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A REMARK ON EULER'S CONSTANT

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ABSTRACT. Let x_0 be any real positive non-natural number which satisfies $\Gamma(x_0).k = \Gamma'(x_0)$ with k a rational number. We prove that either Euler's constant γ is trascendental or x_0 is irrational.

Define for $p, q \in N$,
$\alpha(p,q) := \sum_{i=1}^{\infty}\left(\frac{1}{qi} - \frac{1}{qi+p}\right)$ and $F(x) := \sum_{i=1}^{\infty}\left(\frac{x^{qi}}{qi} - \frac{x^{qi+p}}{qi+p}\right)$
Obviously $F(1) = \alpha(p,q)$ and $\frac{dF}{dx} = x^{q-1}\frac{(1-x^p)}{(1-x^q)}$. Thus $\alpha(p,q) = \int_0^1 x^{q-1}\frac{(1-x^p)}{(1-x^q)}dx$ and one obtains, for example, $\alpha(1,2) = 1 - ln2$, $\alpha(1,3) = 1 - \frac{ln3}{2} - \frac{\pi}{6\sqrt{3}}$, etc. Indeed one can compute $\alpha(p,q)$ in closed form with the following formula due to Gauss ([1] pg. 35):

$$\alpha(p,q) = -\frac{1}{2q}\pi cot(\frac{p}{q}\pi) - \frac{1}{q}ln(q) + \frac{S}{q} + \frac{1}{p} \tag{1}$$

where $S = \sum_{r=1}^{(q-1)/2} cos(2\pi rp/q)\ ln[4sin^2(\pi r/q)]$, $(q\ odd)$,
$S = \sum_{r=1}^{(q-2)/2} cos(2\pi rp/q)\ ln[4sin^2(\pi r/q)] + (-1)^p ln2$, $(q\ even)$.

Lemma 1. *$\alpha(p,q) - \frac{1}{p} \neq 0$ for $p, q \in N$, $0 < \frac{p}{q} < 1$.*

Proof. Suppose $\alpha(p,q) = \frac{1}{p}$. Then as $0 < p < q$, $\frac{1}{p} = \int_0^1 x^{q-1}\frac{(1-x^p)}{(1-x^q)}dx \leqslant \int_0^1 x^{q-1}dx = \frac{1}{q}$, a contradiction. ■

The following theorem, proved in 1966, is due to Baker (see [2] pg.11):

Baker's Theorem. *$e^{\beta_0}.\theta_1^{\beta_1}\dots\theta_n^{\beta_n}$ is trascendental for any non-zero algebraic numbers $\beta_0, \dots, \beta_n, \theta_1, \dots, \theta_n$.*

We use this theorem to prove the following result.

Theorem 1. *$\alpha(p,q)$ is trascendental for every pair $p, q \in N$, $\frac{p}{q}$ non-integer.*

Proof. There is no loss of generality if we assume p, q coprime. It is enough to prove the theorem for $0 < \frac{p}{q} < 1$, because $\alpha(p,q)$ and $\alpha(p+q.n,q)$, $n \in N$, differ by a rational number. Thus assume p, q are coprime and verify $0 < \frac{p}{q} < 1$. Moreover one can assume $\frac{p}{q} \neq \frac{1}{2}$ for $\alpha(1,2) = 1 - ln2$ and $ln2$ is trascendental by Lindemann's theorem ([2], pg. 6).

Recall that the set of algebraic numbers is a field. First observe that $sin(\frac{p}{q}\pi)$ and $cos(\frac{p}{q}\pi)$ are algebraic because $sin(\frac{1}{q}\pi)$ and $cos(\frac{1}{q}\pi)$ are algebraic, and this last assertion follows from De Moivre formula $e^{inx} = (cos\, x + isin\, x)^n$ with $x = \frac{1}{q}\pi$ and $n = q$.
Thus from (1) for one sees that $\alpha(p,q) = \pi.\zeta_0 + \sum_{j=1}^{n} \delta_j log(\zeta_j) + \frac{1}{p}$ with $\zeta_0, \ldots, \zeta_n$, $\delta_1, \ldots, \delta_n$ algebraic and non-zero.
Assume that $\alpha(p,q)$ is algebraic. Then $\beta_0 = \frac{\alpha(p,q)-1/p}{\zeta_0} i = i\pi + \sum_{j=1}^{n} \frac{i\delta_j log(\zeta_j)}{\zeta_0}$ is algebraic and non-zero by lemma 1. Therefore

$$e^{\beta_0}.\zeta_1^{-i(\frac{\delta_1}{\zeta_0})}\zeta_2^{-i(\frac{\delta_2}{\zeta_0})}\ldots\ldots\zeta_n^{-i(\frac{\delta_n}{\zeta_0})} = -1$$

which contradicts Baker's theorem. ■
The point $x_0 \in R$ stands for a non integer positive number which satisfies $\Gamma(x_0).k = \Gamma'(x_0)$ where k is a rational number. Then we have
Theorem 2. *Either x_0 is irrational or γ is trascendental.*
Proof. If x_0 is irrational then the theorem is true. Thus assume $x_0 = p/q$ is a positive rational non-integer number. Recall the well-known formula $\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+x}\right) - \frac{1}{x} = \frac{\Gamma'(x)}{\Gamma(x)} + \gamma$. Then, replacing x by x_0 in this formula we get $q\alpha(p,q) - q/p = k + \gamma$ and therefore γ is trascendental by theorem 1.■
NOTE: One such point x_0 could be the point where the minimun of $\Gamma(x)$ is attained.

REFERENCES

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