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A NEW SEQUENCE OF LINEAR POSITIVE OPERATORS FOR HIGHER ORDER L_p-APPROXIMATION

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ABSTRACT. The purpose of this paper is to develop some direct results in the L_p-approximation by a linear combination of a new sequence of linear positive operators. We estimate the error in the approximation in terms of the higher order integral modulus of smoothness using the properties of another sequence of linear approximating functions e.g. Steklov means.

1 INTRODUCTION

For $f \in L_p[0,\infty)$ $(p \geq 1)$, Agrawal and Thamer [1] introduced a new sequence of linear positive operators defined as:

$$(1.1) \quad M_n(f;x) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k-1}(u)\, f(u)\, du + (1+x)^{-n} f(0), x \in [0,\infty),$$

where

$$p_{n,k}(u) = \binom{n+k-1}{k} u^k (1+u)^{-(n+k)} .$$

Alternatively, (1.1) may be written as

$$M_n(f(u);x) = \int_0^{\infty} W_n(x,u) f(u) du,$$

where

$$W_n(x,u) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x) p_{n,k-1}(u) + (1+x)^{-n} \delta(u),$$

$\delta(u)$ being the Dirac-delta function.
We observe that, howsoever smooth the function may be, the order of approximation by these operators is, at its best, $O(n^{-1})$. To improve this order of approximation, we apply the technique of linear combination due to May [5] and Rathore [6] to (1.1). The *linear*

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combination is described as follows:

$$(1.2)\qquad M_n(f,k,x)=\begin{vmatrix}1 & d_0^{-1} & \cdots & d_0^{-k}\\ 1 & d_1^{-1} & \cdots & d_1^{-k}\\ \cdots & \cdots & \cdots & \cdots\\ 1 & d_k^{-1} & \cdots & d_k^{-k}\end{vmatrix}^{-1}\begin{vmatrix}M_{d_0 n}(f,x) & d_0^{-1} & \cdots & d_0^{-k}\\ M_{d_1 n}(f,x) & d_1^{-1} & \cdots & d_1^{-k}\\ \cdots & \cdots & \cdots & \cdots\\ M_{d_k n}(f,x) & d_k^{-1} & \cdots & d_k^{-k}\end{vmatrix},$$

where $d_o, d_1, ..., d_k$ are $(k+1)$ arbitrary, fixed and distinct positive integers.
On a simplification of (1.2), we are led to

$$(1.3)\qquad M_n(f,k,x)=\sum_{j=0}^{k} C(j,k)\, M_{d_j n}(f;x)\ ,$$

where

$$C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i},\ \text{for}\ \ k\neq 0\ \text{and}\ C(0,0)=1.$$

The aim of this paper is to show that the linear combinations of the operators (1.1) converge faster to the function provided the function is sufficiently smooth . The estimate of error in L_p-approximation is obtained in terms of the $(2k+2)$th order integral modulus of smoothness of the function.

2 HIGHER ORDER L_p-APPROXIMATION

Throughout this paper, let $0<a_1<a_3<a_2<b_2<b_3<b_1<\infty$, $I_i=[a_i,b_i]$, i =1,2,3 and $[\beta]$ denote the integral part of β. Furthermore, C denotes a positive constant not necessarily the same at each occurrence.
For $f\in L_p[a,b]$, $1\le p<\infty$, the integral modulus of smoothness of order m is defined as:

$$\omega_m(f,\tau,p,[a,b])=\sup_{0<\delta\le\tau}\left\|\Delta_\delta^m f(t)\right\|_{L_p[a,b-m\delta]}.$$

We prove the following main result:
THEOREM. Let $f\in L_p[0,\infty)$, $p\ge 1$. Then for all n sufficiently large

$$\left\|M_n(f,k,.)-f\right\|_{L_p(I_2)}\le M_{k,p}\left(\omega_{2k+2}(f,n^{-1/2},p,I_1)+n^{-(k+1)}\|f\|_{L_p[0,\infty)}\right),$$

where $M_{k,p}$ is a constant that depends on k and p but is independent of f and n.
We require the following results.
Let $A.C.([a,b])$ denote the class of absolutely continuous function on $[a,b]$.
Let $1\le p<\infty$, $f\in L_p[0,\infty)$. Then, for sufficiently small $\eta>0$, the Steklov mean $f_{\eta,m}$ of m th order corresponding to f is defined as

$$f_{\eta,m}(u)=\eta^{-m}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}\left(f(u)+(-1)^{m-1}\Delta^{m}_{\sum_{i=1}^{m}u_i}f(u)\right)du_1\cdots du_m,$$

where $u\in I_1$ and $\Delta_h^m f(u)$ is the mth order forward difference of the function f with step length h.

Lemma 2.1. For the function $f_{\eta,m}(u)$ as defined above, we have

(a) $f_{\eta,m}$ has derivatives up to order m over I_1, $f_{\eta,m}^{(m-1)}\in A.C.(I_1)$ and $f_{\eta,m}^{(m)}$ exists a.e. and belongs to $L_p(I_1)$;

(b) $\left\|f_{\eta,m}^{(r)}\right\|_{L_p(I_2)}\le M_r\eta^{-r}\omega_r(f,\eta,p,I_1)$, $r=1(1)m$;

(c) $\left\|f-f_{\eta,m}\right\|_{L_p(I_2)}\le M_{m+1}\omega_m(f,\eta,p,I_1)$;

(d) $\left\|f_{\eta,m}\right\|_{L_p(I_2)}\le M_{m+2}\|f\|_{L_p(I_1)}$;

(e) $\left\|f_{\eta,m}^{(m)}\right\|_{L_p(I_2)}\le M_{m+3}\eta^{-m}\|f\|_{L_p(I_1)}$,

where M_i's are certain constants that depend on i but are independent of f and η.

By a repeated application of [4,Theorem 18.17], Jensen's inequality and Fubini's theorem, the proof of this lemma easily follows and hence is omitted.

Lemma 2.2 [1]. Let the function $T_{n,m}(x)$, $m\in N^0$ (the set of nonnegative integers) be defined as

$$T_{n,m}(x)=(n-1)\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k-1}(t)\,(t-x)^m\,dt+(-x)^m(1+x)^{-n}.$$

Then,

$$T_{n,0}(x)=1\ ,\ T_{n,1}(x)=\frac{2x}{n-2}\ \text{and}$$

$$(n-m-2)T_{n,m+1}(x)=x(1+x)T'_{n,m}(x)+[(2x+1)m+2x]T_{n,m}(x)+2mx(1+x)T_{n,m-1}(x),$$

$m\in N$.

Hence,

(i) $T_{n,m}(x)$ is a rational function in n and a polynomial in x of degree m,

(ii) For every $x\in[0,\infty)$, $T_{n,m}(x)=O(n^{-[(m+1)/2]})$.

Lemma 2.3. Let the function $V_{n,m}(t)(m\in N)$ be defined as

$$V_{n,m}(t)=\int_0^{\infty}W_n(x,t)\,(x-t)^m\,dx\ ,\ t\in(0,\infty).$$

Then, we have

$$V_{n,0}(t)=1\ ,\ V_{n,1}(t)=\frac{2(1+t)}{n-2}$$

and there holds the following recurrence relation
$(n-m-2)\, V_{n,m+1}(t) = t(1+t)\, V'_{n,m}(t) + [(2t+1)(m+1)+1]\, V_{n,m}(t) + 2mt(1+t)\, V_{n,m-1}(t)$.
Consequently,
(i) $V_{n,m}(t)$ is a rational function in n and a polynomial in t of degree m.
(ii) For every $t \in (0,\infty)$, $V_{n,m}(t) = O(n^{-[(m+1)/2]})$.
The proof of this lemma follows easily on proceeding along the lines of the proof of Lemma 2.2 and hence is omitted.
Lemma 2.4. For $p \in N$ and n sufficiently large there holds

$$M_n\left((u-x)^p, k, x\right) = n^{-(k+1)}\{Q(p,k,x) + o(1)\},$$

where $Q(p,k,x)$ is a certain polynomial in x of degree p and $x \in [0,\infty)$ is arbitrary but fixed. Further, we have

$$Q(2k+1,k,x) = \frac{(-1)^k (2k+1)!}{k!\prod_{j=0}^{k} d_j}\{(k+1)(1+2x)-1\}\{x(1+x)\}^k$$

and

$$Q(2k+2,k,x) = \frac{(-1)^k (2k+2)!\{x(1+x)\}^{k+1}}{(k+1)!\prod_{j=0}^{k} d_j}.$$

The proof of this lemma follows from [2,Theorem 1] .
Let $B.V.([a,b])$ denote the class of functions of bounded variation on $[a,b]$.The seminorm $\|f\|_{B.V.([a,b])}$ is defined by the total variation of f on $[a,b]$.
Lemma 2.5. Let $f \in B.V.(I_1)$. Then,

$$\left\| M_n\left(\int_x^u (u-w)^{2k+1}\, df(w)\,\phi(u), x\right)\right\|_{L_1(I_2)} \le C\, n^{-(k+1)} \| f \|_{B.V.(I_1)},$$

where $\phi(u)$ is the characteristic function of I_1.
Proof. Following the proof of Proposition 2.2.5 [7, p. 50-52], for each n there exists a nonnegative integer $r = r(n)$ such that $rn^{-1/2} < \max(b_1 - a_2, b_2 - a_1) \le (r+1)n^{-1/2}$. Then, we have

$$K \equiv \left\| M_n\left(\int_x^u (u-w)^{2k+1}\, df(w)\,\phi(u), x\right)\right\|_{L_1(I_2)}$$

$$\le \sum_{l=0}^{r} \int_{a_2}^{b_2} \left[\int_{x+ln^{-1/2}}^{x+(l+1)n^{-1/2}} \phi(u) W_n(x,u)\, |u-x|^{2k+1} \left\{ \int_x^{x+(l+1)n^{-1/2}} \phi(w) |df(w)| \right\} du \right.$$

$$+\int_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}} \phi(u)\, W_n(x,u)|u-x|^{2k+1}\left\{\int_{x-(l+1)n^{-1/2}}^{x} \phi(w)|df(w)|\right\} du \Bigg] dx .$$

Let $\phi_{x,c,d}(w)$ denote the characteristic function of the interval $[x-cn^{-1/2}, x+dn^{-1/2}]$ where c,d are nonnegative integers. Then, we have

$$K \le \sum_{l=1}^{r}\Bigg[l^{-4}n^{2}\int_{a_2}^{b_2}\left(\int_{x+ln^{-1/2}}^{x+(l+1)n^{-1/2}} \phi(u)\, W_n(x,u)|u-x|^{2k+5}\left(\int_{a_1}^{b_1}\phi_{x,0,l+1}(w)\,|df(w)|\right) du\right.$$

$$\left.+\int_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}} \phi(u)\, W_n(x,u)|u-x|^{2k+5}\left(\int_{a_1}^{b_1}\phi_{x,l+1,0}(w)\,|df(w)|\right) du\right) dx\Bigg]$$

$$+\int_{a_2}^{b_2}\int_{-n^{-1/2}}^{a_1+n^{-1/2}} \phi(u) W_n(x,u)|u-x|^{2k+1}\left(\int_{a_1}^{b_1}\phi_{x,l,1}(w)\,|df(w)|\right) du\, dx .$$

By using Lemma 2.2 and Fubini's theorem we obtain

$$K \le C n^{-(2k+1)/2}\Bigg[\sum_{l=1}^{r} l^{-4}\Bigg\{\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\phi_{x,0,l+1}(w)\,dx\right)|df(w)| + \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\phi_{x,l+1,0}(w)\,dx\right.$$

$$\left.\times\, |df(w)|\right)\Bigg\} + \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\phi_{x,l,1}(w)\,dx\right)|df(w)|\Bigg]$$

$$\le C n^{-(k+1)}\|f\|_{B.V.(I_1)}$$

This completes the proof of the Lemma.

For $1 \le p < \infty$, let $L_p^{(2k+2)}(I_1) := \{f \in L_p[0,\infty) : f^{(2k+1)} \in A.C.(I_1)$ and $f^{(2k+2)} \in L_p(I_1)\}$.

For $f \in L_p[a,b]$, $1 < p < \infty$, the Hardy-Littlewood majorant [8, p.244] of f is defined as:

$$h_f(x) = \sup_{\xi \ne x} \frac{1}{\xi - x}\int_x^{\xi} f(t)\,dt, \quad (a \le \xi \le b) .$$

In order to prove our main result, we first discuss the approximation in the smooth subspace $L_p^{(2k+2)}(I_1)$ of $L_p[0,\infty)$.

Proposition. Let $1 < p < \infty$ and $f \in L_p^{(2k+2)}(I_1)$, then for all n sufficiently large

(2.1) $$\|M_n(f,k,.)-f\|_{L_p(I_2)} \le C_1\, n^{-(k+1)}\left\{\left\|f^{(2k+2)}\right\|_{L_p(I_1)} + \|f\|_{L_p[0,\infty)}\right\},$$

where $C_1 = C_1(k,p)$.

If $f \in L_1[0,\infty)$, f has $2k+1$ derivatives in I_1 with $f^{(2k)} \in A.C.(I_1)$ and $f^{(2k+1)} \in B.V.(I_1)$, then for all n sufficiently large

(2.2) $\|M_n(f,k,.)-f\|_{L_1(I_2)} \le C_2 n^{-(k+1)} \left\{ \left\|f^{(2k+1)}\right\|_{B.V.(I_1)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)} \right\}$,

where $C_2 = C_2(k)$.

Proof. Let $p > 1$. With the given assumptions on f, for $x \in I_2$ and $u \in I_1$ we can write

$$f(u) = \sum_{j=0}^{2k+1} \frac{(u-x)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!} \int_x^u (u-w)^{2k+1} f^{(2k+2)}(w)dw .$$

Hence, if $\phi(u)$ is the characteristic function of I_1, we have

$$f(u) = \sum_{j=0}^{2k+1} \frac{(u-x)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!} \int_x^u (u-w)^{2k+1} \phi(u) f^{(2k+2)}(w)dw$$
$$+ F(u,x)(1-\phi(u)),$$

where $F(u,x) = f(u) - \sum_{j=0}^{2k+1} \frac{(u-x)^j}{j!} f^{(j)}(x)$, for all $u \in [0,\infty)$ and $x \in I_2$.

In view of $M_n(1,k,x) = 1$, we obtain

$$M_n(f,k,x) - f(x) = \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j!} M_n\left((u-x)^j, k, x\right) + \frac{1}{(2k+1)!} M_n\left(\phi(u) \int_x^u (u-w)^{2k+1} \right.$$
$$\left. \times f^{(2k+2)}(w)dw, k, x \right) + M_n\left(F(u,x)(1-\phi(u)), k, x\right)$$
$$= \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say.}$$

It follows from Lemma 2.2 and [3,p.5] that

$$\|\Sigma_1\|_{L_p(I_2)} \le Cn^{-(k+1)} \left(\|f\|_{L_p(I_2)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)} \right).$$

To estimate Σ_2, let h_f be the Hardy-Littlewood majorant of $f^{(2k+2)}$ on I_1. Making use of Hölder's inequality and Lemma 2.2, we have

$$J_1 \equiv \left| M_n\left(\phi(u) \int_x^u (u-w)^{2k+1} f^{(2k+2)}(w)dw, x \right) \right|$$
$$\le M_n\left(\phi(u)|u-x|^{2k+1} \left| \int_x^u \left|f^{(2k+2)}(w)\right| dw \right|, x \right)$$
$$\le M_n\left(\phi(u)(u-x)^{2k+2} |h_f(u)|, x \right)$$
$$\le \left(M_n\left(|u-x|^{(2k+2)q} \phi(u), x \right) \right)^{1/q} \left(M_n\left(|h_f(u)|^p \phi(u), x \right) \right)^{1/p}$$

$$\leq C n^{-(k+1)} \left(\int_{a_1}^{b_1} W_n(x,u) |h_f(u)|^p du \right)^{1/p} .$$

Now, Fubini's theorem, Lemma 2.3 and [8,p.244] imply that

$$\|J_1\|_{L_p(I_2)} \leq C n^{-(k+1)} \left\{ \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} W_n(x,u) dx \right) |h_f(u)|^p du \right\}^{1/p}$$

$$\leq C n^{-(k+1)} \|h_f(u)\|_{L_p(I_1)}$$

$$\leq C n^{-(k+1)} \left\|f^{(2k+2)}\right\|_{L_p(I_1)} .$$

Consequently,

$$\| \Sigma_2 \|_{L_p(I_2)} \leq C n^{-(k+1)} \left\|f^{(2k+2)}\right\|_{L_p(I_1)} .$$

For $u \in [0,\infty) \setminus [a_1, b_1]$ and $x \in I_2$ we can find a $\delta > 0$ in such a way that $|u - x| \geq \delta$. Thus

$$|M_n(F(u,x)(1-\phi(u)), x)| \leq \delta^{-(2k+2)} \left[M_n\left(|f(u)| (u-x)^{2k+2}, x \right) + \sum_{j=0}^{2k+1} \frac{|f^{(j)}(x)|}{j!} M_n\left(|u-x|^{2k+j+2}, x \right) \right]$$

$$= J_2 + J_3 \text{ , say} .$$

It follows from Hölder's inequality and Lemma 2.2 that

$$|J_2| \leq C n^{-(k+1)} (M_n(|f(u)|^p, x))^{1/p} .$$

Again, applying Fubini's theorem, we get

$$\|J_2\|_{L_p(I_2)} \leq C n^{-(k+1)} \|f\|_{L_p[0,\infty)} .$$

Moreover, using Lemma 2.2 and [3,p.5]

$$\|J_3\|_{L_p(I_2)} \leq C n^{-(k+1)} (\|f\|_{L_p(I_2)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)}) .$$

Hence

$$\| \Sigma_3 \|_{L_p(I_2)} \leq C n^{-(k+1)} (\|f\|_{L_p[0,\infty)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)}) .$$

Combining the estimates of $\Sigma_1 - \Sigma_3$, (2.1) follows.

Now, let $p = 1$. By the hypothesis, for almost all $x \in I_2$ and for $u \in I_1$, we can write

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-x)^i}{i!} f^{(i)}(x) + \frac{1}{(2k+1)!} \int_x^u (u-w)^{2k+1} \, df^{(2k+1)}(w) .$$

Hence, if $\phi(u)$ is the characteristic function of I_1 then

(2.3) $$f(u)=\sum_{i=0}^{2k+1}\frac{(u-x)^i}{i!}f^{(i)}(x)+\frac{1}{(2k+1)!}\int_x^u (u-w)^{2k+1}\,df^{(2k+1)}(w)\,\phi(u) + F(u,x)(1-\phi(u)),$$

where $F(u,x)=f(u)-\sum_{i=0}^{2k+1}\frac{(u-x)^i}{i!}f^{(i)}(x)$, for almost all $x\in I_2$ and for all $u\in[0,\infty)$.
Operating on (2.3) by $M_n(.,k,x)$ and breaking the right hand side into three parts J_1, J_2 and J_3, say, corresponding to the three terms on the right hand side of (2.3), we have

$$\|J_1\|_{L_1(I_2)}\le Cn^{-(k+1)}\left(\|f\|_{L_1(I_2)}+\|f^{(2k+1)}\|_{L_1(I_2)}\right),$$

using Lemma 2.2 and [3,p.5].
Next, using Lemma 2.5, we obtain

$$\|J_2\|_{L_1(I_2)}\le Cn^{-(k+1)}\|f^{(2k+1)}\|_{B.V.(I_1)}.$$

For all $u\in[0,\infty)\setminus[a_1,b_1]$, $x\in I_2$, we can choose a $\delta>0$ such that $|u-x|\ge\delta$. Then

$$\|M_n(F(u,x)(1-\phi(u)),x)\|_{L_1(I_2)}\le\int_{a_2}^{b_2}\int_0^{\infty}W_n(x,u)\,|f(u)|\,(1-\phi(u))du dx$$
$$+\sum_{i=0}^{2k+1}\frac{1}{i!}\int_{a_2}^{b_2}\int_0^{\infty}W_n(x,u)\,|f^{(i)}(x)|\,|u-x|^i(1-\phi(u))du dx$$
$$=J_4+J_5,\ \text{say.}$$

For sufficiently large u, we can find positive constants M and C' such that $\frac{(u-x)^{2k+2}}{u^{2k+2}+1}>C'$ for all $u\ge M, x\in I_2$.
By Fubini's theorem,

$$J_4=\left(\int_0^M\int_{a_2}^{b_2}+\int_M^{\infty}\int_{a_2}^{b_2}\right)W_n(x,u)\,|f(u)|\,(1-\phi(u))\,dx\,du$$
$$=J_6+J_7,\ \text{say}.$$

Now, using Lemma 2.3 we have

$$J_6=\delta^{-(2k+2)}\int_0^M\int_{a_2}^{b_2}W_n(x,u)\,|f(u)|\,(u-x)^{2k+2}\,dx\,du$$
$$\le Cn^{-(k+1)}\left(\int_0^M|f(u)|du\right),\ \text{and}$$

$$J_7=\frac{1}{C'}\int_M^{\infty}\int_{a_2}^{b_2}W_n(x,u)\frac{(u-x)^{2k+2}}{u^{2k+2}+1}\,|f(u)|\,dx\,du$$

$$\leq Cn^{-(k+1)} \int_M^{\infty} |f(u)|\, du .$$

Combining the estimates of J_6 and J_7 , we get

$$J_4 \leq Cn^{-(k+1)} \|f\|_{L_1[0,\infty)} .$$

Further, using Lemma 2.2 and [3,p.5] we obtain

$$J_5 \leq Cn^{-(k+1)} \left(\|f\|_{L_1(I_2)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} \right) .$$

Hence,

$$\left\| M_n(F(u,x)(1-\phi(u)),x) \right\|_{L_1(I_2)} \leq Cn^{-(k+1)} \left(\|f\|_{L_1[0,\infty)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} \right) .$$

Consequently,

$$\| J_3 \|_{L_1(I_2)} \leq Cn^{-(k+1)} \left(\|f\|_{L_1[0,\infty)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} \right) .$$

Finally, combining the estimates of $J_1 - J_3$, we obtain (2.2) .
This completes the proof of the Proposition .

Proof of Theorem. Let $f_{\eta,2k+2}$ be the Steklov mean of $(2k+2)$th order corresponding to $f(u)$ where $\eta>0$ is sufficiently small and $f(u)$ is defined as zero outside $[0,\infty)$. Then we have

$$\|M_n(f,k,.)-f\|_{L_p(I_2)} \leq \| M_n(f - f_{\eta,2k+2},k,.) \|_{L_p(I_2)} + \|M_n(f_{\eta,2k+2},k,.) - f_{\eta,2k+2}\|_{L_p(I_2)}$$
$$+ \|f_{\eta,2k+2} - f\|_{L_p(I_2)}$$
$$= \Sigma_1 + \Sigma_2 + \Sigma_3 \text{ , say .}$$

Letting $\phi(u)$ to be the characteristic function of I_3 , we get

$$M_n((f - f_{\eta,2k+2})(u),x) = M_n(\phi(u)(f - f_{\eta,2k+2})(u),x)$$
$$+ M_n((1-\phi(u))(f - f_{\eta,2k+2})(u),x)$$
$$= \Sigma_4 + \Sigma_5 \text{, say .}$$

Clearly, the following inequality holds for $p = 1$, for $p > 1$ it follows from Hölder's inequality

$$\int_{a_2}^{b_2} | \Sigma_4|^p dx \leq \int_{a_2}^{b_2} \int_{a_3}^{b_3} W_n(x,u) \left| (f - f_{\eta,2k+2})(u)\right|^p du\, dx .$$

Using Fubini's theorem and Lemma 2.3, we get

$$\| \Sigma_4 \|_{L_p(I_2)} \leq \|f - f_{\eta,2k+2}\|_{L_p(I_2)} .$$

Proceeding in a similar manner, for all $p \geq 1$

$$\| \Sigma_5 \|_{L_p(I_2)} \leq Cn^{-(k+1)} \|f - f_{\eta,2k+2}\|_{L_p[0,\infty)} .$$

Consequently, by the property (c) of Steklov means, we get

$$\Sigma_1 \le C\ (\omega_{2k+2}(f,\eta,p,I_1)+n^{-(k+1)}\|f\|_{L_p[0,\infty)}).$$

Since

$$\left\|f_{\eta,2k+2}^{(2k+1)}\right\|_{B.V.(I_3)} = \left\|f_{\eta,2k+2}^{(2k+1)}\right\|_{L_1(I_3)}.$$

By our Proposition, for all $p \ge 1$ there follows

$$\Sigma_2 \le C\ n^{-(k+1)}(\left\|f_{\eta,2k+2}^{(2k+2)}\right\|_{L_p(I_3)}+\left\|f_{\eta,2k+2}\right\|_{L_p[0,\infty)})$$

$$\le C n^{-(k+1)}(\eta^{-(2k+2)}\omega_{2k+2}(f,\eta,p,I_1)+\|f\|_{L_p[0,\infty)}),$$

in view of the property (b) of Steklov means.
Finally, by the property (c) of Steklov means

$$\Sigma_3 \le C\omega_{2k+2}(f,\eta,p,I_1)\,.$$

Choosing $\eta = n^{-1/2}$, the result follows .

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References:

[1] P.N. Agrawal and Kareem J. Thamer , Approximation of unbounded functions by a new sequence of linear positive operators, J.Math.Anal.Appl., 225(1998), 660-672.
[2] P.N. Agrawal and Kareem J. Thamer , Degree of approximation by a new sequence of linear operators, preprint.
[3] S. Goldberg and A. Meir, Minimum moduli of ordinary differential operators, Proc. London Math. Soc., 23(3)(1971), 1-15.
[4] E. Hewitt and K. Stromberg, Real and Abstract Analysis, McGraw-Hill, New York (1956).
[5] C.P. May, Saturation and inverse theorems for combinations of a class of exponential type operators, Can. J. Math., *XXVIII* , 6(1976), 1224-1250.
[6] R.K.S. Rathore, Linear Combinations of Linear Positive Operators and Generating Relations in Special Functions, Ph.D.Thesis, IIT Delhi (India) (1973).
[7] T.A.K. Sinha, Restructured Sequences of Linear Positive Operators for Higher Order L_p- Approximation, Ph.D. Thesis, IIT Kanpur (India) (1981) .
[8] A.Zygmund, Trigonometrical Series, Dover Publications, Inc., New York (1955).

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