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# SOLUTIONS TO THE MEAN CURVATURE EQUATION FOR NONPARAMETRIC SURFACES BY FIXED POINT METHODS

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## ABSTRACT

We study the existence of solutions for the equation of prescribed mean curvature when the surface is the graph of $u : \overline{\Omega} \longrightarrow R$, with mean curvature $H(x, y, u(x, y))$. We give conditions on the boundary data in order to obtain at least one solution for the quasilinear Dirichlet problem (1) below, with $H$ a given continuous function.

## INTRODUCTION

We consider the quasilinear Dirichlet problem in a bounded domain $\Omega \subset R^2$ with $\partial\Omega \in C^2$

$$(1)\begin{cases} (1+u_y^2)u_{xx} + (1+u_x^2)u_{yy} - 2u_xu_yu_{xy} = 2H(x,y,u)\left(1+|\nabla u|^2\right)^{\frac{3}{2}} & \text{in} \quad \Omega \\ u(x,y) = \varphi(x,y) \quad \text{on } \partial\Omega \end{cases}$$

where $H : \overline{\Omega} \times [\epsilon, \epsilon] \longrightarrow R$ is continuous for some $\epsilon > 0$ and $\varphi \in W^{2,p}(\Omega)$ is the boundary data.

The problem above is the mean curvature equation for nonparametric surfaces which has been studied in general for hypersurfaces in $R^{n+1}$ by Gilbarg, Trudinger, Simon, Serrin, Díaz, Saa and Thiel among other authors. For $H$ independent of $u$ it has been proved [GT] that there exists a solution for any smooth boundary data if the mean curvature $H'$ of $\partial\Omega$ satisfies:

$$H'(x_1, ..., x_n) \geq \frac{n}{n-1}|H(x_1, ..., x_n)|$$

for any $(x_1, ..., x_n) \in \partial\Omega$, and $H \in C^1(\overline{\Omega}, R)$ satisfying the inequality:

$$|\int_\Omega H\varphi| \leq \frac{1-\epsilon}{n}\int_\Omega |D\varphi|$$

for any $\varphi \in C_0^1(\Omega, R)$ and some $\epsilon > 0$. The sharpness of the geometric condition on the curvature of $\partial\Omega$ is shown by a non-existence result ([GT], corollary 14.13): if $H'(x_1, ..., x_n) < \frac{n}{n-1}|H(x_1, ..., x_n)|$ for some $(x_1, ..., x_n)$ and the sign of $H$ is constant, then for any $\epsilon > 0$ there exists $g \in C^\infty(\overline{\Omega})$ such that $\|g\|_\infty \leq \epsilon$ for which the Dirichlet's problem is not solvable.

On the other hand, Díaz, Saa and Thiel [DST] studied the general quasilinear elliptic equation $div(Q(|\nabla u|)\nabla u) + f(u) = g(x_1, ..., x_n)$ in $R^n$ under Dirichlet and Neumann conditions. They studied existence and uniqueness of the problem for nonincreasing $f$ by finding apriori bounds for $\nabla u$. The case $Q(r) = (1 + r^2)^{-1/2}$ corresponds to the mean curvature equation (1), and the condition on $f$ becomes: $h'(u) \geq 0$.

In the present paper we study the problem by topological methods, obtaining a solution under some restrictions on $\|H\|_\infty$ and $\|\varphi\|_{2,p}$ but avoiding the conditions on the curvature of $\partial\Omega$. The condition $\frac{\partial h}{\partial u} \geq 0$ will not be necessary either.

The general Plateau problem and the Dirichlet asociated problem, have been studied in [AMR],[BC],[H],[LD-M],[MR] [S1],[S2],[WG], etc.

The quasilinear operator associated to problem (1) is strictly elliptic since its eigenvalues are $\lambda = 1$ and $\Lambda = 1 + |p|^2$, where $p = (u_x, u_y)$ (see [GT] chapter 10).

The main result is the following theorem

THEOREM 1

*Let $p > 2$ and assume that $\|\varphi\|_{2,p}$ and $\|H\|_{L^\infty(\overline{\Omega}\times[\epsilon,\epsilon])}$ are small enough with respect to $|\Omega|$, the Sobolev's constant and the apriori bounds for $\Delta$ in $\Omega$. Then there exists at least one solution $u \in W^{2,p}(\Omega)$ of (1).*

## SOLUTIONS BY FIXED POINT METHODS

First we note that $u$ is a solution of (1), if and only if $w = u - \varphi$ is a solution of the following equation:

$$(2)\begin{cases} (1 + (w_y + \varphi_y)^2)w_{xx} + (1 + (w_x + \varphi_x)^2)w_{yy} - 2(w_x + \varphi_x)(w_y + \varphi_y)w_{xy} \\ \qquad = 2H(x, y, w + \varphi)\left(1 + |\nabla(w + \varphi)|^2\right)^{\frac{3}{2}} - (1 + (w_y + \varphi_y)^2)\varphi_{xx} \\ \qquad -(1 + (w_x + \varphi_x)^2)\varphi_{yy} + 2(w_x + \varphi_x)(w_y + \varphi_y)\varphi_{xy} \qquad \text{in } \Omega \\ w = 0 \qquad \text{on } \partial\Omega \end{cases}$$

For each $\overline{v} \in C^1(\overline{\Omega})$ such that $\|\overline{v} + \varphi\|_\infty \leq \epsilon$ we consider the elliptic linear Dirichlet problem associated to equation (2)

$$(3)\begin{cases} L_{\overline{v}}(v) = F(\overline{v}) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where

$$L_{\overline{v}}(v) = (1+(\overline{v}_y+\varphi_y)^2)v_{xx} + (1+(\overline{v}_x+\varphi_x)^2)v_{yy} - 2(\overline{v}_x+\varphi_x)(\overline{v}_y+\varphi_y)v_{xy}$$

and

$$F(\overline{v}) = 2H(x,y,\overline{v}+\varphi)\left(1+|\nabla(\overline{v}+\varphi)|^2\right)^{\frac{3}{2}} - (1+(\overline{v}_y+\varphi_y)^2)\varphi_{xx} - (1+(\overline{v}_x+\varphi_x)^2)\varphi_{yy}$$
$$+2(\overline{v}_x+\varphi_x)(\overline{v}_y+\varphi_y)\varphi_{xy}$$

The linear equation (3) has a unique solution $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [GT], theorem 9.15). Thus, if we consider the Sobolev imbedding $W^{2,p} \hookrightarrow C^1$ with imbedding constant $k$ (i.e. $\|u\|_{1,\infty} \le k\,\|u\|_{2,p}$ ), we may define an operator $T : \overline{B_\epsilon(-\varphi)} \subset C^1(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega})$ given by $T(\overline{v}) = v$ if $v$ is the solution of (3) for $\overline{v}$.

We'll see that the operator $T$ has at least one fixed point in $C^1$, and this will give a solution of the original problem (1).

Our main tool will be the Schauder fixed point theorem (see [GT] theorem 11.1 and corollary 11.2). We prove first the following lemma and proposition.

LEMMA 2

*There exists a constant $C$ (depending only on $|\Omega|$, $p$) and $R > 0$ such that if*

$$\|\overline{v}+\varphi\|_{1,\infty} \le R$$

*then for every $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$*

$$\|w\|_{2,p} \le C\,\|L_{\overline{v}}(w)\|_p$$

**Proof**

We can write

$$L_{\overline{v}}(w) = \Delta w + S_{\overline{v}}(w)$$

where $S_{\overline{v}}(w) = (\overline{v}_y+\varphi_y)^2 w_{xx} + (\overline{v}_x+\varphi_x)^2 w_{yy} - 2(\overline{v}_x+\varphi_x)(\overline{v}_y+\varphi_y)w_{xy}$.

The operator $\Delta$ satisfies the hypotheses of [GT], lemma 9.17, then there exists a constant $C_1$ (independent of $w$) such that

$$\|w\|_{2,p} \le C_1 \|\Delta w\|_p$$

for all $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Then

$$\|L_{\overline{v}}(w)\|_p \ge \|\Delta w\|_p - \|S_{\overline{v}}(w)\|_p \ge \frac{1}{C_1}\|w\|_{2,p} - \|S_{\overline{v}}(w)\|_p$$

and being

$$\|S_{\overline{v}}(w)\|_p \le 4\|\overline{v}+\varphi\|_{1,\infty}^2 \|w\|_{2,p}$$

we obtain

$$\|L_{\overline{v}}(w)\|_p \ge \left(\frac{1}{C_1} - 4\|\overline{v}+\varphi\|_{1,\infty}^2\right)\|w\|_{2,p}$$

The second member of the last inequality is positive if $\|\overline{v}+\varphi\|_{1,\infty} \le R < \frac{1}{2\sqrt{C_1}}$, and setting $C = \dfrac{C_1}{1-4C_1R^2}$ the lemma holds.

In the following proposition we'll find $0 < R < \frac{1}{2\sqrt{C_1}}, \epsilon$ such that $T(\overline{B_R(-\varphi)}) \subset \overline{B_R(-\varphi)} \subset C^1(\overline{\Omega})$.

PROPOSITION 3

*Let $p > 2$ and assume that $\|\varphi\|_{2,p}$ and $\|H\|_{L^\infty(\overline{\Omega}\times[\epsilon,\epsilon])}$ are small enough. Then there exists $R \le \epsilon$ such that if*

$$\|\overline{v}+\varphi\|_{1,\infty} \le R$$

*then*

$$\|T(\overline{v})+\varphi\|_{1,\infty} \le R$$

*Furthermore, the operator $T$ is continuous in the closed ball $\overline{B_R(-\varphi)}$, and its range is a precompact set.*

**Proof**

Assume that $\|\overline{v}+\varphi\|_{1,\infty} \le R < \frac{1}{2\sqrt{C_1}}$. Then

$$\|v+\varphi\|_{1,\infty} \le k\|v\|_{2,p} + \|\varphi\|_{1,\infty} \le Ck\|L_{\overline{v}}(v)\|_p + \|\varphi\|_{1,\infty} = \frac{C_1 k}{1-4C_1R^2}\|F(\overline{v})\|_p + \|\varphi\|_{1,\infty}$$

and

$$\|F(\overline{v})\|_p \le 2(1+\|\overline{v}+\varphi\|_{1,\infty}^2)^{3/2}\|H(x,y,\overline{v}+\varphi)\|_p + 2(1+2\|\overline{v}+\varphi\|_{1,\infty}^2)\|\varphi\|_{2,p} \le$$

$$\leq 2(1+R^2)^{3/2}|\Omega|^{1/p}\,\|H\|_\infty + 2(1+2R^2)\,\|\varphi\|_{2,p}$$

We look for a number $R$ such that

$$\frac{2C_1k}{1-4C_1R^2}((1+R^2)^{3/2}|\Omega|^{1/p}\,\|H\|_\infty + (1+2R^2)\,\|\varphi\|_{2,p}) + \|\varphi\|_{1,\infty} \leq R$$

or, equivalently, such that $f(R)\leq 0$, where

$$f(R) = \frac{2C_1k}{1-4C_1R^2}((1+R^2)^{3/2}|\Omega|^{1/p}\,\|H\|_\infty + (1+2R^2)\,\|\varphi\|_{2,p}) + \|\varphi\|_{1,\infty} - R$$

It is clear that $f \leq \frac{P}{1-4C_1R^2}$ in the interval $(0, \frac{1}{2\sqrt{C_1}})$, where

$$P(R) = 2C_1k((1+\frac{1}{4C_1})^{3/2}|\Omega|^{1/p}\,\|H\|_\infty + (1+\frac{1}{2C_1})\,\|\varphi\|_{2,p}) + \|\varphi\|_{1,\infty} - R + 4C_1R^3$$

$P$ achieves a minimum in $R_0 = \frac{1}{\sqrt{12C_1}}$ and $P(R_0)\leq 0$ if $\|\varphi\|_{2,p}$ and $\|H\|_\infty$ are small enough. Then $f(R_0)\leq 0$.

In order to complete the proof we must see that $T$ is continuous and compact. Indeed, for $\overline{u}$, $\overline{v}\in\overline{B_R(-\varphi)}$:

$$\|u-v\|_{2,p} \leq C\,\|L_{\overline{u}}(u-v)\|_p \leq C\left(\|F(\overline{u})-F(\overline{v})\|_p + \|L_{\overline{v}}(v)-L_{\overline{u}}(v)\|_p\right).$$

But

$$\|F(\overline{u})-F(\overline{v})\|_p \leq \|2H(x,y,\overline{u}+\varphi) - 2H(x,y,\overline{v}+\varphi))\left(1+|\nabla(\overline{u}+\varphi)|^2\right)^{\frac{3}{2}}\|_p$$

$$+\|2H(x,y,\overline{v}+\varphi)(\left(1+|\nabla(\overline{u}+\varphi)|^2\right)^{\frac{3}{2}} - \left(1+|\nabla(\overline{v}+\varphi)|^2\right)^{\frac{3}{2}})\|_p$$

$$+\|((\overline{u}_y+\varphi_y)^2-(\overline{v}_y+\varphi_y)^2)\varphi_{xx}\|_p+\|((\overline{u}_x+\varphi_x)^2-(\overline{v}_x+\varphi_x)^2)\varphi_{yy}\|_p+2\|((\overline{u}_x+\varphi_x)(\overline{u}_y+\varphi_y)-$$

$$(\overline{v}_x+\varphi_x)(\overline{v}_y+\varphi_y))\varphi_{xy}\|_p \leq 2\|H(x,y,\overline{u}+\varphi)-H(x,y,\overline{v}+\varphi)\|_p(1+R^2)^{3/2}$$

$$+6R(1+R^2)^{\frac{1}{2}}(R+\|\varphi\|_{1,\infty})\|\overline{u}-\overline{v}\|_{1,\infty}|\Omega|^{1/p}\|H\|_\infty + 8R\|\overline{u}-\overline{v}\|_{1,\infty}\|\varphi\|_{2,p}$$

and

$$\|L_{\overline{v}}(v)-L_{\overline{u}}(v)\|_p \leq 8R\|\overline{u}-\overline{v}\|_{1,\infty}\|v\|_{2,p}$$

Being $H$ uniformely continuous and $||u-v||_{1,\infty} \leq k||u-v||_{2,p}$, the continuity follows. Moreover, fixing any $\overline{v} \in \overline{B_R(-\varphi)}$ we see that $\overline{T(B_R(-\varphi))}$ is bounded in $W^{2,p}$, and the result follows from the compactness of the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$.

REMARK

*In the situation of proposition 3, if we write* $P(R) = 4C_1R^3 - R + a$, *the smallness of* $H$ *and* $\varphi$ *can be stated in the more precise condition* $a \leq \frac{2}{3\sqrt{12C_1}}$.

**Proof of theorem 1:** From proposition 3 we know that the operator $T$ satisfies the assumptions of Schauder fixed point theorem (see [GT] corollary 11.2). Thus, we obtain a fixed point $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for the operator $T$, which corresponds to a solution of equation (2).

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