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ON THE EISENSTEIN SET

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ABSTRACT. In the number system (-2,D) with base b=−2 and family of ciphers $D=\{0,1,w,\overline{w}\}$ where $w=e^{2\pi i/3}$, $\overline{w}=w^2$, every complex number z is representable: $z=(a_N\ldots a_0.a_{-1}a_{-2}\ldots)_{-2}$, i.e., $z=\sum_{-\infty}^{N} a_j b^j$. (-2,D) has as set of integers $W:=\{a_N\ldots a_1a_0;a_j\in D\}$, the family of Eisenstein numbers $E=\{m+nw:m,n\in Z\}$. The integers of the system are uniquely representable. The set of fractional numbers $F:=\{0.a_{-1}a_{-2}\ldots;a_{-h}\in D\}$ coincides with a copy of the so called Eisenstein set. This set is a fractile of diameter equal to $\sqrt{3}$ that contains a ball of radius 1/8 and whose convex hull is an irregular hexagon contained in a ball of radius $\sqrt{7/9}$. The Lebesgue measure of F is equal to $\sqrt{3}/2$. The family $\{F_g:g\in W\}$ is a tessellation of the plane such that F touches 12 sets $F_g\neq F$, $g\in S\setminus\{0\}$. Here S:=D−D. $F_g\cap F$ contains only one point iff $g\in S' := \{\pm(1-w),\pm(1-\overline{w}),\pm(w-\overline{w})\}\subset S$. ∂F is not a Jordan curve. Moreover, F is a continuum whose interior and exterior have infinitely many components.

I. INTRODUCTION. Let $b\in\mathbf{C}$, $|b|>1$, $D=\{0,d_1,d_2,\ldots,d_k\}\subset\mathbf{C}$. α is said *representable* in base b with ciphers D if there exists $\{a_j\in D: j=M,M-1,\ldots\}$ such that $\alpha=\sum_{-\infty}^{M} a_jb^j$. We write $\alpha=a_M\ldots a_0.a_{-1}a_{-2}\ldots=(e.f)_b$ and call (e) the integral part of α and (f) the fractional part of α. G denotes the set of all representable numbers. F is the set of *fractional numbers*, i.e., those numbers in G with a representation such that $(e)=0$. The set W of *integers* of the system is the subfamily of G with a representation such that $(f)=0$. A number r will be called a *rational* of the number system (b,D) if it has a finite positional representation, that is, $a_j=0$ for $j<J(r)$. U will denote the set of rationals of the system. We wish to represent the whole of **C** using a number system with a real base. To this end we will study the number system with base −2 and the set of ciphers D $\not\subset$ **R**, $D:=\{0,1,w,w^2\}$ where $w=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. D\{0}={third roots of unity}, is a multiplicative group such that $1+w+w^2=0$ (the cyclotomic equation).

DEFINITION I.1. E denotes the Eisenstein's point-lattice: $E\equiv[1,w]:=$ $=\{m.1+n.w:\ m,n\in Z\}$. Let $\sigma=D\cup(-D)=\{0,\pm1,\pm w,\pm\overline{w}\}$. S:=D−D = $=\{0,\pm1,\pm w,\pm\overline{w},\pm(1-w),\pm(1-\overline{w}),\pm(w-\overline{w})\}$, $S':=S\setminus\sigma=\{\pm(1-w),\pm(1-\overline{w}),\pm(w-\overline{w})\}$.•

Then, S and σ are subsets of the set E of Eisenstein "integers". It is easy to verify that the numbers in S\{0} can be written in a unique way as a difference of two numbers in D. The numbers in $\sigma\backslash\{0\}$ have modulus equal to 1 and those in S' have modulus equal to $\sqrt{3}$. Besides, $\alpha \in S \Rightarrow |\alpha| \leq \sqrt{3}$, $|\mathrm{Re}\alpha| \leq 3/2$, $|\mathrm{Im}\alpha| \leq \sqrt{3}$.

NOTATION I 1. x used as a cipher will represent the number $w^2 = \overline{w}$. m(A) will denote the plane Lebesgue measure of A⊂**C** and B(z;r) the open ball of center z and radius r.
The reader can find in [P] a detailed proof of each statement in the following Ths. I 1-3. However, we give in this section for most of them and for the sake of completeness, a reference, a hint or an alternative proof.
Any number in W, the set of integers of the number system $(-2,\{0,1,w,\overline{w}\})$, belongs to E. This follows from the identity: 1+w+x=0. Moreover,

THEOREM I 1. i) W=E.
ii) The integer m+nw is representable with at most $|m|+|n|+1$ ciphers.
iii) If $g \in W$ and $|g| \leq 2^k + 1$ then g is representable with at most k+3 ciphers.
iv) 0 has a unique representation in the number system $(-2,\{0,1,w,x\})$.
v) m+nw has a unique representation in $(-2,\{0,1,w,x\})$. •

PROOF. i) cf. [P] or [E]. ii) can be proved by induction. iii) same proof as in [B], lemma 1,i). iv) If $0.(0) = e.(c)$ and both expressions are different then, after multiplying by an adequate power of the base b we obtain an analogous equality with $e \in D\backslash\{0\}$. Since D is a multiplicative group we can assume that e=1. But then we must have $-1 = 0.(c)$ and this is impossible. v) We have $d_i, d_j \in D, d_i \neq d_j \Rightarrow d_i - d_j \neq b.r, r \in W$. But iv) is the hypothesis H0) in [Z], §2, which together with the preceding statement imply that the ciphers have a unique representation as integers of the number system. Since W is a **Z**-module, an application of the theorem of uniqueness of that paper proves that any number in W has a unique representation in (b,D), QED.

DEFINITION I 2. $F_g := g + F$ where $g \in E$.•

Thus, $F_0 \equiv F$, the fractional set of the number system $(-2,\{0,1,w,x\})$. We shall call it the *Eisenstein set* (Fig 2 §I).. The definition I 2 can be extended in the following way (see Figs. in §VI):

$$F_{a_M \dots a_0 . a_{-1} a_{-2} \dots a_{-n}} := \{x; x = a_M \dots a_0 . a_{-1} \dots a_{-n} \dots\}. \tag{I.1}$$

DEFINITION I 3. For $j \in D = \{0,1,w,x\}$ let us define $\Phi_j(z) = \frac{z}{b} + \frac{j}{b} = -\frac{z+j}{2}$.•

Then, $F = \bigcup_{i \in D} \Phi_i(F)$. Thus, the 4-reptile F is the invariant set of the family $\{\Phi_j\}$.

THEOREM I 2. i) The compact set F ⊂B(0;1) is invariant under rotations of $2\pi/3$ and is the attractor of the family of similarities $\{\Phi_j\}$.
ii) If $z \in \mathbf{C}$ and $|z| \leq 1/8$ then $z \in F$.
iii) G=**C**, i.e., $\mathbf{C} = \bigcup_{g \in E} F_g$.•

PROOF. ii) can be proved as in [B], lemma 1, ii). For iii), cf. [E] or [P]. It follows immediately from ii), QED.

THEOREM 1 3. i) The family $\{F_g : g \in E\}$ defines a *tessellation* in the sense that not only $\mathbf{R}^2 = \bigcup \{F_g : g \in E\}$ but also that any two different sets of the family have an intersection of plane Lebesgue measure zero.
ii) $m(F_0) = \sqrt{3}/2$.•

PROOF. To prove i) observe that $m(b^4 F) = (16)^2 m(F)$ and that $b^4 F = \bigcup_{d,e,f,g \in D} F_{defg}$ is the union of 256 copies of sets congruents to F. Thus, $(defg)_b \neq (d'e'f'g')_b$ implies that $m(F_{defg} \cap F_{d'e'f'g'}) = 0$. We show in section 2 that $F \cap F_g \neq \emptyset \Rightarrow g \in S$. But we know from Table 1 that any number in S is representable with at most 4 ciphers. Therefore, $m(F \cap F_g) \neq 0$ only if g=0 and i) follows.
ii) The tiling of $\mathbf{R}^2$ by the parallelograms defined by the Eisenstein's point-lattice is composed of tiles of area $\sqrt{3}/2$. Therefore, i) implies that $m(F) = \sqrt{3}/2$, (cf. [HW] §3.11 or [Z], §4), QED.

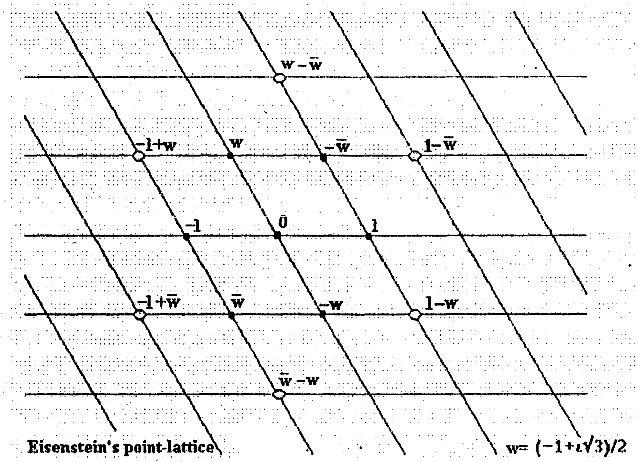


FIG. 1

TABLE 1.

Positional representation of the numbers $\{m+n.w: m,n \in \mathbb{Z}, |m|,|n| \leq 4\}$ in the number system (-2,D).

-4 − 4 .w = x 0 0	-2 − 2 .w = x x 0	0 + 1 .w = w	2 + 3 .w = x w
-4 − 3 .w = x 0 w	-2 − 1 .w = x x w	0 + 2 .w = w w 0	2 + 4 .w = x x 1 0
-4 − 2 .w = 1 1 w 0	-2 + 0 .w = 1 0	0 + 3 .w = w w w	3 − 4 .w = 1 1 x 1 1
-4 − 1 .w = 1 1 w w	-2 + 1 .w = 1 w	0 + 4 .w = w 0 0	3 − 3 .w = 1 w x
-4 + 0 .w = 1 1 0 0	-2 + 2 .w = 1 1 x 0	1 − 4 .w = w w 0 1	3 − 2 .w = w w x 1
-4 + 1 .w = 1 1 0 w	-2 + 3 .w = 1 1 x w	1 − 3 .w = w w x x	3 − 1 .w = 1 0 x
-4 + 2 .w = w w x w 0	-2 + 4 .w = w 1 0	1 − 2 .w = w 1	3 + 0 .w = 1 1 1
-4 + 3 .w = w w x w w	-1 − 4 .w = w w 1 1	1 − 1 .w = 1 1 x	3 + 1 .w = x x w x
-4 + 4 .w = w w x 0 0	-1 − 3 .w = w x	1 + 0 .w = 1	3 + 2 .w = x 1
-3 − 4 .w = x 0 1	-1 − 2 .w = x x 1	1 + 1 .w = x x	3 + 3 .w = x x 0 x
-3 − 3 .w = x x x	-1 − 1 .w = x	1 + 2 .w = w w 1	3 + 4 .w = x x 1 1
-3 − 2 .w = 1 1 w 1	-1 + 0 w = 1 1	1 + 3 .w = x x 1 x	4 − 4 .w = 1 1 x 0 0
-3 − 1 .w = 1 x	-1 + 1 .w = w w x	1 + 4 .w = w 0 1	4 − 3 .w = 1 1 x 0 w
-3 + 0 .w = 1 1 0 1	-1 + 2 .w = 1 1 x 1	2 − 4 .w = 1 1 x 1 0	4 − 2 .w = 1 w 0
-3 + 1 .w = 1 1 x x	-1 + 3 .w = w 0 x	2 − 3 .w = 1 1 x 1 w	4 − 1 .w = 1 w w
-3 + 2 .w = w w x w 1	-1 + 4 .w = w 1 1	2 − 2 .w = w w x 0	4 + 0 .w = 1 0 0
-3 + 3 .w = w 1 x	0 − 4 .w = w w 0 0	2 − 1 .w = w w x w	4 + 1 .w = 1 0 w
-3 + 4 .w = w w x 0 1	0 − 3 .w = w w 0 w	2 + 0 .w = 1 1 0	4 + 2 .w = x x w 0
-2 − 4 .w = w w 1 0	0 − 2 .w = w 0	2 + 1 .w = 1 1 w	4 + 3 .w = x x w w
-2 − 3 .w = w w 1 w	0 − 1 .w = w w	2 + 2 .w = x 0	4 + 4 .w = x x 0 0

Some arithmetic: to prove that $1.\overline{10}=0.\overline{01}$ observe that $0.\overline{10}=-2/3$, $0.\overline{01}=1/3$. Since $(11)_b=-1$, we get (ww)=–w, and from 110=b(11)=2 that 11x=2+x=1–w. That is, $11x.\overline{10}=(1/3)-w=ww.\overline{01}$. From this, after multiplying by b^{-2}, we get the desired equality $1.1w^2\overline{10}=0.ww\overline{01}$.

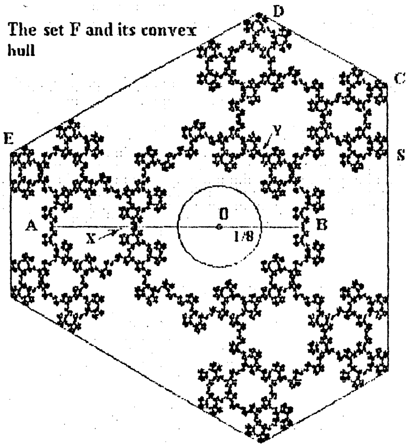


FIG. 1

III. THE CONVEX HULL of the FRACTIONAL SET F of the SYSTEM (-2,{0,1,w,x}). We show in the theorem that $C=\frac{2}{3}+\frac{i}{\sqrt{3}}, D=\frac{1}{6}+\frac{i\sqrt{3}}{2}, E=-\frac{5}{6}+\frac{i}{2\sqrt{3}}$; then $|C-D|=1/\sqrt{3}$, $\Phi_x(E)=S=\frac{2}{3}+\frac{i\sqrt{3}}{6}$.

THEOREM III 1. co(F)=co$\left(CDE\overline{EDC}\right)$.•

PROOF. Let us define H=co$\left(CDE\overline{EDC}\right)$, $\{C\}=F\cap F_{1-x}$, $\{D\}=F\cap F_{w-x}$, $\{E\}=F\cap F_{w-1}$. Then, according to Theorem II 1 ii) and Lemma II 1,

$$C=\left(1-x.\overline{1x}\right)_b=\left(0.\overline{x1}\right)_b=e^{-2\pi i/3}\sum_{j=0}^{\infty}\left(\frac{1}{-2}\right)^{2j+1}+\sum_{j=1}^{\infty}\left(\frac{1}{-2}\right)^{2j}=e^{-2\pi i/3}\left(\frac{-2}{3}\right)+\frac{1}{3}=\frac{2+i\sqrt{3}}{3},$$

$D=w-x.\overline{wx}=0.\overline{xw}=\frac{1+i3\sqrt{3}}{6}$, $E=w-1.\overline{w1}=0.\overline{1w}=\frac{-5+i\sqrt{3}}{6}$. On the other hand, we have, $\Phi_1(C)=\overline{E},\Phi_1(\overline{C})=E$. Since $\Phi_1(z)$ is a conformal mapping, $\Phi_1(H)\subset H$. In general, $d\in D\Rightarrow\Phi_d(H)\subset H$. In consequence, $F\subset H$. Since, C, D, E and its complex conjugates belong to F, we obtain co(F)=H, QED.

The next corollary improves Th. II 2 i).

COROLLARY. i) $F\subset$ cl B(0;$\sqrt{7/9}$); ii) diam F=$\sqrt{3}$; iii) $\sigma\cap$co(F) = {0}.•

Conspicuous points: A=–2/3, $X=\left(x.\overline{xw}\right)_b=(0.\overline{1})=-1/3=w.\overline{wx}$=the fixed point of Φ_1, B=$1/3=0.\overline{01}=1.\overline{10}$, (see Fig. 1).

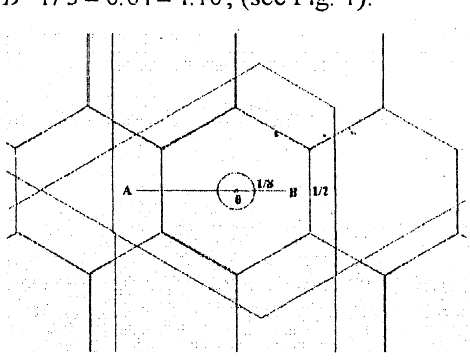


FIG. 2. It shows the relative position of H in the tessellation H defined in Th. IV 2. Observe that the convex hexagon H does not tile the plane.

IV. SOME PROPERTIES of the (EISENSTEIN) FRACTIONAL SET of the SYSTEM (-2,{0,1,w,x}).

DEFINITION IV 1. $F^*:=\left\{z:z=\sum_{-N}^{-1}a_jb^j\right\}=$ the set of rationals in F. •

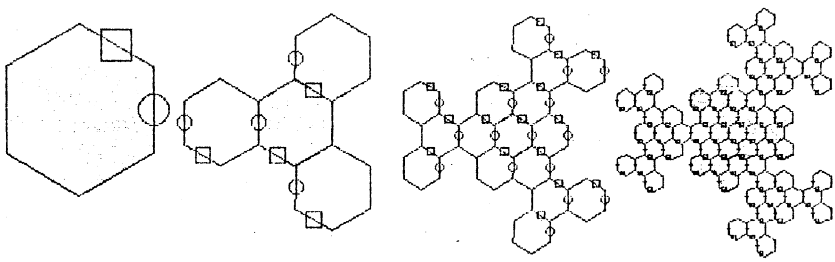


Fig. 1.

THEOREM IV 1. i) F* $\subset$ int F.
ii) cl(F*) = cl(int F) = F.
iii) If $g \neq 0$ then (int F)$\cap F_g = \varnothing$.•

PROOF. i) If $z = 0.c_{-1}\ldots$ and $d \in D$ then $\Phi_d(z) = 0.dc_{-1}\ldots$. Besides, $\Phi_d(\text{int}\, F) \subset \text{int}\, F$. Since 0 is an interior point of F it follows that $\Phi_{a_{-1}} \circ \ldots \circ \Phi_{a_{-N}}(0) = 0.a_{-1}\ldots a_{-N} \in$ int F.

ii) follows from i) since cl(F*)=F.

iii) If $z \in F_g$ then $z = (g)_b.a_{-1}\ldots$. Because of Theorem I 1 v), $z_N = (g)_b.a_{-1}\ldots a_{-N}$ is not in F. Thus, $z = \lim z_N$ does not belong to int(F), QED.

COROLLARY. i) The family $\left\{\text{int}\, F_g : g \in E\right\}$ is pairwise disjoint.
ii) F is a self-similar set that satisfies the open set condition. •

THEOREM IV 2. F is a connected set.•

PROOF. The tessellation H of the plane given by regular hexagons of apotheme 1/2 centered at the points of the Eisenstein's point-lattice (see Figs. 1, 2) is invariant under a translation by a vector of the lattice and under multiplication by $e^{2\pi i/3}=w$, $e^{4\pi i/3}=\overline{w}$ or $e^{\pi i}=-1$. The similarities $\Phi_d, d \in D$, transform the tessellation H into the tessellation H/2. And the similarity $\Phi_{d_n} \circ \ldots \circ \Phi_{d_1}$ transforms $H_0 \equiv H$ into $H_n \equiv H/2^n$. To recognize the position of a hexagon in H_n, it is sufficient to know the coordinates of it's center in E=[1,w]. If T_0 denotes the central hexagon in H then the set of centers of the hexagons in H_n, n>0, contained in $T_{n,d} := \Phi_d(\bigcup\{\Phi_{d_{n-1}} \circ \ldots \circ \Phi_{d_1}(T_0) : (d_{n-1},\ldots,d_1) \in D^{n-1}\})$, for a fixed $d \in D$, is $t_{n,d} := \{\frac{d_{n-1}}{b^2} + \ldots + \frac{d_1}{b^n} + \frac{d}{b}; d_k \in D, k = 1,\ldots,n-1\}$. Applying v) of Th. I 1, we get that $d \neq d' \Rightarrow t_{n,d} \cap t_{n,d'} = \varnothing$. Therefore, $(\text{int}\, T_{n,d}) \cap (\text{int}\, T_{n,d'}) = \varnothing$.

This explains the behaviour of the relevant subsets of the families of hexagons in the preceding diagram. For example, at its extreme right we can distinctly see the families of hexagons in H_3 contained in the sets $T_{3,0}, T_{3,1}, T_{3,w}, T_{3,x}$. If Ω_n denotes the compact set $\bigcup_{d \in D} T_{n,d}$ then, because of a theorem due to Hutchinson (cf. [H]), the sequence $\{\Omega_n\}$ converges to F in the metric of Hausdorff. To prove that F is connected, it is sufficient to prove that Ω_n is a connected set. It suffices to show that given $d \in D$ there exist a $c \in \sigma \setminus \{0\}$ and two families of ciphers, $\{d_{n-1},\ldots,d_1\}$, $\{t_{n-1},\ldots,t_1\}$, such that

(IV 1) $\qquad db^{-1}+d_{n-1}b^{-2}+\dots+d_1b^{-n}+cb^{-n} = t_{n-1}b^{-2}+\dots+t_1b^{-n}$.

In fact, (IV 1) implies that $T'_{n,d}\cap T'_{n,0}\neq\emptyset$. Obviously, it suffices to consider the case d=1. In this case (IV 1) reads: $b^{-1}+cb^{-n}+d_{n-1}b^{-2}+\dots+d_1b^{-n} = t_{n-1}b^{-2}+\dots+t_1b^{-n}$. The problem is reduced to find, for n>1, $\gamma_j=\gamma_j(n)\in S$, j=1,...,n−1, and a $c=c(n)\in\sigma\setminus\{0\}$ such that $b^{n-1}+c=[\gamma_{n-1},\dots,\gamma_1]_b$ (Square brackets in expressions like $[\dots]_b$ have the same meaning as $(\dots)_b$ except for the fact that the numbers inside them may not be ciphers). But, since b=−2 and $2^n-1=\sum_0^{n-1}2^j$, we have $b^n+(-1)^{n+1}=\sum_{j=1}^{n}(-1)^j b^{n-j}=$

$=[-1, 1, -1,\dots,(-1)^n]_b$, QED.

The preceding proof is borrowed from [P]. Fig. 3 of section I illustrates the fact, already proved, that each F_g touches exactly twelve different tiles. This property is shared with the regular tiling by equilateral triangles.

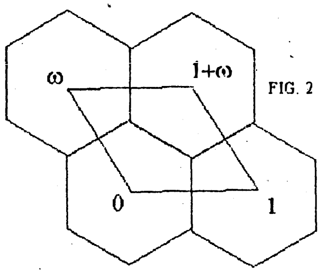


FIG. 2

PROPOSITION IV 1. *The process shown in Fig. 1 of this section provides a tessellation* F_n, n>0, *with central tile* Ω_n, *that uniformly approximates the tiling* $\mathsf{F}=\{F_g: g\in E\}$. *Each tile in* F_n *is in contact with exactly six different congruent tiles.*

PROOF. The centers of the hexagons in Ω_n and Ω_n+g, $g\in E\setminus\{0\}$, are of the form: $d_1b^{-n}+\dots+d_{n-1}b^{-2}+d_0b^{-1}$ and $t_1b^{-n}+\dots+t_0b^{-1}+c_0+c_1b+\dots+c_mb^m$ with a $c_k\neq0$, respectively. After multiplying by b^n, we obtain two different integers of the system because of Th. I 1 v). That is, m($\Omega_n\cap(\Omega_n+g)$)=0 if $g\neq0$. It is easy to prove that each Ω_n has an area equal to the area of a parallelogram in Eisenstein point-lattice, that is, $m(\Omega_n)=\sqrt{3}/2$. A standard result on point-lattices (cf. [HW] or [Z], Prop. 2,2') that makes use of the fact that $\mathsf{F}_n=\{g+\Omega_n: g\in E\}$ and H are composed of compact sets of the same measure associated to the same point-lattice asserts that F_n is a tessellation of the plane. It is not difficult to show that if z belongs to Ω_n then $|z|<\sqrt{3}$. A consequence of this is that $\Omega_n\cap(\Omega_n+g)\neq\emptyset$ only if $g\in\sigma$. Thus, there exists $g\in\sigma\setminus\{0\}$ such that $\Omega_n\cap(\Omega_n\pm g)\neq\emptyset$. $g\in\sigma\Rightarrow\Omega_n\cap(\Omega_n+g)\neq\emptyset$ follows after multiplying by w and x the preceding relation, QED.

Thus, each approximating tessellation behaves like the regular tiling by hexagons.

DEFINITION IV 2. K:=∂F, $\mathrm{K}_n:=\partial\mathsf{F}_n$. {Z}=$F_x\cap F_1$, {Y}=$F_w\cap F_1$, X=−1/3.•

THEOREM IV 3. i) Let $a=\Phi_1(0)=-1/2$ and $M=F_w\cup F_x\cup F_1$. Then, M is a continuum such that $a\in A$ = the infinite open component of C\\M. Besides, 0 $\notin A$.

ii) B(0;1/6) $\subset$ int($\overset{\Delta}{XYZ}$) $\subset$ int(F).•

PROOF. We proved that F_s has one point in common with F_t when t−s∈S' (Th. II 1 iii)). From this and Th. IV 2, it follows that M is a continuum. Using Theorem III 1 we

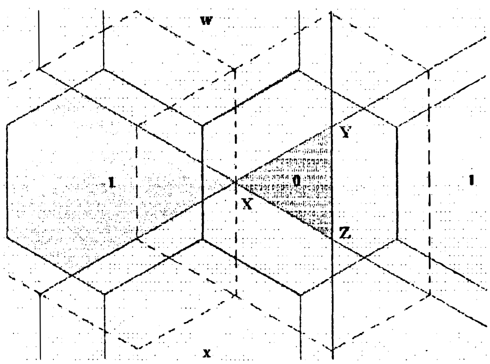


FIG. 3

see also that $\widetilde{M}=\bigcup\{co(F_c);c\in\{1,x,w\}\}$ is a continuum that does not contain 0 and the half line $(-\infty,a]$, (cfr. Fig. 4). Therefore, a belongs to A. Let q be a polygonal joining 0 with a. Then $q\cap M\neq\varnothing$. In fact, clearly $q\cap\widetilde{M}\neq\varnothing$ (cf. Fig. 4) and q intersects one of the polygons $co(F_c)$ included in $\widetilde{M}$ at a vertex or at two different sides. Since the vertices of that polygon are points of F_c, and F_c is connected, we have in either case $q\cap F_c\neq\varnothing$. In consequence, $0\notin A$. It is not hard to prove that the interior of the triangle XYZ has to be contained in the interior of F, QED.

COROLLARY. The theorem holds with $F_w\cup F_x\cup F_1$ replaced by M:= $F_x\cup F_{-w}\cup F_1\cup F_{-x}\cup F_w$.•

PROOF. The convex hulls of F_{-x},F_{-w} have void intersection with the interior of the triangle XYZ and the half line $(-\infty,X)$, QED.

NOTATION IV 1. We shall denote with V^ the complement **C**\V of the set V.

THEOREM IV 4. i) int(F) is the union of an infinite denumerable family of open components

ii) K is not a Jordan curve

iii) $K\subset \mathrm{cl}\bigcup_{d\in S\setminus\{0\}}\mathrm{int}(F_d)$.•

PROOF. i) 0 and $-1/2=(0.1)_b=\Phi_1(0)$ are interior points of F. Because of the preceding theorem they belong to different components since int(F) ⊂ M^. i) follows now from the self-similarity of F. ii) is a consequence of i) and the theorem of Jordan and Veblen.

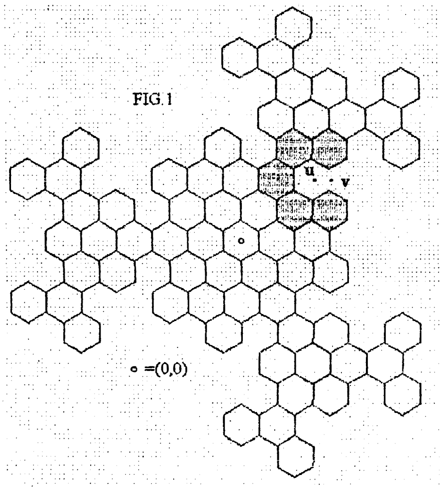
iii) If $z=0.a_{-1}\ldots\in K$ then it is the limit of a sequence of points in **C**\F and therefore, $z=s.d_{-1}\ldots$ with s ∈ S\{0}. It follows from Th. IV 1 that z is the limit of a sequence of points belonging to int(F_s), QED.

V. A LOCKED TESSELLATION. We wish to prove that the complement of F, F^, has infinitely many open components. This explains the existence of holes in the diagram of F shown in Fig. 2 of section I. To prove that F^ is not connected, it is sufficient to demonstrate the next result. In its proof we use again Table 1.

THEOREM V 1. F^ ⊄ Ω = the infinite open component of F^. •

PROOF. u=1.1w1 and v=1.1w11 are rationals not in F. They are not connected in $(F_{0.xx0}\cup F_{0.xxw}\cup F_{0.xxx}\cup F_{0.01x}\cup F_{0.x0w})$^ ⊃ F^. This can be verified by applying the similarity $\Phi(z):=b^3z-(11w1)_b$. In fact, one readily sees that $\Phi(u)=0$, $\Phi(v)=0.1=-1/2$, $\Phi(0.xx0)=(xx0)_b-(11w1)_b=(-2-2w)-(-3-2w)=1$, $\Phi(0.xxw)=-x$, $\Phi(0.xxx)=-w$, $\Phi(0.01x)=w$, $\Phi(0.x0w)=x$. Therefore, $\Phi(F_{0.xx0})=F_1$, $\Phi(F_{0.xxw})=F_{-x}$, $\Phi(F_{0.xxx})=F_{-w}$, $\Phi(F_{0.01x})=F_w$, $\Phi(F_{0.x0w})=F_x$. By ii) of Corollary to Th. IV 3, $\Phi(u)$ and $\Phi(v)$ are not connected in M^ = $\Phi((F_{0.xx0}\cup F_{0.xxw}\cup F_{0.xxx}\cup F_{0.01x}\cup F_{0.x0w})$^$)$. Hence, $u\in$F^\Ω, QED.

This shows that the tessellation by F is different from the usual ones in which the tiles are "glued" side by side. In the present situation the tiles are "locked", each one with six neighbouring tiles out of the twelve with which it has a non void intersection. The hexagons in Fig. 1 belong to H_3.



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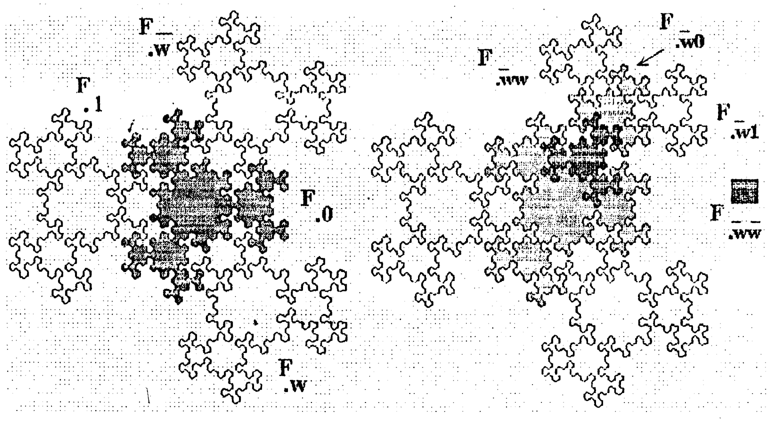

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