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Homogeneous (2,0)-geodesic Submanifolds of Euclidean Spaces

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Abstract

Under the hypothesis that an almost complex submanifold M^n of R^N is $(2,0)$-geodesic and homogeneous, a formula for the canonical covariant derivative of the second fundamental form of the submanifold is obtained. As a consequence of this formula, it is proved that if the submanifold is full then the first normal space coincides with the whole normal space. Other consequence is obtained under more restrictive conditions.

1 Introduction and main results

Let (M, g, J) be a connected Riemannian manifold with metric g and an almost complex structure J. We are not assuming, at least at this point, that the manifold is Hermitian i.e. $g(JX, JY) = g(X,Y)$. Let N be another Riemannian manifold and $\varphi : M \to N$ be an isometric immersion. As usual, we shall denote by α the second fundamental form of the immersion φ. Let $T(M)$ denote the tangent bundle of M and let $T^c(M)$ be its complexification. The almost complex structure J, extended to $T^c(M)$ induces a decomposition of this bundle into its eighenspaces

$$T^c(M) = T^c(M)^{(1,0)} \oplus T^c(M)^{(0,1)}$$

which in turn induces a decomposition of the complexified second fundamental form $\alpha^c$of the isometric immersion φ. This is of course defined as

$$\alpha^c(X_1 + iY_1, X_2 + iY_2) = \alpha(X_1, X_2) - \alpha(Y_1, Y_2) + i[\alpha(X_1, Y_2) + \alpha(Y_1, X_2)]$$

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and if $Z_1,\ Z_2 \in T^c(M)$ then we have, for $k = 1, 2$,

$$Z_k = \frac{1}{2}(Z_k - iJZ_k) + \frac{1}{2}(Z_k + iJZ_k) = Z_k^{(1,0)} + Z_k^{(0,1)}.$$

Then

$$\begin{aligned}\alpha^c(Z_1, Z_2) &= \alpha^c\left(Z_1^{(1,0)}, Z_2^{(1,0)}\right) + \alpha^c\left(Z_1^{(1,0)}, Z_2^{(0,1)}\right)\\ &\quad+\alpha^c\left(Z_1^{(0,1)}, Z_2^{(1,0)}\right) + \alpha^c\left(Z_1^{(0,1)}, Z_2^{(1,0)}\right).\end{aligned}$$

It is usual to define now

$$\begin{aligned}\alpha^{(2,0)}(Z_1, Z_2) &= \alpha^c\left(Z_1^{(1,0)}, Z_2^{(1,0)}\right)\\ \alpha^{(0,2)}(Z_1, Z_2) &= \alpha^c\left(Z_1^{(0,1)}, Z_2^{(1,0)}\right)\\ \alpha^{(1,1)}(Z_1, Z_2) &= \alpha^c\left(Z_1^{(1,0)}, Z_2^{(0,1)}\right) + \alpha^c\left(Z_1^{(0,1)}, Z_2^{(1,0)}\right).\end{aligned}$$

The isometric immersion φ is called (i,k)-geodesic if $\alpha^{(i,k)} \equiv 0$.
In the present paper we want to assume that φ is a $(2,0)$-geodesic.
Now recall that, due to the almost complex structure J_p, the tangent space $T_p(M)$ is a complex vector space which is isomorphic to the *holomorphic tangent space* $T^c(M)^{(1,0)}$ by the correspondence $X \mapsto \frac{1}{2}(X - iJ_pX)$. Then, for $X,\ Y \in T_p(M)$ we get, computing by definition

$$\begin{aligned}&\alpha^c(X - iJX, Y - iJY) =\\ &= \alpha(X, Y) - \alpha(JX, JY) + i[\alpha(X, JY) + \alpha(JX, Y)]\\ &= \alpha^c\left(X^{(1,0)}, Y^{(1,0)}\right) = \alpha^{(2,0)}(X, Y)\end{aligned}$$

Then the condition $\alpha^{(2,0)} \equiv 0$ is clearly amounts to

$$\begin{aligned}&(i) \quad \alpha(X, Y) - \alpha(JX, JY) = 0\\ &(ii) \quad \alpha(X, JY) + \alpha(JX, Y) = 0\end{aligned}$$

and it is clear that (i) and (ii) are equivalent. Then φ is $(2,0)$-geodesic if and only if

$$\alpha(X, Y) = \alpha(JX, JY) \qquad \forall X,\ Y \in T(M). \tag{1}$$

The objective of the present paper is to present the following two results concerning $(2,0)$-geodesic isometric immersions.

Theorem 1 *Let M be a compact homogeneous almost complex Riemannian manifold and $\varphi : M \rightarrow N$ an isometric $(2,0)$-geodesic immersion which is substantial or full (i.e. $\varphi(M)$ is not contained in any proper totally geodesic submanifold of N). Assume furthermore that the immersion φ has the property that its second fundamental form α satisfies Codazzi's equation ($\left(\overline{\nabla}_X \alpha\right)(Y,Z) = \left(\overline{\nabla}_Y \alpha\right)(X,Z)$ where $\overline{\nabla}\alpha$ denotes the usual covariant derivative of the second fundamental form α). Then, at each point, the first normal space of the immersion coincides with the whole normal space. i.e. the space generated by the image of the second fundamental form coincides with the normal space.*

If the Riemannian manifold N is R^n then any isometric immersion has the property that its second fundamental form satisfies Codazzi's equation. In a general Riemannian manifold N this may not be the case.

By a homogeneous almost complex Riemannian manifold we mean a Riemannian manifold M supporting a transitive action of a Lie group G of isometries. and having an invariant almost complex structure J which is *not necessarily compatible with the metric* (when this compatibility exists it is customary to say that the manifold is Hermithian).

Let us denote by $\langle .,. \rangle$ the Riemannian metric in the ambient manifold N. In general an isometry g of the group G does not extend to N but it follows easily from the above theorem, that the necessary and sufficient condition for the existence of these extensions is the invariance, by the group G, of the tensor $\Psi(X,Y,Z,W) = \langle \alpha(X,Y), \alpha(Z,W) \rangle$ (see for instance [7]).

The presence of the transitive action of the Lie group G on M yields the existence on M of a canonical affine connection (see [4] or [3]), usually denoted by ∇^c. The invariance of the metric induced by $\langle .,. \rangle$ on M, by the action of the group G, implies that $\nabla^c \langle .,. \rangle = 0$ i.e. the connection ∇^c is compatible with the metric on M.

Let ∇ denote the Riemannian connection on M associated to the metric and let $D(X,Y) = \nabla_X Y - \nabla^c_X Y$ be the difference tensor. Both, the tensor D and the almost complex structure J, are invariant by the action of the group G and hence $\nabla^c D = 0 = \nabla^c J$. Even when the connection ∇^c is compatible with the Riemannian metric it has, in general, non zero torsion and it is easy to see that it has the form $T(X,Y) = D(Y,X) - D(X,Y)$.

As in [6] and [2] we say that the canonical connection ∇^c satisfies **Axiom 6** (with respect to the immersion φ) if for each $p \in M$ and every $X,\ Y,\ Z \in$

$T_p(M)$ the second fundamental form of φ satisfies the identity

$$\alpha_p(T(X,Y),Z) = \alpha_p(Y, D(X,Z)) - \alpha_p(X, D(Y,Z)). \qquad (2)$$

There are plenty of compact manifolds M and isometric immersions $\varphi : M \to R^N$ such that M admits a canonical connection ∇^c which satisfies Axiom 6. In fact if M is an R-space (also called orbit of an s-representation or real flag manifold) and φ is its canonical imbedding then, for any of the possible canonical connections, Axiom 6 holds (see [6] or [2]).

The following consequence of the proof of Theorem 1 shows that in the case that $N = R^n$, the fact that Axiom 6 holds for a $(2,0)$-geodesic embedding φ of a compact homogeneous almost complex manifold, *implies that M is an R-space and in fact φ must be its canonical imbedding.*

Theorem 2 *Let M be a compact homogeneous almost complex Riemannian manifold and $\varphi : M \to R^n$ an isometric $(2,0)$-geodesic embedding which is substantial or full (i.e. $\varphi(M)$ is not contained in any proper totally geodesic submanifold of N). Assume that the canonical connection satisfies Axiom 6 with respect to φ. Then M is an R-space and φ is its canonical embedding.*

This result generalizes Theorem 4 in [1, p. 88].

The proof of these two results is contained in the next section.

2 Proof of the results.

Proof of Theorem 1.

Let $\varphi : M \to R^N$ be the $(2,0)$-geodesic isometric immersion and recall that in [5] a "canonical" covariant derivative of the second fundamental form was introduced by the formula

$$(\nabla^c_X \alpha)(Y,Z) = \nabla^\perp_X(\alpha(Y,Z)) - \alpha(\nabla^c_X Y, Z) - \alpha(Y, \nabla^c_X Z).$$

This covariant derivative is the key ingredient in the characterization of general R-spaces obtained in [4] (see also [2]).

By recalling the definition of the Riemannian covariant derivative of the second fundamental form we obtain immediately

$$(\nabla^c_X \alpha)(Y,Z) = \left(\overline{\nabla}_X \alpha\right)(Y,Z) + \alpha(D(X,Y),Z) + \alpha(Y, D(X,Z)). \qquad (3)$$

Since the second fundamental form α of the immersion φ satisfies Codazzi's equation, by interchanging the letters X and Y and substracting we get

$$\begin{aligned}&(\nabla^c_Y\alpha)(X,Z)-(\nabla^c_X\alpha)(Y,Z)\\=\;&\alpha(T(X,Y),Z)-[\alpha(Y,D(X,Z))-\alpha(X,D(Y,Z))]\,.\end{aligned}\tag{4}$$

This formula replaces Codazzi's equation for the canonical covariant derivative of the second fundamental form α.

Now since our immersion φ is an isometric $(2,0)$-geodesic immersion we have by the condition (1)

$$\alpha(JX,Y)=-\alpha(X,JY)\,.\tag{5}$$

and this yields very easily

$$(\nabla^c_Y\alpha)(JX,Z)=-(\nabla^c_Y\alpha)(X,JZ)\tag{6}$$

Now starting with the identity (4) we write

$$\begin{aligned}&(\nabla^c_Y\alpha)(X,Z)=\\&=(\nabla^c_X\alpha)(Y,Z)+\alpha(T(X,Y),Z)-\alpha(Y,D(X,Z))+\alpha(X,D(Y,Z))\\&=-(\nabla^c_X\alpha)(Y,J^2Z)+\alpha(T(X,Y),Z)-\alpha(Y,D(X,Z))+\alpha(X,D(Y,Z))\end{aligned}$$

because $J^2=-I$.

Then by (6)

$$\begin{aligned}&(\nabla^c_Y\alpha)(X,Z)=\\&(\nabla^c_X\alpha)(JY,JZ)+\alpha(T(X,Y),Z)-\alpha(Y,D(X,Z))+\alpha(X,D(Y,Z))\,.\end{aligned}$$

Now we may change $(\nabla^c_X\alpha)(JY,JZ)$ using again (4) and then the last equation becomes

$$\begin{aligned}&(\nabla^c_Y\alpha)(X,Z)=\\&=(\nabla^c_{JY}\alpha)(X,JZ)+\alpha(T(X,Y),Z)-\alpha(Y,D(X,Z))+\alpha(X,D(Y,Z))\\&+\alpha(T(JY,X),JZ)-\alpha(X,D(JY,JZ))+\alpha(JY,D(X,JZ))\,.\end{aligned}$$

By using (6) now we get

$$\begin{aligned}&(\nabla^c_Y\alpha)(X,Z)=\\&=-(\nabla^c_{JY}\alpha)(JX,Z)+\alpha(T(X,Y),Z)-\alpha(Y,D(X,Z))+\alpha(X,D(Y,Z))\\&+\alpha(T(JY,X),JZ)-\alpha(X,D(JY,JZ))+\alpha(JY,D(X,JZ))\end{aligned}$$

and (4) again yields

$$(\nabla^c_Y\alpha)\,(X,Z) =$$
$$= -\,(\nabla^c_Z\alpha)\,(JX,JY) + \alpha\,(T(X,Y),Z) - \alpha\,(Y,D(X,Z)) + \alpha\,(X,D(Y,Z))$$
$$+\alpha\,(T(JY,X),JZ) - \alpha\,(X,D(JY,JZ)) + \alpha\,(JY,D(X,JZ))$$
$$-\alpha\,(T(Z,JX),JY) + \alpha\,(JX,D(Z,JY)) - \alpha\,(Z,D(JX,JY))\,.$$

Once more (6) implies

$$(\nabla^c_Y\alpha)\,(X,Z) =$$
$$= (\nabla^c_Z\alpha)\,(X,J^2Y) + \alpha\,(T(X,Y),Z) - \alpha\,(Y,D(X,Z)) + \alpha\,(X,D(Y,Z))$$
$$+\alpha\,(T(JY,X),JZ) - \alpha\,(X,D(JY,JZ)) + \alpha\,(JY,D(X,JZ))$$
$$-\alpha\,(T(Z,JX),JY) + \alpha\,(JX,D(Z,JY)) - \alpha\,(Z,D(JX,JY))\,.$$

Using again the identity $J^2 = -I$ we have

$$(\nabla^c_Y\alpha)\,(X,Z) =$$
$$= -\,(\nabla^c_Z\alpha)\,(X,Y) + \alpha\,(T(X,Y),Z) - \alpha\,(Y,D(X,Z)) + \alpha\,(X,D(Y,Z))$$
$$+\alpha\,(T(JY,X),JZ) - \alpha\,(X,D(JY,JZ)) + \alpha\,(JY,D(X,JZ))$$
$$-\alpha\,(T(Z,JX),JY) + \alpha\,(JX,D(Z,JY)) + \alpha\,(Z,D(JX,JY))$$

and (4) once again yields

$$(\nabla^c_Y\alpha)\,(X,Z) =$$
$$= -\,(\nabla^c_Y\alpha)\,(X,Z) + \alpha\,(T(X,Y),Z) - \alpha\,(Y,D(X,Z)) + \alpha\,(X,D(Y,Z))$$
$$+\alpha\,(T(JY,X),JZ) - \alpha\,(X,D(JY,JZ)) + \alpha\,(JY,D(X,JZ))$$
$$-\alpha\,(T(Z,JX),JY) + \alpha\,(JX,D(Z,JY)) - \alpha\,(Z,D(JX,JY))$$
$$-\alpha\,(T(Y,X),Z) + \alpha\,(X,D(Y,Z)) - \alpha\,(Y,D(X,Z))\,;$$

which obviously becomes

$$\begin{aligned} &2\,(\nabla^c_Y\alpha)\,(X,Z) = \\ &= 2\alpha\,(T(X,Y),Z) - 2\alpha\,(Y,D(X,Z)) + 2\alpha\,(X,D(Y,Z)) \\ &+\alpha\,(T(JY,X),JZ) - \alpha\,(X,D(JY,JZ)) + \alpha\,(JY,D(X,JZ)) \\ &-\alpha\,(T(Z,JX),JY) + \alpha\,(JX,D(Z,JY)) - \alpha\,(Z,D(JX,JY))\,. \end{aligned} \tag{7}$$

In the particular case in which $X = Y = Z$ (7) reduces to

$$\begin{aligned} &(\nabla^c_X\alpha)\,(X,X) = \\ &= \alpha\,(T(JX,X),JX) - \alpha\,(X,D(JX,JX)) + \alpha\,(JX,D(X,JX))\,. \end{aligned} \tag{8}$$

By considering formulas (7) and (3) we see immediately that the covariant derivative of the second fundamental form can be written as a linear combination of elements of the first normal space (which, by definition, is the subspace of the normal space generated by the image of the second fundamental form). This clearly implies that the first normal space coincides with the normal space of our immersion φ. and completes the proof of Theorem 1. ■

Proof of Theorem 2.

It follows immediately from Axiom 6 (formula (2)) and formula (8) that, for each point $p \in M$ and each $X \in T_p(M)$,

$$(\nabla^c_X \alpha)(X, X) = 0. \tag{9}$$

Furthermore it follows from (4) that the canonical covariant derivative of the second fundamental form satisfies the identity

$$(\nabla^c_Y \alpha)(X, Z) = (\nabla^c_X \alpha)(Y, Z)$$

for each point $p \in M$ and each $X, Y, Z \in T_p(M)$. This easily implies that the canonical covariant derivative of the second fundamental form vanishes identically on M and since φ is an isometric embedding into a Euclidean space, it follows from [4] that M is an R-space and φ is its canonical embedding. This completes the proof of Theorem 2. ■

References

[1] Ferus, D.: Symmetric submanifolds of Euclidean spaces, *Math. Ann.* **247** (1980), 81-93.

[2] Hulett, E. and Sánchez, C. U.: An algebraic characterization of R-Spaces, *Geometriae Dedicata* 67: 349-365, 1997.

[3] Kowalaski, O.: *Generalized Symmetric Spaces*, Lecture Notes in Math **805**, Springer Verlag, New York 1980.

[4] Olmos, C. and Sánchez, C. U.: A geometric characterization of the orbits of s-representations *J. Reine Angew. Math.* **420** (1991), 195-202.

[5] Sánchez, C. U.: A characterization of extrinsik k-symmetric submanifolds of R^N. Rev. Unión Mat. Argentina. **38** (1992), 1-15.

[6] Sánchez, C. U., Dal Lago, W., García A. and Hulett, E.: On some properties which characterize symmetric and general R-spaces *Differential Geometry and its Applications* 7 (1997), 291-302.

[7] Spivak, M.: *A Comprehensive Introduction to Differential Geometry,* Publish or Perish 1979

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