Revista de la
Unión Matemática Argentina
Volumen 41, 3, 1999.

A REMARK ON NUMBERS WITH POWERS IN A POINT-LATTICE

A. Benedek R. Panzone G. Paolini
Univ. Nac. del Sur CONICET Univ. Nac. del Sur
INMABB Av Alem 1253 (8000)Bahia Blanca ARGENTINA

ABSTRACT. We prove by using elementary methods that if the positive powers of a given complex nonreal number b belong to a point-lattice Λ then they belong also to the point-lattice L generated by 1 and b and b is a quadratic integer. This settles the following question. Let D be a finite set of rational integers that contains 0 and 1. If the set of values of polynomials with coefficients in D evaluated at b is included in Λ, is it true or not that it is part of L?

I. INTRODUCTION. We shall assume that $b=b_1+ib_2$ is a fixed complex number with $|b|>1$, $b_2=\text{Im}(b)>0$. The point-lattice L:=$[b,1]=\{mb+n;m,n\in Z\}$ is naturally associated with b. Let be $u=u_1+iu_2$, $v=v_1+iv_2$, $v_2\neq 0$ and $\Lambda:=[v,u]=\{mv+nu{:}m,n\in Z\}$ the point-lattice generated by the *linearly independent* numbers u,v. Define P:= $\{b^k;k=1,2,\ldots\}$. The following result holds (cf. [1]):
THEOREM 1. If u=1 then $P\subset\Lambda\Rightarrow P\subset L$.▪

II. THE MAIN RESULT. We shall prove the following generalization of this theorem.
THEOREM 2. If $b^j\in\Lambda$ for j≥N then b is a quadratic integer and P⊂L.▪
If $b^{j+N}\in\Lambda$ for j=0,1,2 and $b^{j+N}=m_ju+n_jv$ then

(1) $b^j=m_j\tilde{u}+n_j\tilde{v}$ for j=0,1,2,... with $\tilde{u}=b^{-N}u$ and $\tilde{v}=b^{-N}v$.

So, we can assume without loss of generality that N=0.
We begin with two auxilliary results.
PROPOSITION 1. If $b^j\in\Lambda$ for j ≥ 0 then $|b|^2\in$Z.▪
PROOF. We know that for j=1,2,..., $b^{j-1}=m_{j-1}v+n_{j-1}u$ with m_{j-1}, n_{j-1} rational integers. Any three of these equations is a homogeneous system in 1,u and v. Then we have for any j=1,2,...

(2) $$\begin{vmatrix}-b^{j-1} & m_{j-1} & n_{j-1}\\ -b^j & m_j & n_j\\ -b^{j+1} & m_{j+1} & n_{j+1}\end{vmatrix}=0.$$

If we *define*: $A_j=m_{j-1}n_j-m_jn_{j-1}$, $B_j=m_{j-2}n_j-m_jn_{j-2}$, then $A_j\neq 0$ and

(3) $$b^2A_j-bB_{j+1}+A_{j+1}=0.$$

Since the coefficients in (3) are real and b is not real, we must have

(4) $$2\operatorname{Re}(b)=\frac{B_{j+1}}{A_j} \text{ and } |b|^2=\frac{A_{j+1}}{A_j} \text{ for j=1,2,....}$$

Multiplying the last identities from j=1 to k, one gets

(5) $$|b|^{2k}=\frac{A_{k+1}}{A_1}.$$

Thus $A_1|b|^{2k}$ is an integer for any k. Therefore $|b|^2$ must be an integer, QED.

PROPOSITION 2. If $b^j \in \Lambda$ for $j \geq 0$ then there are rational integers α_j, β_j such that

(6) $$A_1 b^j=\alpha_j b+\beta_j.$$ ▪

PROOF. Regarding the identities $b^{j-1}=m_{j-1}v+n_{j-1}u$ for j=1,2,k as a homogeneous system in $1,u,v$, one gets that

(7) $$\begin{vmatrix} -1 & m_0 & n_0 \\ -b & m_1 & n_1 \\ -b^k & m_k & n_k \end{vmatrix}=0$$

This yields the thesis with $\alpha_j=m_0n_j-m_jn_0$ and $\beta_j=m_jn_1-m_1n_j$, QED.

PROOF OF THEOREM 2. From (3) and proposition 1 one gets

(8) $$b^2=\frac{p}{q}b-k,$$

where $k=|b|^2$, p,q coprime integers. Using (8) one can prove by induction on j that

(9) $$b^j=\left(\left(\frac{p}{q}\right)^{j-1}+\sum_{h<j-1}\sigma_{jh}\left(\frac{p}{q}\right)^h\right)b+\sum_{h<j-1}\lambda_{jh}\left(\frac{p}{q}\right)^h$$

where small greek letters represent rational integers.
Comparing (6) and (9) we get

(10) $$\alpha_j=A_1\left(\left(\frac{p}{q}\right)^{j-1}+\sum_{h<j-1}\sigma_{jh}\left(\frac{p}{q}\right)^h\right) = \text{rational integer for all } j>2.$$

Thus, $\alpha_j=A_1(p^{j-1}+q\gamma_j)/q^{j-1}$. This can only hold for q=1, QED.

COROLLARY. There is an integer K=K(b,u,v,N) such that if $b^j \in \Lambda$ for K+N≥j≥N then b is a quadratic integer.▪

III. NECESSARY CONDITIONS FOR $b^2 \in[1,b]$. In this section we assume that $b^2=mb+n$, $m,n\in$Z, ($n\neq 0$). We obtain from this hypothesis that

(11) $$k:=|b|^2=-n \in \mathbf{N},\ \operatorname{Re}(b)=m/2 \in Z/2$$

THEOREM 2. If $b,b^2 \in \Lambda=[u,v]$ with $u=u_1+u_2i$ and $v=v_1+v_2i$ then

$$u_1,v_1 \in Q,\ \frac{u_2}{b_2},\frac{v_2}{b_2} \in Q.$$ ▪

PROOF. Solving the following system

(12) $$b = m_0 u + n_0 v \ , \ b^2 = mb - k = m_1 u + n_1 v$$

for u and v, we get that

$$A_1 u = (n_1 - mn_0)b + n_0 k \ , \ A_1 v = (mm_0 - m_1)b - m_0 k \ .$$

Taking real and imaginary parts and using (11) we obtain

$$A_1 u_1 = (n_1 - mn_0)m/2 + n_0 k \ , \ A_1 u_2 = (n_1 - mn_0)b_2$$

$$A_1 v_1 = (mm_0 - m_1)m/2 - m_0 k \ , \ A_1 v_2 = (mm_0 - m_1)b_2 \ ,$$ QED.

NB. The results of the present paper should be compared with Theorem 1 of [3].

REFERENCES.

[1] Benedek A. and Panzone R., On complex bases for number systems with the digit set {0,1}, Actas 4to Congreso "Dr. A.A.R. Monteiro", (1997)17-32.

[2] Hardy G. H. and Wright E. M., AN INTRODUCTION TO THE THEORY OF NUMBERS, Oxford, (1960).

[3] Indlekofer K.-H., Kátai I. and Racskó P., Some remarks on generalized number systems, Acta Sci. Math (Szeged), 57(1993)543-553.

Recibido en Noviembre 1998