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AN INTEGRAL EQUATION FOR A STEFAN PROBLEM WITH MANY PHASES AND A SINGULAR SOURCE

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ABSTRACT. We discuss the self-similar solutions $\theta(x,t) = \theta(\eta) = \theta(x/\sqrt{t})$ of the problem

$$E(\theta)_t - A(\theta)_{xx} = \tfrac{1}{t} B(\eta),\ \eta > 0\ ;\quad \theta(0,t) = C > 0,\ t > 0;\quad E(\theta(x,0)) = 0,\ x > 0.$$

We assume that E and A are monotone increasing functions, A being continuous, with $E(0) = A(0) = 0$ and $\lambda = E(0^+) > 0$. This equation can describe the conservation of thermal energy in a heat conduction process for a semi-infinite material with a "self-similar" source or sink term of the type $B\left(x/\sqrt{t}\right)/t$. Moreover, $E(\theta)$ represents energy per unit volume at level (temperature) θ, $A'(\theta) \geq 0$ is the thermal conductivity and $B(\eta)/t$ represents a singular source or sink depending of the sign of the function B. We generalize results obtained in:

(i) J.L. Menaldi – D.A. Tarzia, Comp. Appl. Math. 12 (1993), 123 – 142, for the particular one-phase case $E(\theta) = \theta + \lambda \quad (\theta > 0)$, $E(0) = 0$ and $A(\theta) = \theta$ is studied where necessary and sufficient conditions were given in order to characterize the source term B which provides a unique solution (a generalized Lamé-Clapeyron solution).

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KEY WORDS and PHRASES: Stefan problem, Free boundary problem, Self-similar solutions, Integral equations, Singular source.

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(ii) J.E. Bouillet, IMA Preprints # 230, Univ. Minnesota (March 1986), for the particular case $B \equiv 0$ for the two-phase Stefan problem, and we obtain some new results in connection to the source term.

We obtain for the inverse function $\eta = \eta(\theta)$ an integral equation equivalent to the above problem and we prove that for certain hypothese over data there exists at least a solution of the corresponding integral equation.

I. INTRODUCTION

In this paper, we discuss the self-similar solutions $\theta(x,t) = \theta(\eta) = \theta(x/\sqrt{t})$ of the equation

$$E(\theta)_t - A(\theta)_{xx} = \tfrac{1}{t} B(\eta), \quad \eta > 0,$$

with the initial and boundary condition given by:

$$\theta(0,t) = C > 0, \qquad t > 0 \quad ; \qquad E(\theta(x,0)) = 0, \qquad x > 0.$$

We assume that E and A are monotone increasing functions, A being continuous, with $E(0) = A(0) = 0$ and $\lambda = E(0^+) > 0$. This equation can describe the conservation of thermal energy in a heat conduction process for a semi-infinite material with a "self-similar" source or sink term of the type $B\left(x/\sqrt{t}\right)/t$. Moreover, $E(\theta)$ represents energy per unit volume at level (temperature) θ, $A'(\theta) \geq 0$ is the thermal conductivity and $B(\eta)/t$ represents a singular source or sink depending of the sign of the function B. A study of sublimation-dehydration within a porous medium as a result of volumetric heating, such as that associated with microwave heating, is presented in [4].

In [3] the particular one-phase case $E(\theta) = \theta + \lambda \quad (\theta > 0)$, $E(0) = 0$ and $A(\theta) = \theta$ is studied where necessary and sufficient conditions are given in order to characterize the source term B which provides a unique solution (a generalized Lamé-Clapeyron solution [2]).

In this paper we follow the analysis presented in [1] for the case $B \equiv 0$, and we obtain some new results in connection to the source term.

II. SELF-SIMILAR SOLUTION

We consider the following problem for the function $\theta = \theta(\eta)$, where $\eta = x/\sqrt{t}$ is the similarity variable:

(1) $$-\tfrac{1}{2}\eta\,(E(\theta))' - (A(\theta))'' = B(\eta), \qquad \eta > 0,$$

(2) $$\theta(0) = C > 0, \qquad E(\theta(+\infty)) = 0,$$

where the prime denotes $\frac{d}{d\eta}$.

If we define the function

$$(3)' \qquad h(\eta) = (A(\theta(\eta)))' + \tfrac{1}{2}\eta E(\theta(\eta)),$$

the differential equation (1) can be written as

$$(4) \qquad h'(\eta) = \tfrac{1}{2}E(\theta(\eta)) - B(\eta).$$

From a distributional version of equation (1) it is easy to see that both $A(\theta(\eta))$ and $h(\eta)$ defined by (3) are absolutely continuous (AC) functions of $\theta > 0$, and equation (4) is satisfied almost everywhere (recall that no asumptions on regularity are made on the monotone functions E and A). We shall assume that side derivatives always exist at $\eta > 0$ in order to avoid technical details which are not essential and will obscure this exposition (cf. [1]).

Observe that if thermal energy is suplied to the medium $x > 0$ (initially at $\theta = 0$) by

(i) heat conduction from $x = 0$, and

(ii) the source $\frac{B(\eta)}{t} \geq 0$ for $t > 0$,

therefore it is natural to expect that $\theta(x,t)$ be zero for $0 < t < t(x)$ while the local energy per unit volume at $x > 0$ grows from $E(\theta(x,0)) = 0$ according to the law

$$(5) \qquad E(\theta(x,0)) := \int_0^t \frac{B\left(x/s^{\frac{1}{2}}\right)}{s}\, ds = 2\int_{x/\sqrt{t}}^{\infty} \frac{B(\eta)}{\eta}\, d\eta,$$

and remains less than $\lambda = E(0^+)$.

The notation $E(\theta(x,t)) = E(\theta(\eta))$ is inadequate in this case $B \geq 0$, and we clearly have in mind a multivalued graph for E, where $E(0) = [0,\lambda]$. This fact is seen in the integral equation (6)-(7) whose fixed point gives a solution to our problem.

Lemma 1. *We obtain the following result for the solution of the problem* (1), (2) *according to the sign of the source/sink term* B.

(i) *If* $B(\eta) \leq 0$, $\forall \eta \geq 0$, *then* $\theta = \theta(\eta)$ is a non increasing function of η.

(ii) If $B(\eta) \geq 0$, $\forall \eta \geq 0$, *and the solution verifies the condition* $\theta'(0^+) < 0$, *then* $\theta = \theta(\eta)$ *is a non increasing function of* η.

Proof. Case (i). If this were not the case, there would exist $0 \leq \eta_1 < \eta_2$ and $D > 0$, such that

$\theta(\eta_1) = \theta(\eta_2) = \mathrm{D}$, $\theta(\eta) > \mathrm{D}$ in (η_1, η_2) and therefore

$$(\mathrm{A} \circ \theta)'(\eta_1^+) \geq 0, \qquad (\mathrm{A} \circ \theta)'(\eta_2^-) \leq 0.$$

Employing (3), (4) and $[\mathrm{A}(\mathrm{D})' + \frac{1}{2}\eta\,\mathrm{E}(\mathrm{D})]' = \frac{1}{2}\mathrm{E}(\mathrm{D})$ we obtain by substraction

$$\left\{(\mathrm{A} \circ \theta)' + \tfrac{1}{2}\eta\,[\mathrm{E}(\theta(\eta)) - \mathrm{E}(\mathrm{D})]\right\}' = \tfrac{1}{2}\,[\mathrm{E}(\theta(\eta)) - \mathrm{E}(\mathrm{D})] - \mathrm{B}(\eta)$$

integrating in (η_1, η_2) gives

$$0 \geq (\mathrm{A} \circ \theta)'(\eta_2^-) - (\mathrm{A} \circ \theta)'(\eta_1^+) \geq \tfrac{1}{2}\int_{\eta_1}^{\eta_2} [\mathrm{E}(\theta(\eta)) - \mathrm{E}(\mathrm{D})]\,d\eta > 0,$$

a contradiction.

Case (ii). Again by contradiction, if $\theta(\eta)$ were not monotone there would be a $\mathrm{D} > 0$, $\mathrm{D} \leq \mathrm{C}$, and $0 \leq \eta_1 < \eta_2$, such that $\theta(\eta_1) = \theta(\eta_2) = \mathrm{D}$, $\theta(\eta) < \mathrm{D}$ in (η_1, η_2) and thus, with an argument similar to Case (i) we would have

$$0 \leq (\mathrm{A} \circ \theta)'(\eta_2^-) - (\mathrm{A} \circ \theta)'(\eta_1^+) \geq \tfrac{1}{2}\int_{\eta_1}^{\eta_2} [\mathrm{E}(\theta(\eta)) - \mathrm{E}(\mathrm{D})]\,d\eta - \int_{\eta_1}^{\eta_2} \mathrm{B}(\eta)\,d\eta < 0,$$

a contradiction.

In both cases we conclude that $\theta(\eta)$ is a decreasing function of η due to the fact that $\lim_{\eta \to +\infty} \theta(\eta) = 0$.

When the thesis of Lemma 1 is verified we can consider the inverse function $\eta = \eta(\theta)$ for $0 < \theta < \mathrm{C}$, which satisfies the following property

Theorem 2. *Assume that* $\int_1^{+\infty} \frac{|\mathrm{B}(s)|}{s}\,ds < +\infty$. *For the inverse function* $\eta = \eta(\theta)$ *we have the integral equation equivalent to* (1) – (4):

$$(6) \qquad \eta(\theta) = \mathrm{T}(\eta)(\theta), \qquad \theta \in (0, \mathrm{C}),$$

where the operator T is defined by

(7) $$(T(\eta(\theta)))^2 = 2\int_{\theta}^{C} \left\{ \tfrac{1}{2} E(\psi) - \int_{\eta(\psi)}^{+\infty} \frac{B(s)}{s}\, ds + \int_{0}^{\psi} \frac{dA(r)}{\eta^2(r)} \right\}^{-1} dA(\psi)\,.$$

Proof. We have

$$h'(\eta) = \tfrac{1}{2}E(\theta) - B(\eta) = \frac{h(\eta)}{\eta} - \frac{(A\circ\theta)'(\eta)}{\eta} - B(\eta)\,,$$

hence

$$\eta\left(\frac{h(\eta)}{\eta}\right)' = -\frac{(A\circ\theta)'(\eta)}{\eta} - B(\eta)\,,$$

that is

$$\left(\frac{h(\eta)}{\eta}\right)' = -\frac{(A\circ\theta)'(\eta)}{\eta^2} - \frac{B(\eta)}{\eta}.$$

Assuming for the moment that $\frac{h(\eta)}{\eta} \to 0$ when $\eta \to +\infty$, we have:

$$-\frac{h(\eta)}{\eta} = -\int_{\eta}^{\infty} \frac{(A\circ\theta)'(s)}{s^2}\, ds - \int_{\eta}^{\infty} \frac{B(s)}{s}\, ds\ ,$$

and hence

$$\frac{(A\circ\theta)'(\eta)}{\eta} - \tfrac{1}{2}E(\theta) = \int_{\eta}^{\infty} \frac{(A\circ\theta)'(s)}{s^2}\, ds + \int_{\eta}^{\infty} \frac{B(s)}{s}\, ds$$

$$\frac{d(A\circ\theta)}{\eta} = \frac{(A\circ\theta)'(\eta)}{\eta}\, d\eta = -\left\{ \tfrac{1}{2}E(\theta) - \int_{\eta(\theta)}^{+\infty} \frac{B(s)}{s}\, ds + \int_{0}^{\theta} \frac{dA(\psi)}{\eta^2(\psi)} \right\} d\eta$$

whence

$$\eta^2(\theta) = 2\int_{\theta}^{C} \left\{ \tfrac{1}{2} E(\psi) - \int_{\eta(\psi)}^{+\infty} \frac{B(s)}{s}\, ds + \int_{0}^{\psi} \frac{dA(r)}{\eta^2(r)} \right\}^{-1} dA(\psi)\,,$$

i.e. (6), (7).

We now prove that $h(\eta)/\eta \to 0$ when $\eta \to +\infty$ (under the hypothesis $\int_0^{+\infty} B(s)\, ds < +\infty$ for $B \geq 0$). From (4) it follows

$$h(\eta) = h(0^+) + \tfrac{1}{2}\int_{0}^{\eta} E(\theta(s))\, ds - \int_{0}^{\eta} B(s)\, ds\ ,$$

i.e.

$$(A \circ \theta)'(\eta) + \tfrac{1}{2}\eta\, E(\theta(\eta)) = (A \circ \theta)'(0^+) + \tfrac{1}{2}\int_0^\eta E(\theta(s))\,ds - \int_0^\eta B(s)\,ds\,.$$

Hence

$$0 = (A \circ \theta)'(\eta) = (A \circ \theta)'(0^+) + \tfrac{1}{2}\int_0^\eta [\,E(\theta(s)) - E(\theta(\eta))\,]\,ds - \int_0^\eta B\,ds\,.$$

Therefore, if $B \le 0$ or if $B \ge 0$ and $\int_0^\infty B(s)\,ds < \infty$, we will have $0 \ge (A \circ \theta)'(\eta) \ge$ constant. The claim now follows due to $\lim_{\eta \to +\infty} E(\theta(\eta)) = 0$. More precisely if $\lim_{\eta \to +\infty} \frac{1}{\eta}\int_0^\eta B(s)\,ds = 0$, we have

$$\frac{h(\eta)}{\eta} = \frac{h(0^+)}{\eta} + \frac{1}{2}\frac{1}{\eta}\int_0^\eta E(\theta(s))\,ds - \frac{1}{\eta}\int_0^\eta B(s)\,ds$$

and therefore $\lim_{\eta \to +\infty} \frac{h(\eta)}{\eta} = 0$, due again to the fact that $E(\theta(+\infty)) = 0$.

Theorem 3. (i) *If* $B(\eta) \le 0$, $\forall\, \eta \ge 0$ *then*

(9) $$T(\eta(\theta)) \le 2\sqrt{\frac{A(c)}{\lambda}},$$

and T *is a monotone operator in the following sense*

(10) $$\eta_1(\theta) \le \eta_2(\theta) \;\Rightarrow\; T(\eta_1(\theta)) \le T(\eta_2(\theta))\,.$$

(ii) *If* $B(\eta) \le 0$, $\forall\, \eta \ge 0$, *and*

(11) $$-\int_0^{+\infty} \frac{B(s)}{s}\,ds = K < +\infty,$$

then, there exists a function $\eta_0 = \eta_0(\theta)$ *which verifies the condition*

(12) $$\eta_0 \le T(\eta_0)\,.$$

Moreover, η_0 *is given by*

$$\eta_0(\theta) = \mu\,[A(C) - A(\theta)] > 0 \quad \text{in } (0, C),$$

where $\mu > 0$ is a parameter to be chosen so that

(14) $$0 < \mu^2 \le \frac{1}{A(C)\,[K + \frac{1}{2}E(C)]} \,.$$

(iii) *Under the previous hypotheses there exists at least a solution $\eta = \eta(\theta)$ of the integral equation*
(6)-(7).

Proof. (i) It is clear that

$$(T\eta(\theta))^2 \le 2 \int_\theta^C \frac{dA(\psi)}{\frac{1}{2}E(\psi)} = 4 \int_\theta^C \frac{dA(\psi)}{E(\psi)} \le 4\, \frac{A(C)}{E(0^+)} \,.$$

The monotonicity is obvious.

(ii) By (11) and (13)

$$(T\eta_0(\theta))^2 \ge 2 \int_\theta^C \Big(\tfrac{1}{2}E(\psi) + K + \frac{1}{\mu^2}\Big(\frac{1}{A(C) - A(\psi)} - \frac{1}{A(C)} \Big)\Big)^{-1} dA(\psi) \,.$$

Select μ so that $\frac{1}{2}E(\psi) + K \le \frac{1}{\mu^2\, A(C)}$, (i.e. (14)). We find

$$(T\eta_0(\theta))^2 \ge 2\,\mu^2 \int_\theta^C (A(C) - A(\psi))\, dA(\psi) = (\eta_0(\theta))^2 \,.$$

Therefore $\eta_0 \le T\eta_0 \le T^2\eta_0 \le \ldots \le 2\Big(\int_\theta^C \frac{dA(\psi)}{E(\psi)}\Big)^{\frac{1}{2}} \le 2\Big(\frac{A(C)}{E(0^+)}\Big)^{\frac{1}{2}}$. It follows that there exists the pointwise limit $\eta(\theta) = \lim_{\eta\to\infty} (T^m\eta_0)(\theta)$. Repeated application of the monotone convergence theorem to the integrals in the definition of T gives $\eta(\theta) = T(\eta(\theta))$, $0 < \theta \le C$.

Theorem 4. *There is at most one solution to* (1) – (4) *that is monotone decreasing in* $(0, +\infty)$.

Proof. Assume $\theta_1(\eta)$, $\theta_2(\eta)$ are two such solutions. Two cases are posible:

(i) There are $0 \le \eta_1 \le \eta_2$ such that $\theta_1(\eta_1^+) = \theta_2(\eta_1^+)$

$$\theta_1(\eta_2^-) = \theta_2(\eta_2^-), \qquad \theta_1(\eta) < \theta_2(\eta) \text{ in } (\eta_1, \eta_2) \,;$$

(ii) for $0 \leq \eta_1 < \eta$, $\theta_1(\eta_1^+) = \theta_2(\eta_1^+)$ and $\theta_1(\eta) < \theta_2(\eta)$.

Case (i). By (4),

$$h_i(\eta_2^-) - h_i(\eta_1^-) = \frac{1}{2}\int_{\eta_1}^{\eta_2} E(\theta_i(\eta))\, d\eta - \int_{\eta_1}^{\eta_2} B(\eta)\, d\eta\ , \quad i = 1, 2.$$

Substracting them we get

$$\begin{aligned} 0 \ &\geq (A \circ \theta_2)'(\eta_2^-) - (A \circ \theta_1)'(\eta_2^-) - ((A \circ \theta_2)'(\eta_1^+) - (A \circ \theta_1)'(\eta_1^+)) + \\ &+ \tfrac{1}{2}\eta_2 (E(\theta_2(\eta_2^-)) - E(\theta_1(\eta_2^-))) + \tfrac{1}{2}\eta_1 (E(\theta_2(\eta_1^+)) - E(\theta_1(\eta_1^+))) = \\ &\frac{1}{2}\int_{\eta_1}^{\eta_2} [E(\theta_2(\eta)) - E(\theta_1(\eta))]\, d\eta > 0 \ . \end{aligned}$$

Case (ii). With analogous considerations ($\eta_2 = +\infty$)

$$(A \circ \theta_2)'(\eta_1^+) - (A \circ \theta_1)'(\eta_1^+)) + \tfrac{1}{2}\eta_1 (E(\theta_2(\eta_1^+)) - E(\theta_1(\eta_1^+))) =$$

$$= \frac{1}{2}\int_{\eta_1}^{\eta_2} [E(\theta_2(\eta)) - E(\theta_1(\eta))]\, d\eta > 0 .$$

Therefore, the assumptions $\theta_1 \neq \theta_2$ leads to contradiction.

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