Revista de la
Unión Matemática Argentina
Volumen 41, 3, 1999.

A POINTWISE ERGODIC THEOREM

María Elena Becker

Abstract. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and φ a nonsingular transformation on $(X, \mathcal{A}, \mu)$. Necessary and sufficient conditions are given in order that for any f in L_1 the average $\frac{1}{n}\sum_{i=0}^{n-1} f \circ \varphi^i(x)$ converges almost everywhere.

INTRODUCTION AND RESULTS.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and φ a nonsingular transformation on $(X, \mathcal{A}, \mu)$, that is, $A \in \mathcal{A}$ and $\mu(A) = 0$ implies $\mu(\varphi^{-1}A) = 0$. We consider the operator T, acting on measurable functions,

$$Tf(x) = f(\varphi x) \ .$$

Associated with T we have the averages

$$M_n(T)f = \frac{1}{n}\sum_{i=0}^{n-1} T^i f \ .$$

Since T maps L_∞ to L_∞ and φ is nonsingular, the adjoint operator S acting on $L_1(\mu)$ can be defined by the relation

$$\int gTf d\mu = \int fSg d\mu \quad ,$$

$f \in L_\infty$, $g \in L_1$. As in [3], in order to extend the domain of S to the space $M^+(\mu)$ of all nonnegative extended real valued measurable functions on X, fix any $f \in M^+(\mu)$ and take $\{f_n\} \subset L_1^+(\mu)$ such that $f_n \uparrow f$ a.e. on X. We then define

$$Sf = \lim_n Sf_n \qquad \text{a.e. on } X \ .$$

It is easily checked that by this process S can be uniquely extended to an operator on $M^+(\mu)$ satisfying $S(\alpha f + \beta g) = \alpha Sf + \beta Sg$, $0 \le \alpha, \beta < \infty$. In the sequel, S will be understood to be defined on $M^+(\mu)$ in this manner and we write

$$M_n(S)f = \frac{1}{n}\sum_{i=0}^{n-1} S^i f \ .$$

1991 Mathematics Subject Classification: Primary 47A35 Secondary 28D05

The purpose of this paper is the following theorem:

Theorem 1. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and φ a nonsingular transformation on X. Then the following conditions are equivalent:

A) For any $f \in L_1^+(\mu)$, $\lim_n M_n(T)f$ exists and is finite a.e. on X.

B) S satisfies the mean ergodic theorem in $L_1(\mu)$ and, further, for any $f \in L_1^+(\mu)$, $\lim_n M_n(S)(fv_0)$ exists and is finite a.e. on X, where v_0 is the pointwise and L_1−norm limit of $M_n(S)1$.

We will need the following well-known fact (see, e.g. [4])

Lemma 1. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and φ a nonsingular transformation on X. The following are equivalent:

(i) For any $f \in L_\infty$, $M_n(T)f$ converges almost everywhere.

(ii) For any $A \in \mathcal{A}$, $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \mu(\varphi^{-i}(A))$ exists.

(iii) S satisfies the mean ergodic theorem in L_1.

Throughout this paper χ_A will be the characteristic function of the set A and we will consider two sets as 'equal' if they agree up to a set of measure zero. A measurable set A will be called *invariant* if $T\chi_A = \chi_A$ a.e.. We denote by $\mathcal{J}$ the σ−field of invariant sets.

THE PROOFS.

In order to prove Theorem 1 we will make some previous considerations. First, we observe that by virtue of Lemma 1 we may and do suppose that φ satisfies:

$$\text{For any } A \in \mathcal{A} \text{ there exists } \overline{\mu}(A) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1} \mu(\varphi^{-i}(A)) .$$

By the Vitali-Hahn-Saks theorem $\overline{\mu}$ is a measure. It is easy to see that $\overline{\mu}$ is absolutely continuous with respect to μ, invariant under φ and $\overline{\mu}(A) = \mu(A)$, $A \in \mathcal{J}$.

Let $v_0 = \frac{d\overline{\mu}}{d\mu}$, $C = \{x : v_0(x) > 0\}$ and $D = X \setminus C$. We have $\mu(C \setminus \varphi^{-1}(C)) = 0$ and hence we may suppose that $C \subset \varphi^{-1}(C)$. Then the set $D_0 = \bigcap_{n\geq 0} \varphi^{-n}(D)$ is invariant and $\mu(D_0) = \overline{\mu}(D_0) = 0$. Thus we have

$$X = \bigcup_{n\geq 0} \varphi^{-n}(C) . \tag{1}$$

It is also easy to see that the validity of L_1−mean ergodic theorem for S implies

$$v_0 = \lim_n M_n(S)1 \ .$$

We prove the following:

Lemma 2. Let $h \in M^+(\mu)$ such that $h^*(x) = \lim_n M_n(T)h(x)$ exists and is finite a.e. on X. Then $h \in L_1(\overline{\mu})$ if and only if $h^* \in L_1(\mu)$.

Proof. If h is in $L_1(\overline{\mu})$, then by Birkhoff's classical ergodic theorem $h^* \in L_1(\overline{\mu})$ and we have

$$\int h d\overline{\mu} = \int h^* d\overline{\mu} = \int h^* d\mu \ ,$$

where the last equality follows from the fact that h^* is $\mathcal{J}$−measurable together with $\mu = \overline{\mu}$ on $\mathcal{J}$.
Conversely, assume $h^* \in L_1(\mu)$ and let $\{h_n\}$ be a sequence of nonnegative simple functions increasing to h. By (1) and the Lebesgue bounded convergence theorem, for all $A \in \mathcal{A}$ we have

$$\int \chi_A^* d\mu = \lim_n \int M_n(T)\chi_A d\mu = \int \chi_A d\overline{\mu} \ .$$

Hence

$$\int h^* d\mu \geq \lim_n \int h_n^* d\mu = \lim_n \int h_n d\overline{\mu} = \int h d\overline{\mu} \ .$$

Proof of Theorem 1. A) ⇒ B). Let f be a function in $L_1^+(\mu)$ and $f^* = \lim_n M_n(T)f$. We consider for each natural N the set $J_N = \{x : f^* \leq N\}$. Since $J_N \in \mathcal{J}$, $(f\chi_{J_N})^* = \chi_{J_N} f^*$ μ−a.e., and from Lemma 2 it follows that $f\chi_{J_N}.v_0 \in L_1(\mu)$.

Then Lemma 1 and the fact that the validity of the L_1−mean ergodic theorem for S implies the validity of the pointwise ergodic theorem for S (see, e.g. [2]) give the almost everywhere convergence of $M_n(S)(f\chi_{J_N} v_0)$. From the relation

$$S(f\chi_{J_N} v_0) = \chi_{J_N} S(fv_0)$$

it follows that $M_n(S)(fv_0)$ converges a.e. on J_N. Letting $N \uparrow \infty$, we obtain B).
B) ⇒ A). Since for all $f \in L_1(\mu)$,

$$|S(\chi_C f)| \leq \chi_C S|f| \qquad \mu - \text{a.e. on } X \ ,$$

S can be considered to be a positive linear contraction on $L_1(C, \mu)$.
Let $\mathcal{J}_C$ be the σ−field of invariant subsets of C. Using, for instance, the Chacon-Ornstein theorem and the identification of the limit function (see [1], p.41) it

follows that for each $h \in L_1(C,\mu)$ there exists a $\mathcal{J}_C$-measurable function $R(h)$ such that

$$\hat{h} = \lim M_n(S)h = R(h)v_0 \qquad \mu - \text{a.e. on } C \ .$$

Furthermore, we have

$$\int_K h d\mu = \int_K R(h) v_0 d\mu \quad , K \in \mathcal{J}_C \ .$$

Now, let $f \in L_1^+(\mu)$ and set $f_n = \min\{f, n\}$, for each natural n. Then

$$\widehat{f_n v_0} = R(f_n v_0)v_0 \leq \widehat{f v_0} < \infty \qquad \mu - \text{a.e. on } C \ ,$$

where $\widehat{fv_0} = \lim_n M_n(S)(fv_0)$.
We take $K_N = \{x \in C : \sup_n R(f_n v_0) \leq N\}$. It follows that

$$\int_{K_N} f v_0 d\mu = \lim_n \int_{K_N} f_n v_0 d\mu \leq N \int_{K_N} v_0 d\mu \ .$$

Therefore, $f \in L_1(K_N, \overline{\mu})$ and by Birkhoff's classical ergodic theorem $M_n(T)f$ converges to a finite limit a.e. on K_N.
Since $K_N \uparrow C$, A) follows from (1).

REFERENCES

[1] A.M. GARSIA, Topics in Almost Everywhere Convergence, Markham, Chicago 1970.

[2] Y. ITO, Uniform integrability and the pointwise ergodic theorem, Proc. Amer. Math. Soc. 16 (1965), 222-227.

[3] R. SATO, On pointwise ergodic theorems for positive operators, Studia Math. 97 (2) (1990), 71-84.

[4] R. SATO, Pointwise ergodic theorems in Lorentz Spaces, Studia Math. 144 (3) (1995), 227-236.

Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad Universitaria - Pabellón I
1428 Buenos Aires ARGENTINA
mbecker@dm.uba.ar

Recibido en Setiembre 1998