Revista de la
Unión Matemática Argentina
Volumen 41, 3, 1999.

On Stochastic Parallel Transport and Prolongation of Connections.

Pedro Catuogno
Departamento de Matemáticas FCEYN.
Universidad Nacional de Mar del Plata.

Abstract

Let $P(M,G)$ be a principal fiber bundle, ∇ a G-invariant CDO of P and Ψ a CDO of M. We prove that ∇ is projetable with projection Ψ iff there is a unique prolongation $\mathbf{H} \to \mathbf{H}^{\mathbf{r}}$ such that satisfies: Every stochastic horizontal lift of a Ψ-martingale is a ∇-martingale. We given an explicit expression for $\mathbf{H}^{\mathbf{r}}$ in terms of $\mathbf{H}$ and ∇.

The stochastic parallel displacement of a tensor along a random curve was considered by K. Itô [6]. Its natural generalization, the stochastic horizontal lifting in principal fiber bundles were studied by I. Shigekawa and others ([1], [8], [11], [10]).

The motivation for the present investigation is the discovery of P. Meyer [9] of a correspondence between the stochastic extensions of the equation of parallel transport of vectors and certain extensions to the tangent bundle TM of the connection ∇ on M. The stochastic parallel transports studied by P. Meyer are induced by 2-connections [1] of BM (the fiber bundle of bases of M) that are prolongations of ∇. These prolongations are of 1-connections to 2-connections of BM, and are given by $Gl(n,\mathbb{R})$-invariant connections of BM with projection ∇.

In this work we study these prolongations of 1-connections to 2-connections in the context of principal fiber bundles.

This paper is organized as follows, in 1. we prepare some notions concerning Schwartz geometry, 2-connections and martingales. In 2. we prove the main result of this work. Let $P(M,G)$ be a principal fiber bundle, ∇

a G-invariant covariant derivative operator without torsion of P and Ψ a covariant derivative operator without torsion of M. Then ∇ is projetable with projection Ψ iff there is a unique prolongation of 1 -connections into 2-connections [1] of $P(M,G)$ such that satisfies: Every stochastic horizontal lift of a Ψ-martingale is a ∇-martingale. We given an explicit expression for $\mathbf{H}^{r}$ in terms of $\mathbf{H}$ and ∇. Finally, in section 3 we apply this results to diffusions given by Stratonovich equations, and discussed the special case of the principal fiber bundle of bases of a differential manifold with the G-invariant connections ∇^{C} and ∇^{H} [3].

1 Schwartz Geometry, 2-Connections and ∇-Martingales.

Throughout this paper, manifolds, maps and functions will always be assumed to be smooth. As to manifolds and stochastic differential geometry, we shall freely concepts and notations of Kobayashi-Nomizu [7] and Emery [4].

Now, we recall some fundamental facts about Schwartz second order geometry ([8], [9], [4], [10]) and martingales.

If x is a point in a manifold M, the second order tangent space to M at x, denoted τ_xM, is the vector space of all differential operators on M, at x, of order at most two, with no constant term. If $dimM = n$, τ_xM has $n+\frac{1}{2}n(n+1)$ dimensions; using a local coordinate system (U,x^i) around x, every $L\in\tau_xM$ can be written in a unique way as

$$L = a^{ij}\frac{\partial^2}{\partial x^i\partial x^j} + a^k\frac{\partial}{\partial x^k} \qquad \text{with } a^{ij}=a^{ji}$$

(we use here and in other expressions in coordinates the convention of summing over the repeated indices). The elements of τ_xM are called second-order tangent vectors at x.

The disjoint union $\tau M = \bigcup_{x\in M}\tau_xM$ is canonically endowed with a vector bundle structure over M, called the second order tangent fiber bundle of M. We denote by $\Gamma(\tau M)$ the space of second order operator on M, that is, the space of sections of τM.

If M and N are manifolds and $\varphi : M \to N$ is a smooth mapping, it is possible to push forward second order tangent vectors by φ, given $L\in\tau_xM$

its image under φ is $\varphi_*(x)L \in \tau_{\varphi(x)}N$ given by

$$\varphi_*(x)L(f) = L(f \circ \varphi)$$

with f an arbitrary smooth function. We says that $\phi : \tau_x M \longrightarrow \tau_y N$ is a Schwartz morphism if there exists a smooth mapping $\varphi : M \longrightarrow N$ with $\varphi(x) = y$ such that $\phi = \varphi_*(x)$.

We know [8] that, we can associate with each covariant derivative operator without torsion (in short, CDO) ∇ of M a morphism $\Phi_\nabla : \tau M \longrightarrow TM$ defined in a local chart (U, x^i) of M by

$$\Phi_\nabla(a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + a^k \frac{\partial}{\partial x^k}) = (a^{ij}\Gamma^k_{ij} + a^k)\frac{\partial}{\partial x^k}$$

where $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}$. We observe that Φ_∇ satisfies $\Phi_\nabla \circ i = Id_{TM}$ where $i : TM \longrightarrow \tau M$ is the inclusion.

Conversely, if $\Phi : \tau M \longrightarrow TM$ is a morphism of vector bundles such that $\Phi \circ i = Id_{TM}$ then we have defined a covariant derivative operator without torsion ∇^Φ by $\nabla^\Phi_X Y = \Gamma(XY)$ for all $X, Y \in \Gamma(M)$. Obviously, $\Phi_{\nabla^\Phi} = \Phi$ and $\nabla^{\Phi_\nabla} = \nabla$.

We remember the following proposition ([4], [5]).

Proposition 1 *Let M and N manifolds be endowed with CDO ∇ and Ψ respectively and $\varphi : M \longrightarrow N$ a smooth mapping. The following statements are equivalent:*

i) For every $x \in M$, $\Phi_\Psi \circ \varphi_*(x) = \varphi_*(x) \circ \Phi_\nabla$

ii) For every geodesic $g : U \longrightarrow M$, $\varphi \circ g : U \longrightarrow M$ is a geodesic.

iii) φ is affine.

Let M be a manifold endowed with a CDO ∇ and $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ a filtered probability space satisfying the usual conditions [4]. A continuous semimartingale X in M, is a ∇-martingale ([8], [4]) if, for every $\theta \in \Gamma(T^*M)$ with compact support

$$\int_0^t \langle \theta, \Phi_\nabla d_2 X\rangle \textit{ is a local martingale.}$$

where $\int\limits_0^t \langle\theta, \Phi_\nabla d_2 X\rangle$ is the Itô integral of θ along X. Martingales, too, can be characterized in local coordinates. In fact, let (U, x^i) be a local chart of M a semimartingale $X = (X^i)$ is a ∇-martingale iff for some real local martingales (N^i),

$$X_t^i - X_0^i = N^i - \frac{1}{2}\int\limits_0^t \Gamma_{jk}^i(X_s)d[N^j, N^k]_s$$

Now, we remember the definition of 2-connection [1]

Definition 2 *Let $P(M, G)$ be a principal fiber bundle. A family of Schwartz morphism $\mathbf{H} = \{H_p : p \in P\}$ is called a 2-connection if*

1) $H_p : \tau_{\pi p}M \longrightarrow \tau_p P$.

2) $\pi_* \circ H_p = id_{\tau_{\pi p}M}$

3) $H_{pg} = R_{g*}H_p$ for all $p \in P$ and $g \in G$ where R_g stands for the right action of G in P.

4)The mapping $p \rightarrow H_p L$ belongs to $\Gamma(\tau P)$ if $L \in \Gamma(\tau M)$.

We observed that by changing in the above definition τ for T, we get the classical definition of connection in principal fiber bundles, that we call 1-connection in this work. Obviously, every 2-connection $\mathbf{H} = \{H_p : p \in P\}$ induces a unique 1-connection $\mathbf{H}_R = \{H_p |_{T_{\pi p}M} : p \in P\}$ by restriction to the tangent space.

Let $P(M, G)$ be a principal fiber bundle, $\mathbf{H} = \{H_p : p \in P\}$ a 2-connection, X an M-valued semimartingale and Z a P-valued $\mathcal{F}_0$-random variable such that $\pi \circ Z = X_0$. We know [1] that the stochastic horizontal lift (s.h.l) of X initialized in Z is a P-valued semimartingale Y such that satisfies the following stochastic differential equation

$$\begin{aligned} d_2 Y &= H_Y d_2 X \\ Y_0 &= Z \end{aligned}$$

2 Prolongation of Connections and Stochastic Horizontal Lifts

Let us first introduce some definitions

Definition 3 *Let $P(M,G)$ be a principal fiber bundle and ∇ a CDO of P. We says that ∇ is G-invariant if $\Phi_\nabla(pg)\circ R_{g*} = R_{g*}\circ\Phi_\nabla(p)$ for all $p\in P$ and $g\in G$.*

Definition 4 *Let $P(M,G)$ be a principal fiber bundle and ∇ a G-invariant CDO of P. We says that ∇ is projectable if*

$$\Phi_\nabla(p)(Ker(\pi_*(p))) \subset Ker(\pi_*(p))$$

Example 5 *Let BM be the principal fiber bundle of bases of M, Ψ a CDO of M and Ψ^C (Ψ^H) the complete (horizontal) lift of Ψ to BM [3]. We have that Ψ^C and Ψ^H are projectable.*

The "projection" of ∇ by π is described in the following proposition.

Proposition 6 *Let $P(M,G)$ be a principal fiber bundle and ∇ a G-invariant CDO of P. Then ∇ is projectable iff there is a unique CDO Ψ of M such that π is affine. We says that Ψ is the projection of ∇.*

Proof: Let $L\in\tau_xM$ and $p\in P$ such that $\pi(p)=x$. Then there is $T\in\tau_pP$ such that $\pi_*(p)(T)=L$, we define $\Phi_\Psi(x)(L)$ by $\pi_*(p)(\Phi_\nabla(p)(T))$. Now, we prove that $\Phi_\Psi(x)(L)$ is well defined. For this let $g\in G$ and $S\in\tau_{pg}P$ such that $\pi_*(pg)(S)=L$, we have that

$$\begin{aligned}\pi_*(pg)(\Phi_\nabla(pg)(S)) &= \pi_*(pg)(\Phi_\nabla(pg)\circ R_{g*}\circ R_{g^{-1}*}(S))\\ &= \pi_*(pg)(R_{g*}\circ\Phi_\nabla(p)\circ R_{g^{-1}*}(S))\\ &= \pi_*(p)(\Phi_\nabla(p)\circ R_{g^{-1}*}(S))\end{aligned}$$

On the other hand, $\pi_*(p)(R_{g^{-1}*}(S))=\pi_*(pg)(S)=\pi_*(p)(T)$, this is $R_{g^{-1}*}(S)-T\in Ker(\pi_*(p))$ and by hypothesis $\Phi_\nabla(p)\circ R_{g^{-1}*}(S)-\Phi_\nabla(p)(T)\in Ker(\pi_*(p))$, thus

$$\pi_*(p)(\Phi_\nabla(p)\circ R_{g^{-1}*}(S)) = \pi_*(p)(\Phi_\nabla(p)(T)$$

We conclude that $\Phi_\Psi(x)(L)$ is well defined. Obviously, $\Phi_\Psi : \tau M \longrightarrow TM$ is a morphism of vector bundles such that $\Phi_\Psi \circ i = Id_{TM}$, and define a CDO Ψ and as

$$\Phi_\Psi(\pi(p)) \circ \pi_*(p) \quad = \quad \pi_*(p) \circ \Phi_\nabla(p)$$

for all $p \in P$, we have that π is affine and Ψ is unique.
Conversely, given $L \in Ker(\pi_*(\mathrm{p}))$ we have that:

$$\pi_*(p)(\Phi_\nabla(p)L) \quad = \quad \Phi_\Psi(\pi(p))(\pi_*(p)L) \quad = \quad 0$$

□

Now, we give the definition of prolongation.

Definition 7 *Let $P(M,G)$ be a principal fiber bundle. An application φ from 1-connections into 2-connections of $P(M,G)$ is called a prolongation if $\varphi(\mathbf{H})_R = \mathbf{H}$ for every 1-connection $\mathbf{H}$ of $P(M,G)$.*

In [8] P. Meyer states that there is a canonical prolongation $\mathbf{H} = \{H_p : p \in P\} \longrightarrow \mathbf{H}^S = \{H_p^S : p \in P\}$, this prolongation is called the Stratonovich prolongation and is characterized by

$$H_p^S\{X,Y\} = \{HX, HY\}_p$$

where X and Y are local vector fields of M.
Now we state our main result.

Theorem 8 *Let $P(M,G)$ be a principal fiber bundle, ∇ a G-invariant CDO of P and Ψ a CDO of M. Then ∇ is projetable with projection Ψ iff there is a unique prolongation $\mathbf{H} \longrightarrow \mathbf{H^r}$ such that satisfies: Every stochastic horizontal lift of a Ψ-martingale is a ∇-martingale.*

Proof: Let $\mathbf{H} = \{H_p : T_{\pi p}M \longrightarrow T_pP\}$ be a 1-connection of $P(M,G)$. Then there is a unique 2-connection $\mathbf{H^r}$ ={$\mathbf{H_p^r}$:¿ ¼pM→¿ pP} of $P(M,G)$ such that

$$H_p^\nabla : \tau_{\pi p}M \longrightarrow \tau_p P \ is \ affine$$

In fact, by [5, Lemma 11] we have that

$$H_p^\nabla = (\exp_p^\nabla \circ H_p \circ (\exp_{\pi p}^\Psi)^{-1})_*(\pi p)$$

then the map $p \to H_p^\nabla$ is smooth,

$$\pi_*(p) \circ H_p^\nabla = Id_{T_{\pi p}M}$$

and

$$\begin{aligned} R_{g*} \circ H_p^\nabla &= (R_g \circ \exp_p^\nabla \circ H_p \circ (\exp_{\pi p}^\Psi)^{-1})_*(\pi p) \\ &= (\exp_{pg}^\nabla \circ R_{g*} \circ H_p \circ (\exp_{\pi p}^\Psi)^{-1})_*(\pi p) \\ &= (\exp_p^\nabla \circ H_{pg} \circ (\exp_{\pi pg}^\Psi)^{-1})_*(\pi pg) \\ &= H_{pg}^\nabla \end{aligned}$$

Therefore, we conclude that $\mathbf{H}^r$ is a 2-connection of $P(M,G)$ and as $H_p^\nabla|_{T_{\pi p}M} = H_p$ for every $p \in P$, we have that $\mathbf{H}^r$ is a prolongation of $\mathbf{H}$. Now, let X be a Ψ-martingale and Y a stochastic horizontal lift of X, then Y is solution of $d_2Y = H_Y^\nabla d_2X$, and by the Itô transfer principle [5, Theorem 12] we have that Y is solution of $d^\nabla Y = H_Y d^\Psi X$ (where d^∇ and d^Ψ are the Itô differential in relation to ∇ and Ψ respectively) and as X is a Ψ-martingale, we obtain that Y is a ∇-martingale.

Let ρ be a prolongation such that every stochastic horizontal lift of a Ψ-martingale is a ∇-martingale. Let (x^λ) and (x^λ, y^i) be local charts of M and P respectively. In these local charts

$$\Phi_\Psi\left(\frac{\partial^2}{\partial x^\mu \partial x^\nu}\right) = \tilde{\Gamma}^\lambda_{\mu\nu}\frac{\partial}{\partial x^\lambda}$$

$$\begin{aligned} \Phi_\nabla\left(\frac{\partial^2}{\partial x^\mu \partial x^\nu}\right) &= \Gamma^\lambda_{\mu\nu}\frac{\partial}{\partial x^\lambda} + \Gamma^i_{\mu\nu}\frac{\partial}{\partial x^i} \\ \Phi_\nabla\left(\frac{\partial^2}{\partial x^\mu \partial x^j}\right) &= \Gamma^\lambda_{\mu j}\frac{\partial}{\partial x^\lambda} + \Gamma^i_{\mu j}\frac{\partial}{\partial x^i} \\ \Phi_\nabla\left(\frac{\partial^2}{\partial x^j \partial x^k}\right) &= \Gamma^\lambda_{jk}\frac{\partial}{\partial x^\lambda} + \Gamma^i_{jk}\frac{\partial}{\partial x^i} \end{aligned}$$

and [1, page 6]

$$\begin{aligned} \rho(\mathbf{H})\left(\frac{\partial}{\partial x^\lambda}\right) &= \frac{\partial}{\partial x^\lambda} + a^i_\lambda\frac{\partial}{\partial x^i} \\ \rho(\mathbf{H})\left(\frac{\partial^2}{\partial x^\lambda \partial x^\mu}\right) &= \frac{\partial^2}{\partial x^\lambda \partial x^\mu} + a^i_{\lambda\mu}\frac{\partial}{\partial x^i} + a^{ij}_{\lambda\mu}\frac{\partial^2}{\partial x^i \partial x^j} + \\ &\quad 2a^{i\nu}_{\lambda\mu}\frac{\partial^2}{\partial x^i \partial x^\nu} \end{aligned}$$

where $\mathbf{H}$ is a 1-connection locally given by $H\left(\frac{\partial}{\partial x^\lambda}\right) = \frac{\partial}{\partial x^\lambda} + a^i_\lambda \frac{\partial}{\partial x^i}$ and

$$\begin{aligned} a^{ij}_{\lambda\mu} &= \tfrac{1}{2}(a^i_\lambda a^j_\mu + a^i_\mu a^j_\lambda) \\ a^{i\nu}_{\lambda\mu} &= \tfrac{1}{2}(a^i_\lambda \delta^\nu_\mu + a^i_\mu \delta^\nu_\lambda) \end{aligned}$$

Let X be a semimartingale in M and Z a stochastic horizontal lift of X. Locally Z is given as (X^λ, Y^i), where

$$dY^i_t = a^i_\lambda dX^\lambda_t + \tfrac{1}{2}a^i_{\mu\nu}d[X^\mu, X^\nu]_t \qquad (1)$$

Now, let X be a Ψ-martingale. In local coordinates X is expressed by

$$dX_t^\lambda = dM_t^\lambda - \tfrac{1}{2}\widetilde{\Gamma}_{\mu\nu}^\lambda(X_t)d[M^\mu, M^\nu]_t \tag{2}$$

where M^λ are local martingales. We obtain from (1) and (2) that

$$dY_t^i = a_\lambda^i dM_t^\lambda + \tfrac{1}{2}(a_{\mu\nu}^i - a_\lambda^i \widetilde{\Gamma}_{\mu\nu}^\lambda)d[M^\mu, M^\nu]_t$$

On the other hand, $Z = (X^\lambda, Y^i)$ is a ∇-martingale iff

$$dY_t^i + \frac{1}{2}\Gamma_{\mu\nu}^i d[X^\mu, X^\nu]_t + \Gamma_{j\mu}^i d[Y^j, X^\mu]_t + \frac{1}{2}\Gamma_{jk}^i d[Y^j, Y^k]_t$$

and

$$dX_t^\lambda + \frac{1}{2}\Gamma_{\mu\nu}^\lambda d[X^\mu, X^\nu]_t + \Gamma_{j\mu}^\lambda d[Y^j, X^\mu]_t + \frac{1}{2}\Gamma_{jk}^\lambda d[Y^j, Y^k]_t$$

are local martingale. A direct computation, using the identities previously obtained leads that

$$a_\lambda^i dM_t^\lambda + \frac{1}{2}\left((a_{\mu\nu}^i - a_\lambda^i \Gamma_{\mu\nu}^\lambda) + \Gamma_{\mu\nu}^i + 2\Gamma_{j\mu}^i a_\nu^j + \Gamma_{jk}^i a_\mu^j a_\nu^k\right) d[M^\mu, M^\nu]_t \tag{3}$$

and

$$dM_t^\lambda + \left(\frac{1}{2}(\Gamma_{\mu\nu}^\lambda - \widetilde{\Gamma}_{\mu\nu}^\lambda) + \Gamma_{j\mu}^\lambda a_\nu^j + \Gamma_{jk}^\lambda a_\mu^j a_\nu^k\right) d[M^\mu, M^\nu]_t \tag{4}$$

are local martingale. We obtain from (3) that

$$a_{\mu\nu}^i = a_\lambda^i \Gamma_{\mu\nu}^\lambda - (\Gamma_{\mu\nu}^i + 2\Gamma_{j\mu}^i a_\nu^j + \Gamma_{jk}^i a_\mu^j a_\nu^k)$$

This is $\rho(\mathbf{H}) = \mathbf{H}^{\mathrm{r}}$. Converselly, we have that

$$\begin{aligned} \Gamma_{\mu\nu}^\lambda &= \widetilde{\Gamma}_{\mu\nu}^\lambda \\ \Gamma_{j\mu}^\lambda &= 0 \\ \Gamma_{jk}^\lambda &= 0 \end{aligned}$$

since (4) is true for every 1-connection $\mathbf{H}$. Therefore, ∇ is projectable with projection Ψ. □

The next proposition give an explicit expression for $\mathbf{H}^{\mathrm{r}}$ in terms of $\mathbf{H}$ and ∇.

Proposition 9 *Let X and Y be local vector fields of M. Then*

$$H^{\nabla}\{X,Y\} = \{HX, HY\} - \omega^{H}\left(\nabla_{HX}HY + \nabla_{HY}HX\right)^{*}$$

where ω^{H} is the form of connection associated with **H** *and $* : \mathcal{G} \longrightarrow \Gamma(TP)$ is the homomorphism defined by the right action of G on P.*

Proof: Let X and Y be local vector fields of M, and set $C(X,Y) = H^{\nabla}\{X,Y\} - \{HX,HY\}$. Then $C(X,Y)$ is a vertical local field. In fact, we have that $\pi_*(H^{\nabla}\{X,Y\}) = \pi_*(\{HX,HY\}) = \{X,Y\}$, hence $C(X,Y)$ is vertical. And as

$$\begin{aligned} QH^{\nabla}\{X,Y\} &= H^{\nabla}_{R} \otimes H^{\nabla}_{R}(Q\{X,Y\}) \\ &= H \otimes H(Q\{X,Y\}) \\ &= H^{S}_{R} \otimes H^{S}_{R}(Q\{X,Y\}) \\ &= QH^{S}\{X,Y\} \end{aligned}$$

where Q is the squared gradient operator (In local coordinates $Q(a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + a^k \frac{\partial}{\partial x^k}) = a^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$), we have that $C(X,Y)$ is a local vector field. Now, since

$$\begin{aligned} H\left(\Psi_X Y + \Psi_Y X\right) &= H^{\nabla}\left(\Phi_{\Psi}\{X,Y\}\right) \\ &= \Phi^{\nabla}\left(H^{\nabla}\{X,Y\}\right) \\ &= \Phi^{\nabla}\left(\{HX,HY\}\right) + C(X,Y) \\ &= \left(\nabla_{HX}HY + \nabla_{HY}HX\right) + C(X,Y) \end{aligned}$$

we have that

$$\begin{aligned} C(X,Y) &= \omega^{H}(C(X,Y))^{*} \\ &= \omega^{H}\left(H(\Psi_X Y + \Psi_Y X) - \nabla_{HX}HY - \nabla_{HY}HX\right)^{*} \\ &= -\omega^{H}\left(\nabla_{HX}HY + \nabla_{HY}HX\right)^{*} \end{aligned}$$

This completes the proof. □

3 Applications

i) Let $P(M,G)$ be a principal fiber bundle, $\mathbf{H} = \{H_p : p \in P\}$ a 1-connection and ∇ a projectable CDO of P. Let $A_0, A_1, \ldots, A_n$ be C^{∞} vector fields on M

and $B_t = (B_t^1, \ldots, B_t^n)$ a standard Brownian motion, and $X_t(x)$ the solution of the following Stratonovich differential equation

$$\begin{aligned} dX_t &= A_0(X_t)dt + \sum_{i=1}^{n} A_i(X_t) \circ dB_t^i \\ X_0 &= x \in M \end{aligned} \tag{5}$$

Then the stochastic horizontal lift $Y_t(p)$ of $X_t(x)$ in relation to $\mathbf{H}^r$ is given by the solution of

$$\begin{aligned} dY_t &= \left(HA_0 - \tfrac{1}{2}\sum_{i=1}^{n} \omega^H(\nabla_{HA_i}HA_i)^*\right)(Y_t)dt \\ &\quad + \sum_{i=1}^{n} HA_i(Y_t) \circ dB_t^i \\ Y_0 &= p \in P \end{aligned} \tag{6}$$

In fact, let Z_t be a solution of (6). Since $\pi \circ Z_t = X_t$ and the infinitesimal generator of Z_t is $H^\nabla\left(A_0 + \frac{1}{2}\sum_{i=1}^{n} A_i^2\right)$, by [2, Lemma 2.1] we have that Z_t is the stochastic horizontal lift of X_t in relation to $\mathbf{H}^r$.

ii) Let $E = E(M, \rho, F)$ be a vector bundle associated to $P(M, G)$ with fibre F, $\mathbf{H} = \{H_p : p \in P\}$ a 1-connection of $P(M, G)$ and ∇^E the CDO of E induced by $\mathbf{H}$. Let $Y_t(p)$ be the stochastic horizontal lift of $X_t(\pi p)$ in relation to $\mathbf{H}^r$, and $\eta_t(\pi p) = Y_t(p) \circ p^{-1} : E_{\pi p} \longrightarrow E_{X_t(\pi p)}$, where p is regarded as linear mapping $p : F \longrightarrow E_{\pi p}$. We have the following Itô formula for cross sections of E,

$$\begin{aligned} \eta_t(x)^{-1}\sigma(X_t(x)) - \sigma(x) &= \sum_{i=1}^{n} \int_0^t \eta_s(x)^{-1}\nabla^E_{A_i}\sigma(X_s(x))dB_s^i + \int_0^t \eta_s(x)^{-1} \\ &\quad \left(\nabla^E_{A_0} + \sum_{i=1}^{n} \tfrac{1}{2}\left(\left(\nabla^E_{A_i}\right)^2 - \tfrac{1}{2}\overline{\omega^H(\nabla_{HA_i}HA_i)}\right)\right)\sigma(X_s(x))ds \end{aligned}$$

Where σ is a cross section of E and $\overline{} : \mathcal{G} \longrightarrow \Gamma(TE)$ is the vertical homomorphism defined by $\overline{A}_e = \frac{d}{dt}\big|_{t=0} p \exp tA \cdot p^{-1}(f)$.

iii) Let BM be the principal fiber bundle of bases of M, ∇ a CDO of M, $\mathbf{H} = \{H_p : p \in P\}$ the 1-connection of BM associated with ∇. We have that ∇^C and ∇^H are projectable with projection ∇. Since $\nabla^H_{HX}HY = H(\nabla_X Y) + \frac{1}{2}\widetilde{R(X,Y)}$, where $R(X, Y)$ is the tensor of type $(1, 1)$ defined by $R(X, Y)(Z) = R(X, Y)Z$ (R is the curvature tensor associated with ∇) and $\widetilde{R(X,Y)}$ is the vertical right invariant vector field of BM defined by

$\widetilde{R(X,Y)}_p = (p^{-1}R(X,Y)p)^*_p$ we have that the stochastic horizontal lift $Y_t(p)$ of $X_t(x)$ (solution of (5)) in relation to $\mathbf{H^r}^{\ H}$ satisfies

$$\begin{aligned} dY_t &= HA_0(Y_t)dt + \sum_{i=1}^{n} HA_i(Y_t) \circ dB_t^i \\ Y_0 &= p \end{aligned}$$

In the case of ∇^C we have that $\nabla^C_{HX} HY = H(\nabla_X Y) + \widetilde{R(-,X)Y}$, where $R(-,X)Y$ is the tensor of type $(1,1)$ defined by $R(-,X)Y(Z) = R(Z,X)Y$ and $\widetilde{R(-,X)Y}$ is the vertical right invariant vector field of BM defined by $\widetilde{R(-,X)Y}_p = (p^{-1}R(-,X)Yp)^*_p$([3, page 94]).

The stochastic horizontal lift $Y_t(p)$ of $X_t(x)$ (solution of (5)) in relation to $\mathbf{H^r}^{\ C}$ satisfies

$$\begin{aligned} dY_t &= \left(HA_0 - \tfrac{1}{2}\sum_{i=1}^{n} \widetilde{R(-,A_i)A_i}\right)(Y_t)dt + \sum_{i=1}^{n} HA_i(Y_t) \circ dB_t^i \\ Y_0 &= p \end{aligned}$$

References

[1] P. Catuogno: *Second Order Connections and Stochastic Calculus.* Relatorio de pesquisa 31/95. IMECC. UNICAMP, 1995.

[2] P. Catuogno: *Composition and Factorization of Diffusions on Principal Fiber Bundles.* 4^{to} Congreso Antonio Monteiro. UNS. Bahia Blanca. Argentina, 1997.

[3] L.A. Cordero, C. T. J. Dobson, M. de León: Differential Geometry of Frame Bundles. Kluwer Academic Publishers, 1989.

[4] M. Emery: Stochastic Calculus in Manifolds. Springer-Verlag, 1989.

[5] M. Emery: *On Two Transfer Principles in Stochastic Differential Geometry.* Sèminaire de Probabilitès XXIV. Lecture Notes in Mathematics **1426**, Springer 1990.

[6] K. Itô: *The Brownian Motion and Tensor Fields on Riemannian Manifolds.* Proc. Inter. Congr. Math. (Stockholm), 536-539, 1962.

[7] S. Kobayashi, K. Nomizu: Foundations of Differential Geometry. Interscience. vol.**1** ,1963.vol. **2**, 1968.

[8] P.A. Meyer: *Géométrie Différentielle Stochastique.* Séminaire de Probabilités XV. Lecture Notes in Mathematics **851**, Springer 1981.

[9] P.A. Meyer: *Géométrie Différentielle Stochastique (bis).* Séminaire de Probabilités XVI. Lecture Notes in Mathematics **921**, Springer 1982.

[10] L. Schwartz: *Géométrie Différentielle du 2e ordre, Semimartingales et Équations Différentielle Stochastiques sur une Variété Différentielle.* Séminaire de Probabilités XVI. Lecture Notes in Mathematics **921**, Springer 1982.

[11] I. Shigekawa: *On Stochastic Horizontal Lifts.* Z. Wahrscheinlichkeitsheorie verw. Geviete **59**, 211-221, 1982.

Recibido en Abril 1998