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VARIANCE UPPER BOUNDS AND A PROBABILITY INEQUALITY FOR DISCRETE α-UNIMODALITY

M.I. AGEEL
King Saud University, Abha,
Department of Math, Abha, Saudi Arabia

Abstract

Variance upper bounds for discrete α-unimodal distributions defined on finite support are established. These bounds depend on the support and the unimodality index α. It is noted that the upper bounds increase as the unimodality index α increases. More information about the underlying distributions yields tighter upper bounds for the variance. A parameter-free Bernstein-type upper bound is derived for the probability that the sum S of n independent and identically distributed discrete α-unimodal random variables exceeds its mean $E(S)$ by the positive value nt. The bound for $P\{S - n\mu \geq nt\}$ depends on the range of the summands, the sample size n, the unimodality index α and the positive number t.

1 Introduction

Unimodalilty concept of distributions are well known for continuous case. Olshen and Savage (1970) generalized this concept to α-unimodality. They defined continuous random variable (r.v.) X as α-unimodal (about the origin), $\alpha > 0$, if and only if (iff), there exists some r.v. Y such that $X \overset{d}{=} U^{1/\alpha}.Y$, where U is a uniform r.v. on (0,1) independent of Y.

α-unimodal distributions have been further studied by many authors, see, Aboummoh and Mashhour (1983), Alamatsaz (1985), Dharmadhikari and Jogdeo (1986) and references therein.

Upper bounds on variance represent an important target, since they have applications in many areas of statistics such as variance estimation and stochastic process, see Dharamadhikari and Joag-Dev (1989) and references therein. They proved that if X is a continuous r.v. having α-unimodal distribution about, M, $0 \leq X \leq 1$, and $\mu = E(X)$, then

$$(\alpha+2)\mathrm{Var}(X) \leq \mu(\alpha+1+2M) - (\alpha+2)\mu^2 - M. \quad (1.1)$$

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Among other results they show that the upper bound for the variance of an α-unimodal distribution on [0,1] is $(\alpha+1)^2/4(\alpha+2)^2$ which yield Jacobson's bound of 1/9 when $\alpha = 1$, see Jocobson (1969).

For continuous independent and identically distributed (i.i.d.) unimodal r.v.'s $X_1, X_2, \ldots, X_n$ with bounded support, Young et al. (1988) have derived a parameter-free Berstein upper bound for $P\{S - n\mu \geq nt\}$, where $S = \sum_{i=1}^{n} X_i$.

Abouammoh et al. (1994) have defined the discrete α-unimodality concept for $\alpha > 0$ as

Definition (1.1): A discrete r.v. X is called α-*unimodal* about $a, a \in I$, I is the set of integer, if its probability mass function (p.m.f.) $(p_n)_{-\infty}^{\infty}$, satisfies

$$(\alpha - n + a)p_n \geq (1 - n + a)p_{n-1} \qquad n \leq a$$

$$(\alpha + n - a)p_n \geq (1 + n - a)p_{n+1} \qquad n \geq a$$

It is noted that if $(p_n)_{-\infty}^{\infty}$ is α-unimodal about a, $\beta > \alpha$ then $(p_n)_{-\infty}^{\infty}$ is β-unimodal about a. Consequently all α-unimodal distributions with $\alpha < 1$, are described by the unimodal distributions. For $\alpha \geq 1$, they introduced the characterization.

Theorem 1.1: The p.m.f. $(p_n)_{-\infty}^{\infty}$, with characteristic function (ch. fn.) $p(t)$ is α-unimodal about $n = a$, iff

$$q(t) = [\{\alpha + a(1 - e^{it})\}p(t) + i(1 - e^{it})p'(t)] / \alpha \qquad (1.2)$$

and

$$r(t) = [\{\alpha + a(e^{-it} - 1)\}p(t) + i(e^{-it} - 1)p'(t)] / \alpha \qquad (1.3)$$

are ch. fns.

Also, Abouammoh and Mashhour (1994), have probed that if the r.v. X has discrete α-unimodal distribution about a, defined on support $\{0,1,2,\ldots,N\}$, with mean μ, then

$$(\alpha + 2)Var(X) \leq -(\alpha + 2)\mu^2 + [(\alpha + 1)N + 2a]\mu - Na + \alpha[\min\{\mu, (N - \mu)\}]. \qquad (1.4)$$

This is a discrete version of (1.1).

For any discrete r.v. defined on the support $\{0,1,2,\ldots,N\}$ with mean μ, Muilwijk (1966) showed that $Var(X) \leq (N - \mu)\mu$. The right side becomes maximum for variations of μ, when $\mu = N/2$. Hence, if the end points of the support of any discrete r.v. are known, an upper bound for its variance may be found as

$$Var(X) \leq N^2/4 \tag{1.5}$$

It may be noted that the equality holds, when X assumes the values 0 or N each with probability 1/2. Thus, the value $\frac{N^2}{4}$ represents the least upper bound for the variance of any discrete r.v. on the support $\{0,1,2,...,N\}$.

The purpose of the present article is twofold. First, in section 2, we apply the result (1.4) due to Abouammoh and Mashhour (1994), to establish upper bounds for the variance of discrete α-unimodal r.v.'s; sharper than that given by (1.5). These upper bounds are discrete versions for their continuous counterpart due to Dharmadhikari and Joag-Dev (1989). The new results, with $\alpha = 1$, yield some interesting upper bounds for the variance of discrete unimodal r.vs. on finite supports. The later case, with $\alpha = 1$, corresponds to results due to Young et al. (1988) in the continuous case. Next, in Section 3, our results of Section 2, are applied to get upper bound for $P\{S - n\mu \geq nt\}$ when the X_i's are discrete i.i.d. α-unimodal r.vs.

2. VARIANCE UPPER BOUNDS

Let X be a discrete r.v. on the support $\{0,1,2,...,N\}$. The case when $N = 1$, implies that X is strongly unimodal r.v. . Furthermore, it assumes only the values 0 or 1 with probabilities q and p respectively, $p + q = 1$. One may easily deduce that $Var(X) = pq \leq 1/4$, where the equality holds when $p = q = 1/2$. Therefore, our results henceforth will be devoted mainly for $N \geq 2$. Assume that X has an α-unimodal, $\alpha \geq 1$, distribution about the modal value a. By virtue of Theorem 1.1, let X_1 and X_2 be the discrete r.v.'s whose ch. fns. are $q(t)$ and $r(t)$ respectively. One can easily show that X_1 and X_2 are defined on the supports $S_1 = \{0,1,2,...,N + 1\}$ and $S_2 = \{-1,0,...,N\}$ respectively. Put $\mu_1 = E(X_1)$ and $\mu_2 = E(X_2)$. Then the ch. fns. $q(t)$ and $r(t)$ given by (1.2) and (1.3) yield

$$\mu_1 = \mu_2 = [(\alpha + 1)\mu - \alpha]/\alpha. \tag{2.1}$$

In view of the fact that S_1 is non-negative and the support $\{-N-1,-N, -N + 1,...,0\}$ of X_2-N is non-positive, one may deduce that $\mu_1 > 0$ and $\mu_2 < N$.

Hence the expectation $\mu = E(X)$ must satisfy

$$a/(\alpha + 1) < \mu < (a + N\alpha)/(\alpha + 1). \tag{2.2}$$

On the other hand (1.4) implies

$$(\alpha + 2)\ Var(X) \leq -(\alpha + 2)\mu^2 + [(\alpha + 1)N + 2a]\mu - Na + \alpha N/2, \tag{2.3}$$

since $\min\{\mu, (N - \mu)\} \leq N/2$.

The right side of (2.3) attains its maximum when

$$\mu = [(\alpha + 1)N + 2a]/2(\alpha + 2). \qquad (2.4$$

It is noted that the value of μ given by (2.4) satisfies the restriction (2.2), iff $\alpha^2 + 2\alpha > |2a/N - 1|$. Thus, obviously the later condition is satisfied for all $\alpha \geq 1$.

By the above discussion, the best upper bound given by the right hand side of (2.3) is obtained substituting the value of μ from (2.4) into (2.3) to get

Theorem 2.1: If X is discrete α-unimodal r.v. about a on the support $\{0,1,2,\ldots,N\}$, then

$$Var(X) \leq \frac{N^2(\alpha+1)^2 - 4a(N-a)}{4(\alpha+2)^2} + \frac{\alpha N}{2(\alpha+2)}. \qquad (2.5)$$

Theorem 2.2: Let X be a discrete α-unimodal r.v. about a on the support $\{0,1,2,\ldots,N\}$.

a) If $a = 0$ or $a = N$, then

$$Var(X) \leq \frac{N^2(\alpha+1)^2}{4(\alpha+2)^2} + \frac{\alpha N}{2(\alpha+2)}. \qquad (2.6)$$

b) If N is even and $a = N/2$, then

$$Var(X) \leq \frac{\alpha N(N+2)}{4(\alpha+2)}. \qquad (2.7)$$

c) If N is odd and $a = (N \pm 1)/2$, then

$$Var(X) \leq \frac{\alpha N(N+2)}{4(\alpha+2)} + \frac{1}{4(\alpha+2)^2}. \qquad (2.8)$$

Proof: Part (a) is immediate from Theorem 2.1, when it is recalled that bound (2.5) becomes maximum when $a = 0$ or N. Part (b) and part (c) follow by setting $a = N/2$ and $a = N(\pm 1)/2$ respectively in (2.5).

Also setting $\mu = N/2$ in (2.3) yield the same upper bound (2.7), whatever the corresponding model value a.

Theorem 2.3: Let X be a discrete α-unimodal r.v. about some mode on the support $\{0,1,2,\ldots,N\}$. If $\mu = E(X) = N/2$, then

$$Var(X) \le \frac{\alpha N(N+2)}{4(\alpha+2)}. \tag{2.9}$$

For $\alpha = 1$, the p.m.f. (p_n) is unimodal on the support $\{0,1,2,\ldots,N\}$, and one get

Corollary 2.4: Let the p.m.f. (p_n) has mode a and variance σ^2, then

a) $$\sigma^2 \le N^2/9 + N/6, \tag{2.10}$$

if $a = 0$ or $a = N$.

b) $$\sigma^2 \le (1/9)[N^2 - a(N-a)] + N/6, \tag{2.11}$$

for any a, $1 \le a \le N-1$.

c) $$\sigma^2 \le N^2/12 + N/6 = Var(\text{U}), \tag{2.12}$$

if N is even and $a = N/2$, where U is the discrete r.v. uniformly distributed on the same support.

d) $$\sigma^2 \le (N+1)^2/12, \tag{2.13}$$

if N is odd and $a = (N+1)/2$.

Note that (2.10) represent the discrete version of Jacobson Theorem (1969).

Remark 1: For $\alpha = 1$, it is noted that the upper bounds (2.10), (2.11), (2.12) and (2.13) are at least as sharp as the bound $N^2/4$ given by (1.5). Thus, the unimodality property of X yield lower upper bounds for its variance. As α get larger than one, the situation is quite different. Regarding that (2.6) is increasing in α, it is not expected that the bound (2.6) will be less than $N^2/4$ for all values of α. Investigation of the bound (2.7) shows that it is less than $N^2/4$ when $\alpha < N$ but this is not true for bounds in (2.6) and (2.8). consequently, as N get larger our results assign for more α-unimodal r.v.'s on the support $\{0,1,2,\ldots,N\}$ sharp upper bounds for their variances. Practically, free-parameter upper bounds are established based on the known information about X. For instance, if the only known information about X is that, it is α-unimodal about a specified mode and $\alpha < N$, our results yield sharper upper bounds than $N^2/4$, as given by (2.6), (2.7) and (2.8). Otherwise, when $\alpha \ge N$, the only available upper bounds is $N^2/4$ which describe the case of any discrete r.v. on the same support.

3. PROBABILITY INEQUALITY

Let $X_1, X_2, \ldots, X_n$ be i.i.d discrete α-unimodal r.v.'s where X_i has the support $\{0,1,2,\ldots,N\}$ with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Let $S = X_1 + X_2 + \ldots + X_n$ and $\overline{X} = S/n$. We derive a parameter-free upper bound for the probability

$$P\{\overline{X} - \mu \geq t\} = P\{S - E(S) \geq nt\}, \qquad (3.1)$$

where $t > 0$.

One method employed to derive a new inequality is attributed to S. N. Bernstein (see Young et al. (1989) and references therein). According to that method

$$P\{S - n\mu \geq nt\} \leq E[\exp\{c(S - n\mu - nt)\}],$$

for any positive constant c. Since the X_i's are independent

$$E[\exp\{c(S - n\mu - nt)\}] = \exp(-cnt)\pi_{i=1}^{n} E[\exp\{c(X_i - \mu)\}],$$

where $c > 0$. The new upper bound is derived by bounding

$$E[\exp\{c(X_i - \mu)\}], \qquad (3.2)$$

from above and then minimizing the resulting bound with respect to c. The following lemma and theorem 3.2 can be proved as lemma 4.1 and theorem 4.1 in Young et al.(1988), using our theorem 2.1 and theorem 2.3.

Lemma 3.1: Let X be a discrete α-unimodal r.v. about a, with mean μ and variance σ^2, on the support $\{0,1,...,N\}$. Let $Z = X - \mu$, then

$$E[\exp\{c(Z)\}] \leq \exp[\theta\{\exp(cN) - cN - 1\}],$$

where c is an arbitrary positive constant and $\theta = \theta(\alpha,a,N)$ represent an upper bound for $(\sigma / N)^2$.

Now, we establish the main results for the case of i.i.d. discrete α-unimodal r.v.'s with finite support.

Theorem 3.2: Let $X_1, X_2,...,X_n$ be i.i.d. discrete α-unimodal r.v.'s on the support $\{0,1,2,...,N\}$. Let $E(X_i) = \mu$ and $S = X_1 + X_2 + ... + X_n$. Then

$$P\{S - n\mu \geq nt\} \leq \exp\{\frac{nt}{N} - (\frac{nt}{N} + n\theta) ln(1 + t / N\theta)\} \qquad (3.3)$$

where

$$\theta = \theta(\alpha, a, N) = \frac{(\alpha+1)^2 - \frac{4}{N^2}a(N - a)}{4(\alpha+2)^2} + \frac{\alpha}{2(\alpha+2)N}, \qquad (3.4)$$

when X_i is α-unimodal about the specified mode a, and

$$\theta = \theta(\alpha, N) = \frac{(\alpha+1)^2}{4(\alpha+2)^2} + \frac{\alpha}{2(\alpha+2)N} = \frac{\alpha(N+2)}{4(\alpha+2)N}, \qquad (3.5)$$

when X_i is α-unimodal about some mode such that $\mu = N/2$.

Corollary (3.3): Let X be a discrete α-unimodal r.v. on the support $\{0,1,2,\ldots,N\}$, then for any $t > 0$

$$P\{X - \mu \geq t\} \leq \exp\{t/N - (t/N + \theta) ln(1 + t/N\theta)\} \tag{3.6}$$

where θ is given by (3.4) or (3.5). of Theorem 3.2.

Note that the upper bound in (3.6) does not depend on any parameter of X other than the range N, the unimodality index α and the mode value a. This feature makes (3.6) very applicable in real-data situation.

Remark (2): If X is a discrete r.v. on the support $\{0,1,\ldots,N\}$ then it can be noted that (3.3) holds with $n = 1$ and $\theta = \frac{1}{4}$. That is

$$P\{X - \mu \geq t\} \leq \exp\{t/N - (t/N + \frac{1}{4}) ln(1 + 4t/N)\}. \tag{3.7}$$

Finally, inequality (2.2) seems to be interesting in the following sense

i) It provides lower and upper bounds for the mean of any discrete α-unimodal r.v., $\alpha \geq 1$, about a modal value a on the support $\{0,1,\ldots,N\}$. Also, it represents a discrete version for its most recent continuous counterpart due to Dharmadhikari and Joag-Dev (1989).

ii) As a consequence of (2.2), one can note that

$$\{X - \frac{a + N\alpha}{\alpha + 1} \geq t\} \text{ implies } \{X - \mu \geq t\}.$$

Hence an upper bound for $P\{X - a \geq t + \frac{\alpha(N - a)}{\alpha + 1}\}$ may be obtained by (3.6).

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