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HARDY-ORLICZ SPACES AND HÖRMANDER 'S MULTIPLIERS

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Abstract: We consider Hörmander 's multipliers of fractional order on Hardy-Orlicz spaces $H_w(\mathbb{R}^n)$. The main tools we used are the atomic and molecular decompositions of these spaces.

1. Introduction

In this paper we study multipliers for the Hardy-Orlicz spaces $H_w(\mathbb{R}^n)$. We consider Hörmander's multipliers of order $t > 0$, where t is not necessarely an integer number. In [5], Taibleson and Weiss, proved that the functions m satisfying a Hörmander's type condition (see (1.2)) are multipliers for the classical Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p \leq 1$. There, they use different techniques to deal with the cases t an integer and t real and non-integer (see theorems 4.2 and 4.9).

The purpose of this work is, on one side, to extend these results to the contect of Orlicz spaces. On the other side we present an approach that allows to deal simultaneously with all positive real values of t. Our main tools in this setting are the atomic and molecular decomposition of the Hardy-Orlicz spaces given in [3] and [6].

In order to introduce the spaces $H_w(\mathbb{R}^n)$ we first give some definitions.

Let g be a positive function defined on $\mathbb{R}^+ = \{x \in \mathbb{R}, x > 0\}$. We shall say that g is of lower type $m \geq 0$ (respectively, upper type m) if there exists a positive constant C such that

$$g(st) \leq Ct^m g(s)$$

for every $0 < t \leq 1$ (respectively, $t \geq 1$).

Given g, a function of positive lower type l, we define

$$g^{-1}(s) = \sup\{t : g(t) \leq s\}.$$

Assume that w is a function of positive lower type l and upper type $d \leq 1$.

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Let $j \in I\!N$ such that $jl > 1$. We define

$$H_w = H_w(I\!R^n) = \left\{ f \in \mathcal{S}' : \int\limits_{I\!R^n} w\left(\frac{f_j^*(x)}{\lambda^{1/l}}\right) dx < \infty \right\},$$

where f_j^* is the j-maximal function of a distribution $f \in \mathcal{S}'$, the dual space of the class of Schwartz functions (see [1]). We denote

$$||f||_{H_w} = \inf\left\{ \lambda > 0 : \int\limits_{I\!R^n} w\left(\frac{f_j^*(x)}{\lambda^{1/l}}\right) dx \leq 1 \right\}$$

It can be seen that H_w is a complete topological vector space with respect to the quasi-distance induced by $||\ ||_{H_w}$. Moreover H_w is continuously included in $\mathcal{S}'$. Clearly, when $w(t) = t^p$, $0 < p \leq 1$, $H_w(I\!R^n) = H^p(I\!R^n)$. Also it can be proved that for every $f \in H_w$, $\widehat{f}$ is a continuous function on $I\!R^n$ which satisfies

$$\text{(1.1)} \qquad \begin{cases} |\widehat{f}(x)| \leq C\dfrac{w^{-1}(|x|^n)}{|x|^n}||f||_{H_w}^{1/l}, & x \neq 0 \\ \\ \widehat{f}(0) = 0 \end{cases}$$

where C is a constant independent of f, see [5] for the case $H^p(I\!R^n)$.

Suppose that m is a measurable function such that the function $m\widehat{f}$ belongs to $\mathcal{S}'$ whenever $f \in H_w$. We say that m is a multiplier on H_w iff there is a constant $C > 0$ satisfying

$$||(m\widehat{f})^{\vee}||_{H_w} \leq C||f||_{H_w}$$

for all $f \in H_w$.

The Hörmander condition is given in terms of the difference operator which is defined by

$$\Delta_h u(x) = u(x) - u(x-h),$$

where u is a real valued function on $I\!R^n$ and $h \in I\!R^n$. We denote

$$\Delta_h^0 u = u \quad \text{and} \quad \Delta_h^k u = \Delta_h^{k-1}\Delta_h u, \quad k \in I\!N.$$

We say that a function m satisfies the Hörmander condition for $t > 0$ if m is bounded, $|m(x)| \leq A$, and for some integer $\bar{t} > t$ and all integers k, we have

$$\text{(1.2)} \qquad 2^{k(2t-n)} \int\limits_{|h|<2^{k-1}} |h|^{-2t} \int\limits_{2^k<|x|\leq 2^{k+1}} |\Delta_h^{\bar{t}} m(x)|^2 dx \frac{dh}{|h|^n} \leq A^2.$$

It can be proved that if t is an integer, condition (1.2) is equivalent to

$$R^{2|\beta|-n} \int_{R<|x|\leq 2R} |D^{\beta}m(x)|^2 dx \leq A^2$$

for $0 \leq |\beta| \leq t$ and all $R > 0$.

The main result in this paper is the following:

Theorem (1.3). *Assume that w is a function of positive lower type l and upper type $d \leq 1$. Suppose that m satisfies the Hörmander condition for $t > n(2/l - 3/2)$. Then, there is a constant $C > 0$, independent of m, such that*

(1.4) $$\|(m\hat{f})^{\vee}\|_{H_w} \leq CA^{l}\|f\|_{H_w}$$

for every $f \in H_w$. When $w(t) = t^p$, $p \in (0,1]$ we have (1.4) with $t > n\ (1/l - 1/2)$.

The proof of theorem (1.3) is developed in section 3. As principal tools we use the atomic and molecular decompositions of H_w which are contained in section 2.

2. Atomic and molecular decompositions of H_w

In this section we shall give the definitions of the atomic and molecuar Hardy-Orlicz spaces and state some of their properties used in the next section. The proofs of these results can be found in [6] and [3]. As in those papers in the sequel we shall assume that:
(2.1) w is a function of positive lower type l and upper type $d \leq 1$, ρ is the function defined by $\rho(t) = t^{-1}/w^{-1}(t^{-1})$ and $N = [n(1/l - 1)]$, where $[x]$ stands for the biggest integer less than or equal to x.

The following definition will be usefull to introduce the atomic and molecular spaces.

Definition (2.2). Suppose that $\mathbf{b} = \{b_j\}_{j\in I\!N_0}$ is a sequence of functions in $L^2(I\!R^n)$, and $\mathbf{c} = \{c_j\}_{j\in I\!N_0}$ is a sequence of positive constants such that

$$\sum_j c_j w(||b_j||_2 c_j^{-1/2}) = L < \infty, \tag{2.3}$$

where $I\!N_0$ denotes the set of non-negative integers. We define

$$\Lambda(\mathbf{b},\mathbf{c}) = \inf\left\{\lambda > 0 : \sum_j c_j w\left(\frac{||b_j||_2 c_j^{-1/2}}{\lambda^{1/l}}\right) \leq 1\right\} \tag{2.4}$$

We observe that

$$\sum_j c_j w\left(\frac{||b_j||_2 c_j^{-1/2}}{\Lambda(\mathbf{b},\mathbf{c})^{1/l}}\right) \leq \tilde{C},$$

where $\tilde{C}$ is the upper type constant of w.

Definition (2.5). Let $\eta \in I\!N_0$, $\eta \geq N$. A (ρ,η) atom is a real valued function a on $I\!R^n$ satisfying

$$\int_{I\!R^n} a(x) x^\beta dx = 0 \tag{2.6}$$

for every multi-index $\beta = (\beta_1,\ldots,\beta_n)$ such that $|\beta| = \beta_1 + \ldots + \beta_n \leq \eta$, where $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$,
(2.7) the support of a is contained in a ball B and

$$||a||_2 |B|^{-1/2} \leq [|B|\rho(|B|)]^{-1}. \tag{2.8}$$

Clearly, when $w(t) = t^p$, $p \in [0,1]$, we have that $\rho(t) = t^{1/p-1}$ and a (ρ,η) atom is a $(p,2,\eta)$ atom in the usual sense (see [5]).

Definition (2.9). We define $H^{\rho,\eta} = H^{\rho,\eta}(I\!R^n)$ as the linear space of distributions f on $\mathcal{S}$ which can be represented by

$$f(\psi) = \sum_j b_j(\psi), \tag{2.10}$$

where $\{b_j\}$ is a sequence of (ρ,η) atoms such that there exists a sequence of balls $\{B_j\}$ satisfying $\text{supp}(b_j) \subset B_j = B(x_j, r_j)$ and (2.3) with $c_j = |B_j|$. We denote $\mathbf{b} = \{b_j\}$, $\mathbf{B} = \{|B_j|\}$ and let

$$||f||_{H^{\rho,\eta}} = \inf \ \Lambda(\mathbf{b}, \mathbf{B}),$$

where $\Lambda(\cdot,\cdot)$ is as in (2.4) and the infimum is taken over all possible representations of f of the form (2.10).

The definition of a molecule in the context of Hardy-Orlicz spaces is the following.

Definition (2.11). Assume that ϵ is admissible, that is $\epsilon > 0$ for the case $w(t) = t^p$ and $\epsilon > 1/l - 1$ for a general w. Let $x_0 \in I\!R^n$. A (ρ, ϵ) molecule centered at x_0 is a real valued function M on $I\!R^n$ satisfying

$$||M||_2 ||M\rho(|\cdot - x_0|^n)| \cdot - x_0|^{n(\epsilon+1/2)}||_2 \le C \tag{2.12}$$

and

$$\int\limits_{I\!R^n} M(x) x^\beta dx = 0 \tag{2.13}$$

for every multi-index β such that $|\beta| \le N$.

We observe that when $w(t) = t^p$, $p \in (0, 1]$, a (ρ, ϵ) molecule is a $(p, 2, \epsilon + 1/p - 1/2)$ molecule in the usual sense (see [2] and [5]).

Remark (2.14). It is not difficult to see that condition (2.12) implies that $Mx^\beta \in L^1$ for every β, $|\beta| \le N$ and consequently $\widehat{M}$ has continuous derivatives up to the order N. Moreover, we get

$$D^\beta \widehat{M}(\xi) = [-2\pi i x^\beta M(\cdot)]^\wedge(\xi), \qquad \xi \in I\!R^n.$$

From this, we clearly have that if M satisfies (2.12), then (2.13) is equivalent to

$$D^\beta \widehat{M}(0) = 0\,, \qquad |\beta| \le I\!N.$$

Given M, a (ρ, ϵ) molecule centered at x_0, and B, a ball with the same center, we denote

$$M^B = M\mathcal{X}_B$$

$$M^{CB} = \frac{\rho(|\cdot - x_0|^n)|\cdot - x_0|^{n(\epsilon+1/2)} M \mathcal{X}_{CB}}{\rho(|B|)|B|^{\epsilon+1/2}}.$$

Definition (2.15). Assume that ϵ is admissible. We define $\mathcal{M}_{\rho,\epsilon} = \mathcal{M}_{\rho,\epsilon}(I\!R^n)$, as the linear space of distributions f on S which can be represented by

$$f(\psi) = \sum_j M_j(\psi), \tag{2.16}$$

where $\{M_j\}$ is a sequence of (ρ,ϵ) molecules centered at $\{x_j\}$, such that there exists a sequence of balls $\{B_j\} = \{B(x_j, r_j)\}$ satisfying

$$\sum_j |B_j| w(||M_j^{B_j}||_2 |B_j|^{-1/2}) + \sum_j |B_j| w(||M_j^{CB_j}||_2 |B_j|^{-1/2}) < \infty.$$

Let $\mathbf{M^B} = \{M_j^{B_j}\}$, $\mathbf{M^{CB}} = \{M_j^{CB_j}\}$ and $\mathbf{B} = \{|B_j|\}$. We define

$$||f||_{\mathcal{M}_{\rho,\epsilon}} = \inf(\Lambda(\mathbf{M^B}, \mathbf{B}) + \Lambda(\mathbf{M^{CB}}, \mathbf{B}))\ ,$$

where $\Lambda(\cdot,\cdot)$ is as in (2.4) and the infimum is taken over all possible representations of f of the form (2.16).

We finally remark that both spaces $H^{\rho,\eta}$ and $\mathcal{M}_{\rho,\epsilon}$ coincide with the Hardy-Orlicz spaces H_w.

3. Proof of the main result.

In order to prove theorem (1.3) we first give two technical results which proofs are contained in [5], necessary modifications can be carried out.

Proposition (3.1). *Assume that b is a function belonging to $L^2(I\!R^n)$ with vanishing moments up to the order $\eta \in I\!N_0$ and $supp(b) \subset B = B(0,r)$. Then for every $h \in I\!R^n$, $0 \le k \le \eta$, $\delta \in I\!R$ and $E > 0$ we get*

$$||(\Delta_h^k \hat{b})^2||_r \le C||b||_2^2 |B|^{\frac{2k}{n}+1-\frac{1}{r}} |h|^{2k}, \quad r \in [1,\infty] \tag{3.2}$$

and

$$
(3.3)\qquad \begin{cases} |\tau_{\delta h}\Delta_h^k \widehat{b}(x)| \leq C\|b\|_2 |B|^{\frac{\eta+1}{n}+\frac{1}{2}} |h|^{\eta+1} & \text{if } |x| < E|h| \\ |\tau_{\delta h}\Delta_h^k \widehat{b}(x)| \leq C\|b\|_2 |B|^{\frac{\eta+1}{n}+\frac{1}{2}} |h|^{k} |x|^{\eta+1-k} & \text{if } |x| \geq E|h|, \end{cases}
$$

where C is a positive constant which does not depend on b, h, and x; and $\tau_h f(x) = f(x+h)$.

We observe that by (3.3) with $\delta = k = 0$, we have

$$
(3.4)\qquad |\widehat{b}(x)| \leq C\|b\|_2 |B|^{\frac{\eta+1}{n}+\frac{1}{2}} |x|^{\eta+1}
$$

for every $x \in I\!R^n$.

Theorem (3.5). *Suppose that m satisfies the Hörmander condition for $t > n/2$. Then there exists a constant C, indepent of m, such that for all integer k,*

$$
(3.6)\qquad 2^{k(2\gamma-\frac{n}{r})} \int_{I\!R^n} |h|^{-2\gamma} [\int_{2^k<|x|\leq 2^{k+1}} |\Delta_h^{\overline{\gamma}} m(x)|^{2r} dx]^{1/r} \frac{dh}{|h|^n} \leq C^2 A^2
$$

whenever $r = 1$ or $n/r > n - 2(t-\gamma)$, $\gamma \in I\!R^+$, and $\overline{\gamma}$ an integer greater than γ. Furthermore, m is bounded and continuous on $I\!R^n - \{0\}$ and $\|m\|_\infty \leq CA$.

Let us remark that, since the results stated in the following are invariant under change of equivalent functions, in proving them we shall assume without loss of generality that w is in addition continuous and strictly increasing.

Lemma (3.7). *Assume that m satisfies the Hörmander condition for t such that $\epsilon \equiv t/n + 1/2 - 1/l$ is admissible. Suppose that b is a function belonging to $L^2(I\!R^n)$ with vanishing moments up to the order $[t]+1$ and $supp(b) \subset B = B(0,r)$. Then $(m\widehat{b})^\vee$ is a (ρ,ϵ) molecule centered at zero and satisfies*

$$
(3.8)\qquad \|(m\widehat{b})^\vee\|_2 \leq CA\|b\|_2 \quad \text{and}
$$

$$
(3.9)\qquad \|(m\widehat{b})^\vee| \cdot |^t\|_2 \leq CA\|b\|_2 |B|^{t/n} \quad .
$$

Proof. Since $t/n+1-1/l>0$ and ρ has upper type $1/l-1$, we can write
$||(m\hat{b})^{\vee}(\cdot)\rho(|\cdot|^n)|\cdot|^{n(\frac{t}{n}+1-\frac{1}{l})}||_2$

$$\leq ||(m\hat{b})^{\vee}(\cdot)\rho(|\cdot|^n)|\cdot|^{n(\frac{t}{n}+1-\frac{1}{l})}\mathcal{X}_{|x|\leq 1}(\cdot)||_2+||(m\hat{b})^{\vee}(\cdot)|\cdot|^t\rho(|\cdot|^n)|\cdot|^{n(1-\frac{1}{l})}\mathcal{X}_{|x|\geq 1}(\cdot)$$

$$\leq C(||(m\hat{b})||_2+||(m\hat{b})^{\vee}|\cdot|^t||_2)\ .$$

Then, by (2.14), in order to prove that $(m\hat{b})^{\vee}$ is a (ρ,ϵ) molecule centered at zero, it will be enough to check (3.8), (3.9) and

$$D^{\beta}(m\hat{b})(0)=0\quad ,\ |\beta|\leq N. \tag{3.10}$$

From the boundedness of m we clearly have (3.8). Let us prove (3.9). Suppose that $\bar{t}=[t]+1$. Then, using the identities

$$\int_{I\!R^n}|(m\hat{b})^{\vee}(x)|^2|x|^{2t}=C\int_{I\!R^n}\frac{||\Delta_h^{\bar{t}}(m\hat{b})(\cdot)||_2^2}{|h|^{n+2t}}dh\ ,$$

which proof is contained in [4] (p. 140), and

$$\Delta_h^{\bar{t}}(fg)=\sum_{k+j=\bar{t}}\binom{\bar{t}}{j}(\tau_{-kh}\Delta_h^j f)(\Delta_h^k g)\quad ,$$

we need to show that

$$\int_{I\!R^n}\frac{||\tau_{-kh}\Delta_h^j\hat{b}(\cdot)\Delta_h^k m(\cdot)||_2^2}{|h|^{n+2t}}dh=\left(\int_{|h|<|B|^{-1/n}}+\int_{|h|\geq|B|^{-1/n}}\right)\int_{I\!R^n}\frac{||\tau_{-kh}\Delta_h^j\hat{b}(\cdot)\Delta_h^k m(\cdot)||_2^2}{|h|^{n+2t}}$$

$$\equiv I_1+I_2\leq CA^2||b||_2^2|B|^{\frac{2t}{n}}$$

for $k+j=\bar{t}$. It is clear that

$$I_2\leq CA^2||b||_2^2\int_{|h|\geq|B|^{-1/n}}\frac{1}{|h|^{n+2t}}dh\leq CA^2||b||_2^2|B|^{\frac{2t}{n}}\ ,$$

because $||m||_\infty\leq CA$. Let us estimate I_1. We first consider the case $k=0$. From (3.2) with $r=1$ we easily obtain

$$I_1\leq CA^2||b||_2^2|B|^{\frac{2\bar{t}}{n}}\int_{|h|<|B|^{-1/n}}|h|^{2(\bar{t}-t)-n}dh\leq CA^2||b||_2^2|B|^{\frac{2t}{n}}.$$

For the estimate of I_1 for the case $k \geq 1$ we choose an integer K such that $2^{nK} < |B|^{-1} \leq 2^{n(K+1)}$ and we get

$$I_1 = \sum_{v=-\infty}^{K-1} \int\limits_{|h|\leq 2^{v+1}} \int\limits_{2^v<|x|\leq 2^{v+1}} \frac{|\tau_{-kh}\Delta_h^j \hat{b}(x)|^2 |\Delta_h^k m(x)|^2}{|h|^{n+2t}} dx dh$$

$$+ \sum_{v=-\infty}^{K-1} \int\limits_{2^{v+1}<|h|<|B|^{-1/n}} \int\limits_{2^v<|x|\leq 2^{v+1}} \frac{|\tau_{-kh}\Delta_h^j \hat{b}(x)|^2 |\Delta_h^k m(x)|^2}{|h|^{n+2t}} dx dh$$

$$+ \sum_{v=K}^{\infty} \int\limits_{|h|<|B|^{-1/n}} \int\limits_{2^v<|x|\leq 2^{v+1}} \frac{|\tau_{-kh}\Delta_h^j \hat{b}(x)|^2 |\Delta_h^k m(x)|^2}{|h|^{n+2t}} dx dh$$

$$= S_1 + S_2 + S_3$$

Estimate for S_1. Let z be a nonnegative constant. Aplying the fact that $|B|^{-1/n}|h|^{-1}$ 1 and (3.3) it follows that

$$S_1 \leq |B|^{\frac{-2z}{n}} \sum_{v=-\infty}^{K-1} \int\limits_{|h|\leq 2^{v+1}} \sup_{|x|\in(2^v,2^{v+1}]} |\tau_{-kh}\Delta_h^j \hat{b}(x)|^2 \cdot \frac{1}{|h|^{n+2t+2z}} \int\limits_{2^v<|x|\leq 2^{v+1}} |\Delta_h^k m(x)|^2 dx$$

$$\leq C|B|^{\frac{2(\bar{t}+1)}{n}+1-\frac{2z}{n}} ||b||_2^2 \sum_{v=-\infty}^{K-1} 2^{v2(\bar{t}+1-j)} \int\limits_{|h|\leq 2^{v+1}} \frac{1}{|h|^{n+2(t+z-j)}} \int\limits_{2^v<|x|\leq 2^{v+1}} |\Delta_h^k m(x)|^2 dx$$

Taking $z = \frac{1}{2}$ when t is an integer and $k = 1$, and $z = 0$ in other case, we can apply (3.5) and we obtain that S_1 is bounded by

$$C|B|^{2\frac{(\bar{t}+1)}{n}+1-\frac{2z}{n}} ||b||_2^2 \sum_{v=-\infty}^{K-1} 2^{v(2\bar{t}+2-2t-2z+n)} \leq C||b||_2^2 |B|^{\frac{2t}{n}},$$

because $2\bar{t} + 2 - 2t - 2z + n > 0$ and $2^K \sim |B|^{-1/n}$.
Estimate for S_3. Take z as in the estimate of S_1. Since $t > n/2$, we can choose $r \geq 1$ such that

$$\frac{n}{r} > n - 2j + 2z$$

and

$$2(t + z - j) - \frac{n}{r} > 0.$$

Thus, from Hölder's inequality, (3.5) and (3.2), we get that S_3 is less than or equal to

$$|B|^{-\frac{2z}{n}}\sum_{v=K}^{\infty}\int_{|h|<|B|^{-1/n}}\left(\int_{I\!R^n}|\tau_{-kh}\Delta_h^j\widehat{b}(x)|^{2r'}dx\right)^{1/r'}\frac{1}{|h|^{n+2t+2z}}\left(\int_{2^v<|x|\le 2^{v+1}}|\Delta_h^k m(x)|^{2r}dx\right.$$

$$\le |B|^{\frac{-2z}{n}+\frac{2j}{n}+\frac{1}{r}}||b||_2^2\sum_{v=K}^{\infty}\int_{|h|<|B|^{-1/n}}\frac{1}{|h|^{n+2(t+z-j)}}\left(\int_{2^v<|x|\le 2^{v+1}}|\Delta_h^k m(x)|^{2r}dx\right)^{1/r}dh$$

$$\le CA^2||b||_2^2|B|^{\frac{-2z}{n}+\frac{2j}{n}+\frac{1}{r}}||b||_2^2\sum_{v=K}^{\infty}2^{-v(2(t+z-j)-\frac{n}{r})}$$

$$\le CA^2||b||_2^2|B|^{\frac{2t}{n}}.$$

because $2^k\sim|B|^{-1/n}$.
Estimate for S_2. By Tonelli, we can write

$$S_2\le\int_{|x|\le 2^k}\int_{|x|<|h|<|B|^{-1/n}}\frac{|\tau_{-kh}\Delta_h^j\widehat{b}(x)|^2|\Delta_h^k m(x)|^2}{|h|^{n+2t}}dhdx$$

$$=\int_{|h|\le 2^k}\int_{|x|<|h|}\frac{|\tau_{-kh}\Delta_h^j\widehat{b}(x)|^2|\Delta_h^k m(x)|^2}{|h|^{n+2t}}dxdh$$

$$+\int_{2^k<|h|<|B|^{-1/n}}\int_{|x|\le 2^k}\frac{|\tau_{-kh}\Delta_h^j\widehat{b}(x)|^2|\Delta_h^k m(x)|^2}{|h|^{n+2t}}dxdh$$

$$=I_3+I_4.$$

From (3.3) it is clear that

$$I_3\le CA^2|B|^{\frac{2(\tilde{t}+1)}{n}+1}||b||_2^2\int_{|h|\le 2^k}|h|^{2(\tilde{t}+1)-2t}dh$$

$$\le CA^2||b||_2^2|B|^{\frac{2t}{n}},$$

since $|B|^{-1/n} \sim 2^K$. On the other hand, applying (3.2) with $r = \infty$, we have that

$$I_4 \leq CA^2 \int_{2^k<|h|<|B|^{-1/n}} ||b||_2^2 |B|^{\frac{2j}{n}} |h|^{2j-n-2t} dh \leq CA^2 ||b||_2^2 |B|^{\frac{2t}{n}} ,$$

which concludes the proof of (3.9).

Finally, in order to prove (3.10) we proceed as in the context of the spaces $H^p(I\!R^n)$ applying (2.14), (3.4) and the restriction on t.

Proof of theorem (1.3): Clearly it is sufficient to prove that there exists a constant C independent of m such that

$$||(m\hat{f})^{\vee}||_{\mathcal{M}_{\rho,\epsilon}} \leq CA^l ||f||_{H^{\rho,\bar{t}}} \tag{3.11}$$

for every $f \in H_w$, with $\bar{t}$ and ϵ as in the previous lemma.. Let $f \in H_w$ and assume that b $= \{b_j\}$ is a sequence of multiples of $(\rho, \bar{t})$ atoms such that $f = \sum_j b_j$ in $\mathcal{S}'$. Since $m \in L^\infty$, applying (1.1) it is clear that

$$(m\hat{f})^{\vee} = \sum_j (m\hat{b}_j)^{\vee} = \sum_j (m\widehat{\tau_{x_j} b_j})^{\vee}(\cdot - x_j) \qquad \text{in } \mathcal{S}'.$$

By (3.7) we have that $M_j = (m\widehat{\tau_{x_j} b_j})^{\vee}(\cdot - x_j)$ is a (ρ, ϵ) molecule centered at x_j which satisfies

$$||M_j||_2 \leq CA||b_j||_2 \quad \text{and} \tag{3.12}$$

$$||M_j| \cdot - x_j|^t||_2 \leq CA||b_j||_2 |B_j|^{t/n}. \tag{3.13}$$

Let α be a positive real constant. Following the notation of (2.15), by (3.12), we obtain

$$\sum_j |B_j| w \left(\frac{||M_j^{B_j}||_2 |B_j|^{-1/2}}{[\alpha\Lambda(\mathrm{b},\mathbf{B})]^{1/l}} \right) \leq \sum_j |B_j| w \left(\frac{CA||b_j||_2 |B_j|^{-1/2}}{[\alpha\Lambda(\mathrm{b},\mathbf{B})]^{1/l}} \right). \tag{3.14}$$

On the other hand, applying (3.13) and the fact that ρ is of upper type $1/l - 1$ we get

$$\sum_j |B_j| w \left(\frac{||M_j^{CB_j}||_2 |B_j|^{-1/2}}{[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}} \right) =$$

$$(3.15) \quad = \sum_j |B_j| w \left(\frac{||M_j \mathcal{X}_{CB_j} \rho(|\cdot - x_j|^n)|\cdot - x_j|^{n(1-1/l)}|\cdot - x_j|^t||_2 |B_j|^{-1/2}}{\rho(|B_j|)|B_j|^{\frac{t}{n}+1-\frac{1}{l}}[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}} \right)$$

$$\leq \sum_j |B_j| w \left(\frac{C||M_j|\cdot - x_j|^t||_2 |B_j|^{-1/2}}{|B_j|^{t/n}[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}} \right)$$

$$\leq \sum_j |B_j| w \left(\frac{CA||b_j||_2 |B_j|^{-1/2}}{[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}} \right).$$

Since we can assume that the constants C in (3.14) and (3.15) coincide, taking $\alpha = (CA)^l$ it follows that

$$||(m\hat{f})^{\vee}||_{\mathcal{M}_{\rho,\epsilon}} \leq \Lambda(\mathbf{M^B},\mathbf{B}) + \Lambda(\mathbf{M^{CB}},\mathbf{B}) \leq CA^l\Lambda(\mathbf{b},\mathbf{B})$$

where C is a constant independent of m, which proves (3.11)[2].◊

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