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TWO-WEIGHT INEQUALITIES FOR CERTAIN MAXIMAL FRACTIONAL OPERATORS ON SPACES OF HOMOGENEOUS TYPE

by

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Abstract: The study of mean oscillation properties for the fractional integral is naturally conneted with the study of boundedness properties for the composition of the sharp maximal function with the fractional integral. Here, an operator that generalizes that composition on spaces of homogeneous type is considered. Sufficient conditions on pairs of weights are given for which strong and weak weighted inequalities hold for that operator. The work includes a study about the necessity of the conditions on the weights.

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§1. Introduction and statements of the main results

The spaces of homogeneous type were introduced by R. Coifman and G. Weiss in [CW] and they were studied and used by several authors (see [AM], [BS], [C], [MS1], [MS2], [MT], [SW], [W]). Let us recall some definitions and properties relative to them.

Let X be a set. A function $d : X \times X \to I\!R_0^+$ is called a quasi-distance on X if the following conditions are satisfied:

(1.1) for every x and y in X, $d(x,y) = 0$ if and only if $x = y$,
(1.2) for every x and y in X, $d(x,y) = d(y,x)$ and
(1.3) there exists a constant K such that

$$d(x,y) \leq K(d(x,z) + d(z,y))$$

holds for every x, y and z in X.

The subsets $\{(x,y) : d(x,y) < \varepsilon\}$ of $X \times X$ define a base of metrizable uniform structure on X. Moreover, from this fact, it can be proved that always it is possible to find a distance δ, defined on X, and a number $\alpha \geq 1$ such that d is equivalent to δ^α, i.e.: there exist two constants, D_1 and D_2, such that

$$D_1 \delta(x,y)^\alpha \leq d(x,y) \leq D_2 \delta(x,y)^\alpha \tag{1.4}$$

holds for every x and y in X (see [MS2]).

Let μ be a positive measure on a σ-algebra of subsets of X which contains the balls $B(x,r) = \{y : d(x,y) < r\}$, for every x in X and every finite positive r. We assume that μ satisfy a doubling condition, that is, there exists a constant D such that

$$0 < \mu(B(x,2r)) \leq D\mu(B(x,r)) < \infty \tag{1.5}$$

holds for every ball B in X.

A structure (X,d,μ), with d and μ as above, is called a space of homogeneous type. By keeping in mind (1.4), we can assume (replacing d by δ^α, if it would be necessary) that d is a continuous quasi-distance.

A space of homogeneous type (X,d,μ) is named normal if there exist four constants A_1, A_2, K_1 and $K_2, A_1 \leq A_2$, $K_2 \leq 1 \leq K_1$, such that

(1.6) $A_1 r \leq \mu(B(x,r))$ if $r \leq K_1\mu(X)$,
(1.7) $B(x,r) = X$ if $r > K_1\mu(X)$,
(1.8) $A_2 r \geq \mu(B(x,r))$ if $r \geq K_2\mu(\{x\})$, and
(1.9) $B(x,r) = \{x\}$ if $r < K_2\mu(\{x\})$,

holds for every x in X and $r > 0$.

Let w be a positive and locally integrable function defined on a space of homogeneous type (X, d, μ). We denote by $w(E)$ the measure with density w with respect to the measure μ, i.e.: $w(E) = \int_E w\, d\mu$. The density w will be called a weight with respect to μ. We shall say that a pair of weights (u, v) belongs to the class $A(p, q), 1 \leq p \leq \infty$ and $1 \leq q < \infty$, if there exists a constant C such that

$$\left(\frac{u^q(B)}{\mu(B)}\right)^{\frac{1}{q}} \left(\frac{v^{-p'}(B)}{\mu(B)}\right)^{\frac{1}{p'}} \leq C \tag{1.10}$$

holds for every ball B in X, where $p' = p/(p-1)$. In particular, if $q = \infty$ the condition $(u, v) \in A(p, \infty)$ becomes

$$(\operatorname*{ess\,sup}_B u) \left(\frac{v^{-p'}(B)}{\mu(B)}\right)^{\frac{1}{p'}} \leq C \tag{1.11}$$

for every ball B.

Let (X, d, μ) be a space of homogeneous type. It is not difficult to see that the function $\rho : X \times X \to \mathbb{R}_0^+$ defined as

$$\rho(x,y) = \begin{cases} (\mu(B(x, d(x,y))) + \mu(B(y, d(x,y))))/2 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \tag{1.12}$$

satisfies (1.1), (1.2) and (1.3). If there exists $\alpha \geq 1$ such that $\rho^{\frac{1}{\alpha}}$ is a distance we define $\delta = \rho^{\frac{1}{\alpha}}$. In the case that such α does not exist, we reason as before to obtain δ and α such that (1.4) holds. With this choice of δ and α, we introduce, for each γ in $(0, 1)$, the function

$$K_\gamma(x,z,y) = \begin{cases} \delta(x,y)^{\alpha(\gamma-1)} - \delta(z,y)^{\alpha(\gamma-1)} & \text{if } x \neq y \text{ and } z \neq y \\ \mu(\{x\})^{\gamma-1} - \delta(z,y)^{\alpha(\gamma-1)} & \text{if } x = y \text{ and } z \neq y \\ \delta(x,y)^{\alpha(\gamma-1)} - \mu(\{z\})^{\gamma-1} & \text{if } x \neq y \text{ and } z = y \\ 0 & \text{if } x = y = z \end{cases} \tag{1.13}$$

for x, z and y in X. Now, for each γ and each $s \geq 1$, we define the following operator

$$T_\gamma^s f(x) = \sup_{x \in B} \left(\frac{1}{\mu(B)^2} \int_B \int_B \left| \int_X K_\gamma(x,z,y) f(y) d\mu(y) \right|^s d\mu(z)\, d\mu(x) \right)^{\frac{1}{s}} \tag{1.14}$$

for every $x \in X$ and every measurable function f defined on X, where the sup is taken over all the balls B in X containing x.

In the euclidean case (i.e.:$X = \mathbb{R}^n$ with the usual distance and the Lebesgue measure), this operator appears naturally connected to the study of mean oscillation properties for the fractional integral I_γ, defined as

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n(1-\gamma)}}\, dy,$$

for $0 < \gamma < n$. In fact, this study involves, formally, the study of the behaviour of

$$\left(\sup_B \frac{1}{|B|^2} \int_B \int_B |I_\gamma f(x) - I_\gamma f(y)|^s \, dx\, dy \right)^{\frac{1}{s}} \tag{1.15}$$

where the sup is taken over all balls B in $\mathbb{R}^n$, for $1 \le s < \infty$ (see, for instance, [MW], p. 269). A more correct mathematical formulation (in order to avoid some problems relative to the convergence of $I_\gamma f$) implies to replace $I_\gamma f(x) - I_\gamma f(y)$ by

$$\int_{\mathbb{R}^n} \left(\frac{1}{|x-z|^{n(1-\gamma)}} - \frac{1}{|y-z|^{n(1-\gamma)}} \right) f(z)\, dz$$

in (1.15). But the kernel between parenthesis in the above integral coincides with the euclidean case of (1.13) for $\alpha = n$.

In a general space of homogeneous type (X, d, μ), an extension of the fractional integral can be defined as

$$I_\gamma f(x) = \int_X Q_\gamma(x,y) f(y)\, d\mu(y) \tag{1.16}$$

with

$$Q_\gamma(x,y) = \begin{cases} \delta(x,y)^{\alpha(\gamma-1)} & \text{if } x \neq y \\ \mu(\{x\})^{\gamma-1} & \text{if } x = y \end{cases}$$

for $0 < \gamma < 1$, where δ and α are as in (1.13). So, obviously, the operator associated to the corresponding version of (1.15) is exactly our operator T^s_γ.

The operator T^s_γ was first considered by E. Harboure, R. Macías and C. Segovia, in the euclidean case, in [HMS2], in order to get the boundedness of the fractional operator I_γ from weighted $L^{\frac{n}{\alpha}}$ into weighted BMO. From this result, the authors, as an application of a theorem of extrapolation, proved weigted L^p-norm inequalities for T^s_γ. The purpose of this work is to extend those results to the general setting of spaces of homogeneous type. Our first main result is the following theorem.

(1.17) Theorem: *Let (X, d, μ) be a space of homogeneous type and let $0 < \gamma < 1$. If $(a, b) \in A(1/\gamma, \infty)$, then, for each s in $[1, 1/(1-\gamma))$, there exists a constant C, independent of f, such that*

$$ess \sup_{x \in X} \left(a(x) T^s_\gamma f(x) \right) \le C \left(\int_X \left(|f(x)|\, b(x) \right)^{\frac{1}{\gamma}} d\mu(x) \right)^\gamma, \tag{1.18}$$

for every measurable function f.

The techniques that we are going to use in order to prove the above theorem are extensions of those used in [HMS2] for the euclidean case. In particular, we will need to know that the left hand side of (1.18) behaves like

$$(1.19)\quad \sup_B \left(ess \sup_{x\in B} \ a(x) \left(\frac{1}{\mu(B)^2} \int_B \int_B \left| \int_X K_\gamma(x,z,y) f(y)\, d\mu(y) \right|^s d\mu(z)\, d\mu(x) \right)^{\frac{1}{s}} \right)$$

where the sup is taken over all the balls B in X. The proof of this fact follows a similar reasoning, with obvious changes, to that given for the euclidean case (see [HMS1]) and it is omitted here. With this result, Theorem (1.15) can be considered as a result on boundedness of fractional integrals on spaces of homogeneous type. Moreover it is easy to see that Theorem 7, p. 269, in [MW] of B. Muckenhoupt and R. Wheeden, for fractional integral operators in $I\!R^n$, is the euclidean case for $s = 1$ of (1.17). Actually, the techniques in [HMS2] for the euclidean case of (1.17) have been taken from [MW].

Now, we state an extrapolation theorem which will allow us to derive further results about T^s_γ from (1.17). Let us first introduce some notation. For (X, d, μ) be a space of homogeneous type, we denote by $\mathcal{M}$ the set of measurable functions defined on X, and by $\mathcal{M}_0$ the subset of bounded functions. Now, we state the theorem

(1.20) Theorem: *Let T be an operator defined on $\mathcal{M}_0$ with values in $\mathcal{M}$. Let us assume that T satisfies*

(1.21) $|T(\lambda f)| = |\lambda||Tf|$ *and* $|T(f+g)| \le |Tf| + |Tg|$ *for every scalar* λ *and every* f *and* g *in* $\mathcal{M}_0$

(1.22) *for a fix pair of numbers r and $\beta, 1 \le r < \beta \le \infty$, and for every pair of such that (a^r, b^r) in $A(\beta/r, \infty)$ the operator T satisfies*

$$ess \sup_{x\in X} \ (a(x)|Tf(x)|) \le C \left(\int_X (|f(x)|b(x))^\beta \, d\mu(x) \right)^{\frac{1}{\beta}},$$

for any f in $\mathcal{M}_0$, and where C is a finite constant independent of f (for $\beta = \infty$, the left member of the above inequality becomes $ess \sup_{x\in X}(b(x)|f(x)|)$)

Then, for any p, $r < p < \beta$, $1/q = 1/p - 1/\beta$ and $(u^r, v^r) \in A(p/r, q/r)$, there exists a constant C, independent of f, such that

$$u^q(\{x \in X : |T(f(x)| > \lambda\}) \le C \left(\lambda^{-p} \int_X (|f(x)|v(x))^p \, d\mu(x) \right)^{\frac{q}{p}}$$

holds for every $\lambda > 0$.

This theorem can be proved using an argument similar to that of the euclidean case, with only minor modifications. For the euclidean case, see [HMS2].

Now, from theorems(1.17) and (1.20), we easily obtain

(1.23) **Theorem:** *Let (X, d, μ) be a space of homogeneous type and let $0 < \gamma < 1$, $1 < p < 1/\gamma$ and $1/q = 1/p - \gamma$. If $(u, v) \in A(p, q)$, then, for each s in $[1, 1/(1-\gamma))$, there exists a constant C, independent of f, such that*

$$(1.24) \qquad u^q\left(\{x \in X : T^s_\gamma f(x) > \lambda\}\right) \leq C\left(\lambda^{-p}\int_X (|f(x)|v(x))^p d\mu(x)\right)^{\frac{q}{p}}$$

holds for every $\lambda > 0$ and every measurable function f.

Note that in theorems (1.17) and (1.23) we only give sufficient conditions on the weights to ensure that (1.18) and (1.24) hold. One can wonder whether or not they are also neccesary. The answer in both cases is negative, as we can see from following example.

(1.25) *Example*: Let $X = \{0, 1\}$, $d(x, y) = |x - y|$ and μ be the measure defined as $\mu(\{0\}) = \mu(\{1\}) = 1$. It is clear that (X, d, μ) is a space of homogeneous type. On the other hand, it is obvious that ρ, defined as in (1.12), gives that $\rho(x, y) = 1$, if $x \neq y$, and $\rho(x, y) = 0$, if $x = y$, so it is a distance. Therefore, we have $\delta = \rho$ in (1.17), which implies $K_\gamma \equiv 0$, and, as a consequence, $T^s_\gamma f \equiv 0$ for every function f. Now it is evident that (1.18) and (1.24) hold for every pair of weights, in particular, we can take $a \equiv u \equiv 1$ and $b \equiv v \equiv 0$. Since that pair is not in $A(p, q)$ for every p and q, with $1 \leq p \leq \infty$ and $1 \leq q < \infty$, we have, as we said, that the condition on the weights is not neccessary in (1.17) neither in (1.23).

The above example proves that the reverse implications for (1.15) and (1.21) do not hold in the general case. However, in a more restrictive class of spaces, the normal spaces, we can obtain a result very close to that. In fact, we have

(1.26) **Theorem:** *Let (X, d, μ) be a normal space of homogeneous type. There exist a constant K_0, only depending on the constants of the space, such that*

(1.27) *if* (1.18) *holds, then the pair (a, b) satisfies*

$$(\text{ess}\sup_B a)\left(\frac{1}{\mu(B)}\int_B b^{-\frac{1}{1-\gamma}}\right)^{1-\gamma} \leq C$$

for every ball B with finite radius less than or equal to $K_1 K_0^{-1}\mu(X)$.

(1.28) *if* (1.24) *holds, then the pair* (u,v) *satisfies*

$$\left(\frac{1}{\mu(B)}\int_B u^q\right)^{\frac{1}{q}}\left(\frac{1}{\mu(B)}\int_B v^{-p'}\right)^{\frac{1}{p'}} \leq C$$

for every ball B *with finite radius less or equal than* $K_1K_0^{-1}\mu(X)$.

In each occurrance, K_1 *is the constant of (1.6) and* C *depends only on the constants of the space and the constant involved in the assumptions.*

From the above theorem, it follows clearly that the reverse implications of (1.17) and (1.23) hold whenever (X,d,μ) is a normal space with $\mu(X)=\infty$. But, when $\mu(X)<\infty$, a result like (1.26) is the best that one can expect without further assumptions. Example (1.25) can be used again to see this. In fact, it is very easy to check that the space (X,d,μ) involved is normal, with constants $K_1 = K_2 = 1$ and $A_2 = A_1^{-1} = 2$. As (1.26) and the example suggest, for $\mu(X)<\infty$, the difficulty to get the necessity of the conditions on the weights relies on the existence of points with too large measure. Actually, we can prove

(1.29) **Corollary:** *Let* (X,d,μ) *be as in (1.24) and such that* $\mu(\{x\}) \leq 2KK_1K_2^{-1}K_0^{-1}\mu(X)$ *for every* x *in* X, *where* K_0 *is the same constant of the theorem, and* K, K_1 *and* K_2 *are the constants of (1.3), (1.6) and (1.8), respectively. With these assumptions we get*

(1.30) *if* (1.18) *holds, then* $(a,b)\in A(1/\gamma,\infty)$;

(1.31) *if* (1.22) *holds, then* $(u,v)\in A(p,q)$.

The proofs of (1.17), (1.26) and (1.29) are in the next section.

§2. Proofs

The proof of theorem (1.17) requires the following result concerning a weak type inequality for the operator I_γ, defined in (1.16). This result extends a well known property for the usual fractional integral in $\mathbb{R}^n$.

(2.1) **Lemma:** *Let* $0<\gamma<1$. *The operator* I_γ *is of weak type* $(1,(1-\gamma)^{-1})$, *i.e.: there exists a constant* C, *such that*

$$\mu(\{x\in X : |I_\gamma f(x)|>\lambda\}) \leq C\left(\frac{1}{\lambda}\int_X |f|\,d\mu\right)^{\frac{1}{1-\gamma}}, \tag{2.2}$$

holds for every $\lambda>0$ *and every* f *in* $L^1(X,d\mu)$.

Proof. It is clear, from the definition of Q_γ, that we only need to prove (2.2) for I_γ defined using the kernel $K_\gamma(x,y) = \mu(\bar{B}(x,d(x,y)))^{\gamma-1}$, where $\bar{B}(x,r)$ denotes the set $\{y \in X : d(x,y) \le r\}$, instead of Q_γ. Let $R > 0$. We define

$$I_\gamma^i f(x) = \int_X K_\gamma^i(x,y) f(y) d\mu(y) \quad , i = 1,2 \ ,$$

where $K_\gamma^1(x,y) = K_\gamma(x,y)\chi_{\{(x,y):\mu(\bar{B}(x,d(x,y)))<R\}}$ and $K_\gamma^2(x,y) = K_\gamma(x,y) - K_\gamma^1(x,y)$. Now, let $y \in X$. Let us consider the sets $\Omega_j = \{x \in X : \mu(\bar{B}(y,d(x,y))) < 2^{-j-1}R\}$ for $j = 0,1,\ldots.$ By defining $R_j = \sup\{d(y,x) : \mu(\bar{B}(y,d(y,x))) < 2^{-j-1}R\}$, where the sup is taken over all $x \in X$, it can be proved that $\Omega_{j+1} \subset \bar{B}(y,R_j)$ and $\mu(\bar{B}(y,R_j)) \le C2^{-j}R$ (see [MT], Lemma (2.5), p. 9). Then, we get

$$\int_X K_\gamma^1(x,y)\, d\mu(x) \le C \sum_{j=0}^{\infty} \int_{\Omega_{j+1}-\Omega_j} K_\gamma^1(y,x) d\mu(x)$$

$$\le C \sum_{j=0}^{\infty} \left(\frac{2^j}{R}\right)^{1-\gamma} \mu(\Omega_{j+1})$$

$$\le CR^\gamma \sum_{j=0}^{\infty} 2^{-j\gamma} = CR^\gamma$$

The above inequality and Tonelli's theorem allow us to obtain

$$\int_X |I_\gamma^1 f(x)|\, d\mu(x) \le \int_X \int_X K_\gamma^1(x,y) |f(y)|\, d\mu(y)\, d\mu(x)$$

$$= \int_X |f(y)| \left(\int_X K_\gamma^1(x,y)\, d\mu(x)\right) d\mu(y) \qquad (2.3)$$

$$\le CR^\gamma \int_X |f(y)|\, \mu(y).$$

On the other hand, we have

$$\int_X |I_\gamma^2 f(x)|\, d\mu(x) \le R^{\gamma-1} \int_X |f(y)|\, d\mu(y). \qquad (2.4)$$

Finaly, given $\lambda > 0$, (2.2) follows from the obvious inequality

$$\mu(\{x \in X : |I_\gamma f(x)| > \lambda\}) \le \mu(\{x \in X : |I_\gamma^1 f(x)| > \frac{\lambda}{2}\}) + \mu(\{x \in X : |I_\gamma^2 f(x)| > \frac{\lambda}{2}\})$$

and the estimates (2.3) and (2.4) with $R = \lambda^{\frac{1}{\gamma-1}}$.■

Proof of Theorem (1.17). As we said after the statement of the theorem in § 1, the left member of (1.18) is equivalent to (1.19). Hence we only need to prove that (1.19) is bounded by the right member of (1.18). To do that we first notice that the expression between the inner parenthesis in (1.19) is bounded by the sum of the following terms

$$\left(\frac{1}{\mu(B)^2}\int_B\int_B\left|\int_{\tilde{B}} K_\gamma(x,z,y)f(y)d\mu(y)\right|^s d\mu(z)d\mu(x)\right)^{\frac{1}{s}} \tag{2.5}$$

and

$$\left(\frac{1}{\mu(B)^2}\int_B\int_B\left|\int_{X-\tilde{B}} K_\gamma(x,z,y)f(y)d\mu(y)\right|^s d\mu(z)d\mu(x)\right)^{\frac{1}{s}} \tag{2.6}$$

Here we denote with $\tilde{B}$ to the ball with same center that B and radius equal to $2K$ times the radius of B, where K is the constant of (1.3).

Now, we consider the extension of the fractional integral operator I_γ defined in (1.16). From lemma (2.1) and Kolmogorov's and Hölder's inequalities, we have that (2.5) is bounded by

$$\begin{aligned} 2\left(\frac{1}{\mu(B)}\int_B |I_\gamma(f\chi_{\tilde{B}})|^s\, d\mu\right)^{\frac{1}{s}} &\leq C\frac{1}{\mu(\tilde{B})^{1-\gamma}}\int_{\tilde{B}}|f|\,d\mu \\ &\leq C\left(\frac{1}{\mu(\tilde{B})}\int_{\tilde{B}} b^{\frac{-1}{1-\gamma}}d\mu\right)^{1-\gamma}\left(\int_X (|f|b)^{\frac{1}{\gamma}}\,d\mu\right)^{\gamma} \end{aligned} \tag{2.7}$$

for each s in $[1\,,(1-\gamma)^{-1})$, where C is independent of f and B. On the other hand, from the definitions of K_γ and δ and applying the mean value theorem, we have that

$$\begin{aligned} |K_\gamma(x,y,z)| &= \left|\delta(x,y)^{\alpha(\gamma-1)} - \delta(z,y)^{\alpha(\gamma-1)}\right| \\ &\leq C\frac{\left|\delta(x,y)^{\alpha(1-\gamma)} - \delta(z,y)^{\alpha(1-\gamma)}\right|}{\mu(B(x_0,d(x_0,y)))^{2(1-\gamma)}} \\ &\leq C\frac{\delta(x,z)}{\mu(B(x_0,d(x_0,y)))^{1-\gamma+\frac{1}{\alpha}}} \\ &\leq C\frac{\mu(B)^{\frac{1}{\alpha}}}{\mu(B(x_0,d(x_0,y)))^{1-\gamma+\frac{1}{\alpha}}} \end{aligned}$$

holds for every x, z in B and every y in $X - \tilde{B}$, where x_0 is the center of B and C is independent of B. Therefore, (2.6) is bounded by

(2.8)

$$C\mu(B)^{\frac{1}{\alpha}} \int_{X-\tilde{B}} \frac{|f(y)|}{\mu(B(x_0, d(x_0,y)))^{1-\gamma+\frac{1}{\alpha}}} d\mu(y)$$

$$\leq C\mu(B)^{\frac{1}{\alpha}} \left(\int_X (|f(y)|\, b(y))^{\frac{1}{\gamma}}\, d\mu(y)\right)^{\gamma} \left(\int_{X-\tilde{B}} \frac{b(y)^{-\frac{1}{1-\gamma}}}{\mu(B(x_0,d(x_0,y)))^{1+\frac{1}{\alpha(1-\gamma)}}} d\mu(y)\right)^{1-\gamma}$$

Note that $\mu(B(x_0, d(x_0, y))) \geq R_0$, with $R_0 = C\mu(B)$ for every y in $X - \tilde{B}$, where C is independent of B. Now, let $\Omega_j = \left\{y \in X : \mu\left(\overline{B}(x_0, d(x_0, y))\right) < 2^j R_0\right\}$ for $j = 0, 1, \ldots$, where the sets $\overline{B}(x, r)$ are defined as in the proof of Lemma (2.1). By using Lemma (2.5) of [MT], as in the above Lemma, we get $\Omega_{j+1} \subset \bar{B}(x_0, R_j)$ and $\mu(\bar{B}(x_0, R_j)) \leq C2^{j+1}R_0$, where C only depends on the constants of the space and $R_j = \sup_{y \in X}\{d(x_0, y) : \mu(\bar{B}(x_0, d(x_0, y))) \leq 2^{j+1}R_0\}$. Then, we can bound the integral over $X - \tilde{B}$ in the right member of (2.8) by

$$C \int_{X-\Omega_0} \frac{b(y)^{-\frac{1}{1-\gamma}}}{\mu\left(\bar{B}(x_0, d(x_0,y))\right)^{1+\frac{1}{\alpha(1-\gamma)}}} d\mu(y)$$

$$= C\sum_{j=0}^{\infty} \int_{\Omega_{j+1}-\Omega_j} \frac{b(y)^{-\frac{1}{1-\gamma}}}{\mu\left(\bar{B}(x_0, d(x_0,y))\right)^{1+\frac{1}{\alpha(1-\gamma)}}} d\mu(y)$$

$$\leq C\sum_{j=0}^{\infty} (2^j R_0)^{-1-\frac{1}{\alpha(1-\gamma)}} \int_{\Omega_{j+1}} b(y)^{-\frac{1}{1-\gamma}}\, d\mu(y)$$

$$\leq C\sum_{j=0}^{\infty} (2^j R_0)^{-\frac{1}{\alpha(1-\gamma)}} \frac{1}{\mu(B(x_0, 2R_j))} \int_{B(x_0,2R_j)} b(y)^{-\frac{1}{1-\gamma}}\, d\mu(y).$$

Now, combining this inequality with (2.8), we get that (2.6) is bounded by a constant times the following expresion

$$\left(\sum_{j=0}^{\infty} \frac{2^{-\frac{j}{\alpha(1-\gamma)}}}{\mu\left(B(x_0, 2R_j)\right)} \int_{B(x_0,2R_j)} b(y)^{-\frac{1}{1-\gamma}}\, d\mu(y)\right)^{1-\gamma} \left(\int_X (|f(y)|\, b(y))^{\frac{1}{\gamma}}\, d\mu(y)\right)^{\gamma}$$

Finally, since $B \subset \bar{B}(x_0, R_j)$ for every $j \geq 0$, from the hypothesis on (a, b) we obtain the wished boundedness for (1.19) from the above result and the estimate (2.7) of (2.5). This concludes the proof.∎

In order to prove Theorem (1.26) we need the next lemma.

(2.9) **Lemma:** *Let (X, d, μ) be a normal space of homogeneous type. For each γ in $(0,1)$ and each $s \geq 1$, there exist two constants, K_0 and C, depending only on s, γ and the constants of the space such that for every ball $B = B(x_0, R)$ satisfying $K_2(2K)^{-1}\mu(\{x_0\}) \leq R \leq K_1 K_0^{-1}\mu(X)$, where K, K_1 and K_2 are the constants of (1.3), (1.6) and (1.8), respectively, the inequality*

$$\left(\frac{1}{\mu(B^*)^2}\int_{B^*}\int_{B^*}\left|\int_B K_\gamma(x,z,y)f(y)d\mu(y)\right|^s d\mu(z)d\mu(x)\right)^{\frac{1}{s}} \tag{2.10}$$

$$\geq C\frac{1}{\mu(B)^{1-\gamma}}\int_B f(x)d\mu(x)$$

holds with $B^* = B(x_0, K_0 R)$ for every non negative function f.

Proof. Let $B = B(x_0, R)$ be a ball in X and let θ be fixed in $(0, A_1 A_2^{-1})$, where A_1 and A_2 are the constants of (1.6) and (1.8). From these conditions, it follows that $B(x_0, L\theta^{-1}R) - B(x_0, LR) \neq \emptyset$ whenever $K_2\mu(\{x_0\}) \leq LR \leq \theta K_1 \mu(X)$, where L is a constant to be chosen later. Let x_1 be in the above annulus. Using (1.3) it is not difficult to prove that

$$\left(\frac{L}{K^2} - \frac{1}{K} - 1\right) R \leq d(z,y) \leq \left(K + K^2\left(\frac{L}{\theta} + 1\right)\right) R \tag{2.11}$$

holds for every y in B and every z in $B(x_1, R)$. On the other hand, we know that there exist two constants, D_1 and D_2, such that

$$D_1\mu(B(x, d(x,y))) \leq \delta(x,y)^\alpha \leq D_2\mu(B(x, d(x,y))) \tag{2.12}$$

holds for every x and y in X, where δ is the quasi-distance of (1.13). Then, from this relation, (2.11), (1.6) and (1.8), and choosing $L = K^2(A_2 D_2 A_1^{-1} D_1^{-1} 4K + K^{-1} + 1)$, we get

$$\mu(B(y, d(z,y))) \geq A_1\left(\frac{L}{K^2} - \frac{1}{K} - 1\right) R$$

$$\geq A_1 \frac{A_2}{A_1}\frac{D_2}{D_1} 4KR \tag{2.13}$$

$$\geq 2\frac{D_2}{D_1}\mu(B(y, 2KR)),$$

for every $y \in B$ and every $z \in B(x_1, R)$, whenever x_0 and R satisfy.

$$\frac{K_2}{2K}\mu(\{x_0\}) \leq R \leq (K + K^2(L\theta^{-1} + 1))^{-1} K_1 \mu(X)$$

Therefore, under this restriction, from (2.12), (2.13) and definition of K_γ, we have

$$K_\gamma(x, z, y) \geq (D_2\mu(B(y, 2KR)))^{\gamma-1} - (D_1\mu(B(y, d(z, y))))^{\gamma-1}$$

$$\geq (D_2\mu(B(y, 2KR)))^{\gamma-1}\left(1 - 2^{\gamma-1}\right)$$

$$\geq C\mu(B)^{\gamma-1},$$

for every x and y in B and every z in $B(x_1, R)$. Finally, (2.10) follows inmediately from the above inequality and the fact that $B(x_1, R) \subset B^* = B(x_0, K_0 R)$ with $K_0 = K + K^2(L\theta^{-1} + 1)$.∎

Proof of Theorem (1.26). First let us see (1.27). Let $B = B(x_0, R)$ be a ball such that $R \leq K_1 K_0^{-1}\mu(X)$, where K_0 is the same constant of the above lemma. If $K_2(2K)^{-1}\mu(\{x_0\}) \leq R$ from that lemma, (1.18) and the equivalence between (1.19) and the left member of (1.18), it follows inmediately that

$$(\operatorname{ess}\sup_B a)\frac{1}{\mu(B)^{1-\gamma}}\int_B b_k^{-\frac{1}{1-\gamma}}\,d\mu \leq C\left(\int_B \left(b_k^{-\frac{1}{1-\gamma}} b\right)^{\frac{1}{\gamma}} d\mu\right)^{\gamma}$$

$$\leq C\left(\int_B b_k^{-\frac{1}{1-\gamma}}\,d\mu\right)^{\gamma}$$

holds for every $k \in I\!N$, where $b_k = b + 1/k$. Thus, we get

$$(\operatorname{ess}\sup_B a)\left(\frac{1}{\mu(B)}\int_B b_k^{-\frac{1}{1-\gamma}}\,d\mu\right)^{1-\gamma} \leq C,$$

and letting $k \to \infty$ we obtain the wished inequality. Now, suppose $R < K_2(2K)^{-1}\mu(\{x_0\}) \leq K_1 K_0^{-1}\mu(X)$. If $K_1 K_0^{-1}\mu(X) < K_2\mu(\{x_0\})$ the result clearly follows from the above case since, according (1.9), $B(x_0, R) = \{x_0\}$ for every $R \leq K_2\mu(\{x_0\})$. On the other hand, if $K_2\mu(\{x_0\}) \leq K_1 K_0^{-1}\mu(X)$, we get the result taking $\tilde{R}$ such that $K_2(2K)^{-1}\mu(\{x_0\}) \leq \tilde{R} \leq K_2\mu(\{x_0\})$ and applying the first case again since $B(x_0, R) = B(x_0, \tilde{R}) = \{x_0\}$. This completes the proof of (1.27).

In order to proof (1.28), let us consider a ball $B = B(x_0, R)$ such that, as before, $R \leq K_1 K_0^{-1}\mu(X)$. If $K_2(2K)^{-1}\mu(\{x_0\}) \leq R$, from lemma (2.6), the definition of T_γ^s we have

$$T_\gamma^s f(x) > C\frac{1}{\mu(B)^{1-\gamma}}\int_B f\,d\mu,$$

for every $x \in B$ and every non negative function f. Then, by taking $f = v_k^{-p'}\mathcal{X}_B$, where $v_k = v + 1/k$, $k \in I\!N$, inequality (2.4) allows us to obtain

$$u^q(B) \leq C\left(\lambda^{-p}\int_B \left(v_k^{-p'}v\right)^p d\mu\right)^{\frac{q}{p}}$$

$$\leq C\mu(B)^{(1-\gamma)q}\left(\int_B v_k^{-p'}d\mu\right)^{-\frac{q}{p'}}$$

for every $k \in I\!N$. This yields

$$\frac{u^q(B)}{\mu(B)^{(1-\gamma)q}}\left(\int_B v_k^{-p'}d\mu\right)^{-\frac{q}{p'}} \leq C$$

and letting $k \to \infty$ we achieve the inequality of (1.28) in the considered case. Finally, in the other cases, the result can be obtained by applying a reasoning as in the proof of (1.27).∎

The proof of Corollary (1.29) requires the following lemma concerning to the geometry of the spaces of homogeneous type.

(2.14) **Lemma:** *Let (X, d, μ) be a space of homogeneous type and let θ belonging to $(0,1)$. There exist a number N only depending on θ and the constants of the space, such that, for each $x_0 \in X$ and each $R > 0$, it can be founded a set $\{x_i\}_{i\in I}$ satisfying*

(2.15) *$x_i \in B(x_0, R)$ for every $i \in I$,*

(2.16) *$B(x_0,R) \subset \cup_{i\in I}B(x_i, \theta R)$,*

(2.17) *the cardinal of I is less or equal than N.*

Proof. The lemma is a straightforward consequence of the fact that, given θ in $(0,1)$, there exists a number N, only depending on θ and the constants of the space, such that in each ball B the amount of points whose mutual distance are bigger or equal than θ times the radius of B is, at most, N (see [CW], p. 68).∎

Proof of Corollary (1.29). If $\mu(X) = \infty$, (1.30) and (1.31) are obvious from (1.27) and (1.28), respectively. So, let us assume $\mu(X) < \infty$. In order to prove (1.30), since (1.27) holds, we just need to show that (1.18) implies the inequality of (1.27) in the case $B = X$. Let x_0 be a point in X and let $R = 2K_1\mu(X)$ where K_1 is the constant of (1.18). Then $B(x_0, R) = X$. Now, applying lemma (2.14) with $\theta = (2K_0)^{-1}$, where K_0 is the constant in (1.24), we obtain a finite family of balls, $\{B_i\}_{i\in I}$ with radius $K_1K_0^{-1}\mu(X)$, such that $B(x_0,R) \subset \cup_{i\in I}B_i$. Moreover, from (1.18) and (2.9), it follows that

$$\left(\operatorname*{ess\ sup}_X a\right)\frac{1}{\mu(B_i)^{1-\gamma}}\int_{B_i} b_k^{-\frac{1}{1-\gamma}}d\mu \leq C\left(\int_{B_i} b_k^{-\frac{1}{1-\gamma}}d\mu\right)^{\gamma}$$

holds for every $k \in I\!N$ and every $i \in I$, where $b_k = b + 1/k$. By reasoning as in the proof of (1.26), we get

$$(ess \sup_X a) \left(\frac{1}{\mu(B_i)} \int_{B_i} b^{-\frac{1}{1-\gamma}} d\mu \right)^{1-\gamma} \leq C$$

for every $i \in I$. Therefore, since μ verifies (1.5), an standard argument allows us to obtain

$$(ess \sup_X a) \left(\frac{1}{\mu(X)} \int_X b^{-\frac{1}{1-\gamma}} d\mu \right)^{1-\gamma} \leq C \; .$$

This concludes the proof of (1.30). Let us see (1.31). As before, we only need to consider the case $B = X$. Let $\{B_i\}_{i \in I}$ be as above. From the definition of T_γ^s and Lemma (2.6), it follows that there exists a constant C, only depending on s, γ and the constant of the space, such that

$$\frac{C}{\mu(B_i)^{1-\gamma}} \int_{B_i} v_k^{-p'} < T_\gamma^s \left(\mathcal{X}_{B_i} v_k^{-p'} \right)(x)$$

holds for every $k \in I\!N$ and every $i \in I$ and every $x \in X$, where $v_k = v + 1/k$. Then, from (1.22) with λ equal to the left member of the above inequality, we have

$$\frac{u^q(X)}{\mu(B_i)^{(1-\gamma)q}} \left(\int_{B_i} v_k^{-p'} \right)^{\frac{q}{p'}} \leq C$$

for every $i \in I$. Then, taking $k \to \infty$, we get

$$\frac{u^q(X)}{\mu(B_i)^{(1-\gamma)q}} v_k^{-p'}(B_i)^{\frac{q}{p'}} \leq C$$

for any $i \in I$. Finally, applying an argument like the used in the proof of (1.28), it follows that

$$\frac{u^q(X)}{\mu(X)^{(1-\gamma)q}} v_k^{-p'}(X)^{\frac{q}{p'}} \leq C$$

holds, as we wanted to prove.■

References

[AM] Aimar, H. and Macías, R.: "*Weighted norm inequalities for the Hardy-Littlewood maximal operator on spaces of homogeneous type*", Proc. Amer. Math. Soc. 91 (1984), 213-216.

[BS] Bernardis, A. and Salinas, O.: "*Two-weight norm inequalities for the fractional maximal operator on spaces of homogeneous type*", Studia Math. 108 (1994), 201-207.

[C] Calderón, A.: "Inequalities for the maximal function relative to a metric", Studia Math. 57 (1976), 297-306

[CW] Coifman, R and Weiss, G.: "*Analyse harmonique non-conmutative certains espaces homogènes*", Lectures Notes in Math., 242, Springer-Verlag 1971.

[HMS1] Harboure, E.; Macías, R. and Segovia, C.: "*Extrapolation results for classes of weights*", Amer. J. of Math. 110 (1988), 383-397.

[HMS2] Harboure, E.; Macías, R. and Segovia, C.: "*An extrapolation theorem for pairs of weights*", to appear in Revista de la Unión Matemática Argentina.

[MS1] Macías, R. and Segovia, C.: "*A decomposition into atoms of distributions on spaces of homogeneous type*", Advances in Math. 33 (1979), 271-309.

[MS2] Macías, R. and Segovia, C.: "*Lipschitz functions on spaces of homogeneous type*", Advances in Math. 33 (1979), 257-270.

[MT] Macías, R. and Torrea, J.: "*L^2 and L^p boundedness of singular integrals on non necessarly normalized spaces of homogeneous type*", Cuadernos de Matemática y Mecánica, No. 1-88, PEMA-INTEC-GTM, Santa Fe (Argentina).

[MW] Muckenhoupt, B. and Wheeden, R.: "*Weighted norm inequalities for fractional integrals*", Trans. of the A.M.S. 192 (1974), 261-274.

[SW] Sawyer, E. and Wheeden, R.: "*Wheighted inequalities for fractional integrals on euclidean and homogeneous spaces*", Amer. J. Math. 114 (1992), 813-874.

[W] Wheeden, R.: "*A characterization of some weighted norm inequalities for the fractional maximal function*", Studia Math. 107 (1993) 257-272.

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