $Q = W^{-1}A\partial W$, we have

$$\frac{\partial Q}{\partial t_i^j} = \frac{\partial W^{-1}}{\partial t_i^j}W.Q - Q\frac{\partial W^{-1}}{\partial t_i^j}W = (P_i^j - R_j^i)Q - Q(P_i^j - R_j^i)$$
$$= [Q, [R_j^i]_-] = -[Q, [R_j^i]_+].$$

Proposition A.1.2. *The operator Q satisfies the multicomponent K.P. hierarchy.*

Returning to the Euler Top case, we have seen that starting with a given set $\{\psi_\sigma\}$ of Baker functions, we have the representation $V_i\psi_\sigma = \mathbb{D}_i\psi_\sigma$. Here, the ψ_σ's correspond to a certain element W in the Lie group $\mathbf{G}=I + \mathcal{G}_-$ where $\mathcal{G}_-$ is the space of matrix pseudodifferential operators $\sum_{-\infty}^{-1} w_i\partial^{-i}$.

By Proposition 2.12 $\mathbb{D}_i = K_i r_i\partial K_i$, so for some conjugation of the operators $\mathbb{D}_i$ by elements S_i of $\mathbf{G}$, we get $S_i^{-1}\mathbb{D}_i S_i = W^{-1}r_i\partial W$. Therefore, if $r_i = diag(\lambda_{i1}, \lambda_{i2}, \lambda_{i3}, \lambda_{i4})$, then the differential operators $[S_i^{-1}\mathbb{D}_i S_i]_+ = [W^{-1}r_i\partial W]_+$ equal $\sum_1^4 \lambda_{ij}P_1^j$ and this corresponds to the multicomponent **KP** flow $\sum_1^4 \lambda_{ij}\partial_{t_1^j}$.

Appendix 2. Weierstrass $\wp$-functions on abelian varieties.

We consider a generalization of Weierstrass $\wp$-functions to the case of an ample divisor $\mathcal{D}$ on an abelian variety A. We assume the divisor $\mathcal{D} = \Sigma\mathcal{D}_\alpha$ ($\mathcal{D}_\alpha$ irreducible) has a symmetry group G of a certain order. This group is usually given by translations τ_x such that $\tau_x^{-1}\mathcal{D} = \mathcal{D}$ and (-1) involution ι, $\iota\mathcal{D} = \mathcal{D}$ and so they belong to the finite translation group associated to the divisor $\mathcal{D}$,

$H([\mathcal{D}]) = \{x \in A : \tau_x^{-1}\mathcal{D}$ is linearly equivalent to $\mathcal{D}\}$, unless the variety A has nontrivial automorphisms (which is not a generic case). If ϑ is the theta-function describing the zero locus $\mathcal{D} = \{\vartheta = 0\}$ then ϑ changes whith G by automorphy factors, and so the differentials of 2^{nd} kind $d(\frac{\partial}{\partial z_i} \log \vartheta), i = 1, \dots, g$, which have a pole of order two at $\mathcal{D}$, are invariant. These differentials and the higher order ones $d(\partial_i^n \log \vartheta)$ can be used to get a definition of Baker-Akhiezer functions for abelian varieties similar to that of Manin-Kapranov [Ma-Ka] and Nakayashiki [Na 1].

Let $\{(U_\alpha, f_\alpha)\}$ be a local data for the divisor $\mathcal{D}$ on the abelian variety A, and $\partial_i := \frac{\partial}{\partial z_i} : \mathcal{O}_A \to \mathcal{O}_A$ the usual derivations with respect to the complex coordinates z_i of $\mathbb{C}^g$ = universal covering of A. The line bundle $[\mathcal{D}]$ is given by transition functions $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha} \in \mathcal{O}_A^*$. Now $\partial_i \log g_{\alpha\beta} = \frac{\partial_i f_\beta}{f_\beta} - \frac{\partial_i f_\alpha}{f_\alpha}$ is a 1 cochain in $\mathcal{O}_A$ wich defines an element $[\partial_i \log g_{\alpha\beta}] \in H^1(A, \mathcal{O}_A) \simeq H^{0,1}_{\overline{\partial}}(A)$.

Lemma A.2.1. *The derivations* $\frac{\partial}{\partial z_i}$ *induce the zero map in* $H^1(A, \mathcal{O}_A)$.

Proof. Let $\{\tau_{\alpha\beta}\} \in H^1(A, \mathcal{O}_A)$. Then, by Dolbeault isomorphism there is a form $\omega' \in H^{0,1}_{\overline{\partial}}(A)$ such that $\delta^*(\omega') = \{\tau_{\alpha\beta}\}$ through the sequence

$$H^0(A, \mathcal{A}^0) \to H^0(A, \mathcal{Z}^{0,1}_{\overline{\partial}}) \xrightarrow{\delta^*} H^1(A, \mathcal{O}_A) \to 0$$

where $0 \to \mathcal{O}_A \to \mathcal{A}^0 \xrightarrow{\overline{\partial}} \mathcal{Z}^{0,1}_{\overline{\partial}} \to 0$ is the sheaf exact sequence in which $\mathcal{A}^0$ are C^∞ functions and $\mathcal{Z}^{0,1}_{\overline{\partial}}$ the $(0,1)$ $\overline{\partial}$-closed forms. Let ω denote the $(0,1)$-form such that $\delta^*(\omega) = \{\frac{\partial}{\partial z_i}\tau_{\alpha\beta}\}$, $\omega = \overline{\partial}\Omega_\alpha$, $\Omega_\alpha = C^\infty$ functions such that $\delta\{\Omega_\alpha\} = \Omega_\beta - \Omega_\alpha \overset{homologous}{\sim} \frac{\partial}{\partial z_i}\tau_{\alpha\beta}$. Thus, there are holomorphic functions $\{\mu_\alpha\}$ such that $\Omega_\beta - \mu_\beta = \frac{\partial}{\partial z_i}\tau_{\alpha\beta} + (\Omega_\alpha - \mu_\alpha)$. Analogously there are functions Ω'_α and μ'_α such that $\overline{\partial}\Omega'_\alpha = \omega'$ and $\Omega'_\beta - \mu'_\beta = \tau_{\alpha\beta} + (\Omega'_\alpha - \mu'_\alpha)$. We get the C^∞ function $f = \Omega_\alpha - \mu_\alpha - \frac{\partial}{\partial z_i}(\Omega'_\alpha - \mu'_\alpha)$ such that $\overline{\partial} f = \omega - \frac{\partial}{\partial z_i}\omega' - \overbrace{\overline{\partial}(\mu_\alpha - \frac{\partial}{\partial z_i}\mu'_\alpha)}^{=0}$. However, ω' can be chosen to be harmonic. Indeed, it follows from the Hodge decomposition (see [G-H]), that $\omega' = \mathcal{H}(\omega') + \overline{\partial}(u)$, where $\mathcal{H}(\omega')$ is the harmonic piece and u a C^∞ function on A. But on an abelian variety a harmonic form has constant coefficients. Thus $\frac{\partial}{\partial z_i}\mathcal{H}(\omega') = \frac{\partial}{\partial z_i}\sum a_i d\overline{z}_i = 0$ and we obtain $\omega = \overline{\partial}(f + \frac{\partial}{\partial z_i}(u))$, which proves the lemma. □

Now we get that $\frac{\partial}{\partial z_j}\left(\frac{\partial}{\partial z_i}\log g_{\alpha\beta}\right) = \delta\{\mu_\alpha\} = \mu_\alpha - \mu_\beta$ for functions $\mu_\alpha \in \mathcal{O}_A(U_\alpha)$, thus obtaining the function

$$(44) \qquad p_{ij} = \frac{\partial^2}{\partial z_i \partial z_j}\log f_\alpha - \mu_\alpha = \frac{\partial^2}{\partial z_i \partial z_j}\log f_\beta - \mu_\beta,$$

which is a holomorphic function blowing up twice at $\mathcal{D}$. Namely, $p_{ij} \in \Gamma(A, \mathcal{O}(2\mathcal{D})) = L(2\mathcal{D})$. By taking further derivatives we get

$$\frac{\partial^{n-2}}{\partial z_1^{\alpha_1}\cdots\partial z_g^{\alpha_g}} p_{ij} \in \Gamma(A, \mathcal{O}(nD)) = L(nD).$$

These are the so called generalized Weierstrass functions.

As $\{f_\alpha\}$ represents $\mathcal{D}$, then $\{\tau_x^* f_\alpha\}$ represents $\tau_x^{-1}\mathcal{D} = \mathcal{D}$. Now, for such $x \in \frac{1}{n}\Lambda$ (Λ the principally polarized lattice) then $\tau_x^* f_\alpha = e^{L_\alpha(z)} f_\alpha$, where $L_\alpha(z)$ is linear. See for instance the proof of Weil in [We]. Thus, it follows that $d\frac{\partial}{\partial z_i}\log f_\alpha$ is invariant under such a τ_x. If $\lambda \in \Lambda$ then $\tau_\lambda^* f_\alpha = e^{L_\lambda(z)} f_\alpha$, and, $\frac{\partial}{\partial z_i}\log \tau_\lambda^* f_\alpha = \frac{\partial}{\partial z_i}\log f_\alpha + \frac{\partial}{\partial z_i}L_\lambda(z) \Longrightarrow d\frac{\partial}{\partial z_i}\log\tau_\lambda^* f_\alpha = d\frac{\partial}{\partial z_i}\log f_\alpha$, which means that $d\frac{\partial}{\partial z_i}\log f_\alpha$ is really a form on A, invariant under the action of $G = \{x : \tau_x^{-1}\mathcal{D} = \mathcal{D}\}$.

As for the Weierstrass functions:

Lemma A.2.2. *The Weierstrass functions (44) are invariant under G.*

Proof. This follows because, as above, the functions $\frac{\partial^2}{\partial z_i \partial z_j}\log f_\alpha$ are invariant under $\tau_x \in G$; namely, $\frac{\partial^2}{\partial z_i \partial z_j}\log \tau_x^* f_\alpha = \frac{\partial^2}{\partial z_i \partial z_j}\log f_\alpha$. In particular, the cocycle $\mu_\alpha - \mu_\beta$ is invariant by any $\tau_x \in G$, thus $\tau_x\mu_\alpha - \mu_\alpha = \varphi_x$ has to be a function on A without poles, so $\varphi_x \in \mathbb{C}$. Moreover, $\varphi : G \to (\mathbb{C}, +)$ is a homomorphism of a finite group into the additive complex numbers. So $\varphi(x) = \varphi_x = 0 \ \forall \tau_x \in G$, and this implies p_{ij} is invariant under G. □

Also, the higher order Weierstrass functions are invariant.

In this way we have an arsenal of differential forms and functions that blow up at $\mathcal{D}$ with a certain order and are invariant by G.

In the case A is an elliptic curve we can choose a local parameter z around a point of $\mathcal{D}$ (e.g., the time evolution parameter) so that the local expansion of the function p around this point has the form $p = \frac{a}{z^2} + O(1)$. Around smooth points of the divisor $\mathcal{D}$ on an abelian variety A^g we can pick coordinates (x, z) so that

$x = (x_1, \cdots, x_{g-1})$ are coordinates of $\mathcal{D}$, and $z = 0$ defines the reduced divisor $\mathcal{D}$ locally. In a neighborhood of those points we can write $p_{ij} = \frac{a_{ij}(x)}{z^2} + O(1)$.

Let us consider now a (holomorphic) derivation D on A. We can think of such an object as an element in $\mathbb{C}^g = Lie(A)$. One has the isogeny (whose kernel is the translation group $H(\mathcal{D})$)

$$\begin{array}{rccc} \pi: & A & \longrightarrow & Pic^0(A) \\ & x & \longmapsto & \{\tau_x^{-1}\mathcal{D} - \mathcal{D}\} \end{array}$$

As we can think of $x = \tau_x = \exp(D)$ for some derivation D, we get the application

$$\begin{array}{ccccccc} Lie(A) & \xrightarrow{exponential\ map} & A & \xrightarrow{\pi} & Pic^0(A) & & \\ D & \longmapsto & x & \longmapsto & \{\tau_x^{-1}\mathcal{D} - \mathcal{D}\} & = & \{\frac{\tau_x^* g_{\alpha\beta}}{g_{\alpha\beta}}\} \end{array}$$

Now, by the exponential sheaf sequence $0 \to \mathbb{Z} \to \mathcal{O}_A \xrightarrow{\exp} \mathcal{O}^* \to 0$ the cocycle $[D \log g_{\alpha\beta}] \in H^1(A, \mathcal{O}_A)$ goes into the element of $Pic^0(A)$ given by the cocycle $\{\exp(D \log g_{\alpha\beta})\}$.

Since $H^1(A, \mathcal{O}_A) = Lie(Pic^0(A))$, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^g = Lie(A) & \xrightarrow{d\pi} & H^1(A, \mathcal{O}_A) = Lie(Pic^0(A)) \\ \exp \downarrow & & \downarrow \exp \\ A & \xrightarrow{\pi} & Pic^0(A) \end{array} \tag{45}$$

and one can see that the cocycle $\{\exp(D \log g_{\alpha\beta})\}$ corresponds precisely (up to coboundary) to the cocycle $\left\{\frac{\tau_x^* g_{\alpha\beta}}{g_{\alpha\beta}}\right\}$.

Lemma A.2.3. *$d\pi(D) = [D \log g_{\alpha\beta}]$ and $\left\{\frac{\tau_x^* g_{\alpha\beta}}{g_{\alpha\beta}}\right\} = \{\exp(D \log g_{\alpha\beta})\}$.*

Proof. If $\pi(x) = \frac{g_{\alpha\beta}(x+y)}{g_{\alpha\beta}(y)}$ on $U_{\alpha\beta}$, we can determine the directional derivative of π in the direction D. One has

$$\begin{aligned} d\pi(D) &= \lim_{t\to 0} \frac{\pi(\exp(tD)) - 1}{t} = \lim_{t\to 0} \frac{g_{\alpha\beta}(\exp(tD) + y) - g_{\alpha\beta}(y)}{t g_{\alpha\beta}(y)} \\ &= \lim_{t\to 0} \sum \frac{1}{g_{\alpha\beta}} \frac{\partial g_{\alpha\beta}}{\partial z_i} D_i + o(1) = D \log g_{\alpha\beta}. \end{aligned}$$

(Here $D = \sum D_i \frac{\partial}{\partial z_i}$, and z_i are coordinates in A.) Thus the conclusion follows from the commutativity of (45). □

This lemma shows that the cycle $[D \log g_{\alpha\beta}] \in H^1(A, \mathcal{O}_A)$ inducing the line bundle of cocycle $\frac{\tau_x^* g_{\alpha\beta}}{g_{\alpha\beta}} \sim \exp(D \log g_{\alpha\beta})$, can be thought of as a derivation D via the map $d\pi$, which by exponentiating corresponds to the line bundle $L = [\tau_x^{-1}\mathcal{D} - \mathcal{D}]$. In other words, a direction in the abelian variety A maps to the point $L = [\tau_x^{-1}\mathcal{D} - \mathcal{D}]$ in $Pic^0(A)$.

Note A.2.4. Let now $s_0, s_1, \ldots, s_n$ be linearly independent sections of $\Gamma(\mathcal{D})$. The zero divisor $(s_i)_0 = \mathcal{D}_i$ has to be linearly equivalent to $\mathcal{D} = (s_0)_0$. If D is a chosen holomorphic derivation (i.e. a linear combination of $\frac{\partial}{\partial z_i}$ with constant coefficients) then we can define, as we did, the associated Weierstrass function

$$\wp = D^2 \log(s_i)_\alpha + \mu_\alpha.$$

This function blows up at $2\mathcal{D}_i$, which is linearly equivalent to $2\mathcal{D}$.

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Universidad Nacional del Sur – Av. Alem 1253 – 8000 Bahía Blanca – Argentina
E-mail address: impiovan@criba.edu.ar

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