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DAMEK-RICCI SPACES SATISFYING THE OSSERMAN-P CONDITION

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ABSTRACT. We show that the only Damek-Ricci space S satisfying the Osserman-p condition, for some $1 < p < \dim S$, is the real hyperbolic space.

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Let M^n be a riemannian manifold, R its curvature tensor and R_X the Jacobi operator, defined by $R_XY = R(Y,X)X$, X a unit tangent vector. We say that M satisfies the *Osserman-p condition*, (Oss-p, for short), p a natural number, if for any set of p orthonormal vectors $E = \{X_1, X_2, ..., X_p\}$ in the tangent bundle, the symmetric operator

$$J_E = R_{X_1} + R_{X_2} + ... + R_{X_p}$$

has constant eigenvalues, counting multiplicities. Note that this definition does not depend on the orthonormal basis chosen on the space spanned by E.

This notion, introduced in [SV] or [SP], generalizes the Osserman condition ($p = 1$). It is immediate to see that if $p = n$ then M^n has constant scalar curvature. Moreover, if p satisfies $1 \leq p < n$ then the riemannian manifold is Einstein and there exist a duality between p and $n-p$, that is, M^n satisfies the Osserman-p condition if and only if it satisfies the Osserman-$(n-p)$ condition (See [G]).

We note that spaces of constant sectional curvature satisfy the Oss-p condition for all $p \geq 1$. The curvature formula for spaces of constant curvature c,

$$R(X,Y)Z = c\left(\langle Y,Z\rangle X - \langle Z,X\rangle Y\right),$$

gives $R_XY = cY$ for orthogonal tangent vectors X and Y. Hence, if $E = \{X_1, ..., X_p\}$ is an orthonormal set, we have that

$$J_E\,|_E = c(p-1)\ \mathrm{Id}, \quad J_E\,|_{E^\perp} = cp\ \mathrm{Id},$$

P. Gilkey raised the question whether a nonpositively curved homogeneous manifold M^n satisfying the Oss-p condition, for some $1 < p < n$ must have constant curvature. We note that in [DD1] we have proved that a homogeneous space of nonpositive curvature satisfying the Osserman condition is a rank one-symmetric space of noncompact type. Moreover, in [DD2] we study the class of riemannian manifolds coming from solvable Lie groups S of Iwasawa type and two-step nilpotent radical, with a left invariant metric satisfying the Oss-p condition (without the assumption of nonpositive curvature). In [DD2] we prove that under these conditions, S must be a Damek-Ricci space.

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The purpose of the present paper is to give an affirmative answer to P. Gilkey's question above, in a subclass of the class of homogeneous manifolds of nonpositive curvature. More precisely we shall prove:

Theorem. If M is a Damek-Ricci space of dimension n satisfying the Osserman-p condition for some $1 < p < n$, then M is the real hyperbolic n-space.

Using the main result in [DD2] together with the above theorem we obtain:

Corollary. If S is solvable Lie group of Iwasawa type and dimension n, with $[S, S]$ two step nilpotent, satisfying the Osserman-p condition for some $1 < p < n$, then S is a space of constant negative sectional curvature.

While the paper was submitted the authors learnt that P. Gilkey proved, using methods from vector bundle theory, that a riemannian manifold satisfying the Oss-p condition has constant sectional curvature. The proof presented here, although restricted to our class of spaces, is more elementary.

1. Curvature formulas on Damek-Ricci spaces

We start by recalling the definition of Damek-Ricci spaces (D-R spaces). Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra endowed with an inner product $\langle\,,\,\rangle$. Assume $\mathfrak{n}$ has an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$. Define a linear mapping $j : \mathfrak{z} \rightarrow \mathrm{End}\ \mathfrak{v}$ by

$$\langle j_Z X, Y\rangle = \langle Z, [X, Y]\rangle \tag{1}$$

(note that j_Z is skew-symmetric). Now $\mathfrak{n}$ is said to be an *H-type algebra* if for any $Z \in \mathfrak{z}$

$$j_Z^2 = -\langle Z, Z\rangle \mathrm{Id}. \tag{2}$$

The corresponding H-type group is the simply connected Lie group N with Lie algebra $\mathfrak{n}$ endowed with the left invariant metric induced by the inner product $\langle\,,\,\rangle$ in $\mathfrak{n}$. It is easily seen that if $\mathfrak{n}$ is H-type and non abelian then $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}$.

The class of solvable extensions of H-type groups which we will consider in this section are constructed as follows. Let $\mathfrak{n}$ be an H-type algebra with corresponding simply connected Lie group N. If $A = \mathrm{R}^+$ acts on N by the dilations $(z, x) \rightarrow (tz, t^{\frac{1}{2}}x)$, we let S be the semidirect product AN. Let $\mathfrak{s}$ be the Lie algebra of S. If D is the derivation of $\mathfrak{n}$ given by $D|_{\mathfrak{v}} = \frac{1}{2}\,\mathrm{Id}$ and $D|_{\mathfrak{z}} = \mathrm{Id}$ and $\mathfrak{a} = \mathrm{R}H$, then $\mathfrak{s}$ is the semi-direct product $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{a}$ acts on $\mathfrak{n}$ via $\mathrm{ad}_H|_{\mathfrak{n}} = D$. We endow $\mathfrak{s}$ with the only inner product extending the given one in $\mathfrak{n}$ such that $|H| = 1$, $\langle H, \mathfrak{n}\rangle = 0$. Finally, we give to S the left invariant metric associated to the inner product on $\mathfrak{s}$. The riemannian manifold obtained will be called a *Damek-Ricci space.*

The Levi Civita connection and the curvature tensor associated to the metric can be computed by,

$$\begin{aligned} 2\,\langle \nabla_X Y, Z\rangle &= \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle \\ R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \end{aligned}$$

for any X, Y, Z in $\mathfrak{s}$.

Let $n = \dim \mathfrak{s}$ and $1 \leq p < n$. We will say that the metric Lie algebra $\mathfrak{s}$ satisfies the Oss-p condition, if the corresponding Lie group S satisfies it.

In the next proposition we express the curvature operators of the Lie group S and we summarize the basic curvature formulas that will be used throughout the article.

These formulas can be deduced easily using the expression of the connection and curvature tensor appearing, for example in [DD2] or [BTV].

Proposition 1.1. *If $\mathfrak{s}$ is the Lie algebra of a D-R space S then the following formulas hold.*

(*i*) $R_H = -\mathrm{ad}_H{}^2$.

(*ii*) If either $Z \in \mathfrak{z}$, $|Z| = 1$ or $X \in \mathfrak{v}$, $|X| = 1$ then

$$R_Z H = -H\,, \quad R_X H = -\frac{1}{4}H.$$

(*iii*) If $Z \in \mathfrak{z}$, $|Z| = 1$ and $Z^* \in \mathfrak{z}$, then

$$R_Z Z^* = \langle Z, Z^* \rangle Z - Z^*.$$

(*iv*) If $Z \in \mathfrak{z}$, $|Z| = 1$, and $X \in \mathfrak{v}$, then

$$R_Z X = -\frac{1}{4}X.$$

(*iv*) If $X \in \mathfrak{v}$, $|X| = 1$, and $Z \in \mathfrak{z}\,, Y \in \mathfrak{v}$, then

$$\begin{aligned} R_X Z &= -\frac{1}{4}Z \\ R_X Y &= -\frac{3}{4}j_{[X,Y]}X + \frac{1}{4}\langle X, Y\rangle X - \frac{1}{4}Y. \end{aligned}$$

2. Proof of the main result

Let S be a Damek-Ricci space and let N be its nilpotent radical; let $\mathfrak{s}$ and $\mathfrak{n}$ be the corresponding Lie algebras. Then $\mathfrak{n}$ is an H-type Lie algebra. If $\mathfrak{n}$ is not abelian, given $X \in \mathfrak{v}$ let $j_{\mathfrak{z}}X = \{j_Z X : Z \in \mathfrak{z}\}$. Clearly (1) gives $(j_{\mathfrak{z}}X)^{\perp} = \ker ad_X|_{\mathfrak{v}}$, thus we may consider, for every $X \in \mathfrak{v}$, the orthogonal decomposition

$$\mathfrak{v} = j_{\mathfrak{z}}X \oplus \ker ad_X|_{\mathfrak{v}}.$$

Furthermore (2) gives

$$[X, j_Z X] = \mathrm{ad}_X(j_Z X) = |X|^2 Z,$$

showing in particular that the dimensions of $\mathfrak{z}$ and $j_{\mathfrak{z}}X$ coincide if $X \neq 0$ in $\mathfrak{v}$.

We set $k = \dim \mathfrak{z}$. If $\mathfrak{v} \neq 0$ let $d = \dim \mathfrak{v} - k$ $(= \dim(\ker ad_X|_{\mathfrak{v}}),\ X \neq 0)$, $2k + d = \dim \mathfrak{n}$.

Assume S satisfies the Osserman-p condition, $1 < p < n$. As we recalled in the introduction we may (and will) assume $1 < p \leq \frac{1}{2}n$ since conditions Oss-p and Oss-$(n-p)$ are equivalent.

The following lemma was proved in [DD2, Lemma 4.1]. We include its proof here for the sake of completeness.

Lemma 2.1. *If S is a Damek-Ricci space of dimension n satisfying the Oss-p condition, with $1 < p \leq k + 1$, then S has constant sectional curvature -1.*

Proof. Let $E = \{H, Z_1, ..., Z_{p-1}\}$ be an orthonormal set of p-vectors with $Z_i \in \mathfrak{z}$. We compute the eigenvalues of $j_E = -ad_H^2 + \sum_{i=1}^{p-1} R_{Z_i}$ obtaining

$$j_E H = -(p-1)H, \quad j_E Z_i = -(p-1)Z_i, \ i = 1, ..., p.$$

Hence $j_E \mid_{\mathfrak{z}+\mathbf{R}H} = -(p-1)\mathrm{Id}$ and $j_E \mid_{\mathfrak{v}} = -\frac{1}{4}p\mathrm{Id}$, since $R_{Z_i} \mid_{\mathfrak{v}} = -\frac{1}{4}\mathrm{Id}$.

Assume there exists a non zero $X \in \mathfrak{v}$ and consider the orthonormal set $E' = \{X, Z_1, ..., Z_{p-1}\}$. Then $j_{E'} H = -(p-\frac{3}{4})H$, which shows that the eigenvalue $-(p-\frac{3}{4})$ is not attained by j_E, giving a contradiction.

Lemma 2.2. *Let S be a Damek-Ricci space of dimension n satisfying the Oss-p condition. If $k+1 < p \leq \frac{1}{2}n$, then $2k+1 \leq p \leq d$.*

Proof. Note that $\mathfrak{v} \neq 0$ $(k+1 < p)$. Since $\frac{1}{2}n = \frac{1}{2}(2k+d+1) \leq k+d$ we will set $p = k+r$ with $1 < r \leq d$.

Let X_1 be a non zero element of $\mathfrak{v}$ and let E and E' be the following p-basis,

$$E = \{H, Z_1, ..., Z_k, X_1, ..., X_{r-1}\}, \quad E' = \{Z_1, ..., Z_k, X_1, ..., X_r\},$$

where $Z_1, ..., Z_k$ is an orthonormal basis of $\mathfrak{z}$ and $X_1, ..., X_r$ is an orthonormal set of vectors in $\ker \mathrm{ad}_{X_1}|_{\mathfrak{v}}$. Using the curvature formulas given in Proposition 1.1, applied to

$$J_E = -\mathrm{ad}_H^2 + \sum_{i=1}^{k} R_{Z_i} + \sum_{i=1}^{r-1} R_{X_i} \text{ and } J_{E'} = \sum_{i=1}^{r} R_{Z_i} + \sum_{i=1}^{r-1} R_{X_i},$$

respectively, a straightforward computation shows that

$$J_E H = -\frac{1}{4}(4k+r-1)H, \ J_E Z_i = -\frac{1}{4}(4k+r-1)Z_i, \ J_E X_1 = -\frac{1}{4}(k+r-1)X_1$$

and if $Y \in \mathfrak{v}$, $|Y| = 1$ and $\langle Y, X_1 \rangle = 0$ then

$$\langle J_E Y, Y \rangle = -\frac{1}{4}(k+r) - \frac{3}{4}\sum_{i=1}^{r-1} |[X_i, Y]|^2 + \frac{1}{4}\sum_{i=1}^{r-1} \langle X_i, Y \rangle^2.$$

Since $J_{E'} H = -\frac{1}{4}(4k+r)H$ it follows from the Oss-p condition that $-\frac{1}{4}(4k+r)$ is also an eigenvalue of J_E. Then there exists an eigenvector Y in $\mathfrak{v}$ of J_E satisfying $\langle Y, X_1 \rangle = 0$ with eigenvalue $\langle J_E Y, Y \rangle = -\frac{1}{4}(4k+r)$. Hence

$$-\frac{1}{4}(4k+r) = -\frac{1}{4}(k+r) - \frac{3}{4}\sum_{i=1}^{r-1} |[X_i, Y]|^2 + \frac{1}{4}\sum_{i=1}^{r-1} \langle X_i, Y \rangle^2$$

or equivalently,

$$3(k - \sum_{i=1}^{r-1} |[X_i, Y]|^2) = -\sum_{i=1}^{r-1} \langle X_i, Y \rangle^2.$$

Hence

$$k \leq \sum_{i=1}^{r-1} |[X_i, Y]|^2 \leq r-1$$

since in a D-R space one has that $|[U, V]| \leq 1$ for any unit vectors U and V in $\mathfrak{v}$. Thus $k+1 \leq r = p-k$, or equivalently $2k+1 \leq p$. Since by hypothesis $2p \leq 2k+1+d$ it must hold $p \leq d$ as claimed.

Proposition 2.3. *Let S be a Damek-Ricci space of dimension n satisfying the Oss-p condition with $2k+1 \leq p \leq d$. Then, for any $X \in \mathfrak{v}$, $X \neq 0$, the subspace $\ker ad_X|_{\mathfrak{v}}$ is abelian.*

Proof. Let $E = \{X_1, ..., X_p\}$ be an orthonormal set of p vectors in $\ker ad_{X_1}|_{\mathfrak{v}}$. If $E' = \{H, X_1, ..., X_{p-1}\}$ then one easily computes

$$J_E H = -\frac{p}{4}H,\ J_E Z = -\frac{p}{4}Z,\ Z \in \mathfrak{z},\ J_E X_1 = -\frac{p-1}{4}X_1$$

and

$$J_{E'} H = -\frac{p-1}{4}H,\ J_{E'} Z = -\frac{p+3}{4}Z, Z \in \mathfrak{z},\ J_{E'} X_1 = -\frac{p-1}{4}X_1.$$

Note that the eigenvalue $-\frac{1}{4}(p-1)$ has multiplicity at least two in $J_{E'}$. The Oss-p condition implies there exists a unit vector $Y \perp X_1$ in $\mathfrak{v}$, eigenvector of J_E, satisfying $\langle J_E Y, Y\rangle = -\frac{1}{4}(p-1)$. Since $J_E = \sum_{i=1}^{p} R_{X_i}$, we have

$$-\frac{p-1}{4} = \langle J_E Y, Y\rangle = -\frac{p}{4} - \frac{3}{4}\sum_{i=1}^{p} |[X_i, Y]|^2 + \frac{1}{4}\sum_{i=1}^{p}\langle X_i, Y\rangle^2.$$

It then follows that

$$0 \leq 1 - \sum_{i=1}^{p}\langle X_i, Y\rangle^2 = -3\sum_{i=1}^{p}|[X_i, Y]|^2$$

and consequently

$$\sum_{i=1}^{p}\langle X_i, Y\rangle^2 = 1 \text{ and } \sum_{i=1}^{p}|[X_i, Y]|^2 = 0.$$

In particular, we obtain that $Y = \sum_{i=1}^{p}\langle Y, X_i\rangle X_i$ and $[Y, X_i] = 0$ for $i = 1, ..., p$. Thus we may take $\{X_1, ..., X_p\}$ an orthonormal basis of E satisfying $[X_1, X_i] = [X_2, X_i] = 0$ for all $i = 1, ..., p$. If we repeat the previous argument with E and E' as above, computing

$$J_E H = -\frac{p}{4}H,\ J_E Z = -\frac{p}{4}Z, Z \in \mathfrak{z},\ J_E X_i = -\frac{p-1}{4}X_i,\ i = 1, 2$$

$$J_{E'} H = -\frac{p-1}{4}H,\ J_{E'} Z = -\frac{p+3}{4}Z, Z \in \mathfrak{z},\ J_{E'} X_i = -\frac{p-1}{4}X_i,\ i = 1, 2$$

we have that the eigenvalue $-\frac{1}{4}(p-1)$ is attained with multiplicity at least three for $J_{E'}$. Hence there exists $Y \in \mathfrak{v}$, $|Y| = 1$, such that $\langle Y, X_i\rangle = 0$, $i = 1, 2$ satisfying $\langle J_E Y, Y\rangle = -\frac{1}{4}(p-1)$. Using the expression for $J_E Y$ as before, we have

$$-\frac{p-1}{4} = -\frac{p}{4} - \frac{3}{4}\sum_{i=1}^{p}|[X_i, Y]|^2 + \frac{1}{4}\sum_{i=1}^{p}\langle X_i, Y\rangle^2,$$

obtaining again that $Y = \sum_{i=1}^{p}\langle Y, X_i\rangle X_i$, $\langle Y, X_i\rangle = 0$, $i = 1, 2$, and $[Y, X_i] = 0$ for $i = 1, ..., p$. Consequently we get an orthonormal set of p vectors $\{X_1, X_2, ..., X_p\}$, basis of the space spanned by E, satisfying $[X_1, X_i] = [X_2, X_i] = [X_3, X_i] = 0$ for all $i = 1, ..., p$. By repeating this procedure we obtain $[X_i, X_j] = 0$ for all $i, j = 1, ..., p$. Since $X_1 = X$ and the orthonormal $p-$vectors $\{X_1, X_2, ..., X_p\}$ in $\ker ad_X|_{\mathfrak{v}}$ was arbitrarily chosen, it follows that $[\ker ad_X|_{\mathfrak{v}}, \ker ad_X|_{\mathfrak{v}}] = 0$.

Theorem 2.4. *If S is a Damek-Ricci space of dimension n satisfying the Oss-p condition, $1 < p < n$, then S is the real hyperbolic space of constant curvature -1.*

Proof. Let k denote the dimension of $\mathfrak{z}$, the center of the nilradical. If $1 < p \leq k+1$ it follows from Lemma 2.1 that S has constant sectional curvature -1. Now, we assume that $k+1 < p \leq \frac{1}{2}n$, hence $\mathfrak{v} \neq 0$. Setting $n = 2k+d+1$ with $d = \dim \mathfrak{v} - k$ we will show that the Oss-p assumption gives $\mathfrak{v}$ abelian, hence a contradiction, since $\mathfrak{z} = [\mathfrak{v}, \mathfrak{v}]$.

If $k+1 < p \leq \frac{1}{2}n$, it follows from Lemma 2.2 and Proposition 2.3 that $\ker \mathrm{ad}_X|_{\mathfrak{v}}$ is abelian for any non zero $X \in \mathfrak{v}$. Let X, Y be orthonormal vectors in $\mathfrak{v}$. Since $\ker \mathrm{ad}_X \cap \ker \mathrm{ad}_Y \neq 0$ has dimension greater than $d-k$ and $d-k \geq 1$ (by Lemma 2.2) we can choose $U \neq 0$ in $\mathfrak{v}$ so that $[U, X] = 0 = [U, Y]$. Hence X, $Y \in \ker \mathrm{ad}_U$, and since it is abelian, it follows that $[X, Y] = 0$ and $\mathfrak{v}$ abelian.

Now as consequence of Theorem 5.2 of [DD2], we get

Corollary 2.5. *If S is solvable Lie group of Iwasawa type and dimension n, with $[S, S]$ two step nilpotent, satisfying the Osserman-p condition for some $1 < p < n$, then S is a space of constant negative sectional curvature.*

3. References

[BTV] Berndt J., Tricerri F., Vanhecke L., Generalized Heisenberg Groups and Damek-Ricci harmonic spaces. Lecture Notes in Mathematics 1598, Springer, 1995.

[DD1] Dotti I., Druetta M.J., Negatively curved homogeneous Osserman spaces. To appear in Differential Geometry and Applications.

[DD2] Dotti I., Druetta M.J., Osserman p-spaces of Iwasawa type. Proceedings of the 7th. International Conference on Differential Geometry and Applications, Brno, Masaryk University (1998), 61-72.

[G] Gilkey P., Generalized Osserman manifolds, Abh. Math. Sem. Univ. Hamburg, 68 (1998) 125-127.

[SV] Stanilov G., Videv V., Four dimensional pointwise Osserman manifolds, Abh. Math. Sem. Univ. Hamburg, 68 (1998) 1-6.

[SP]-Stanilov G., Petrova I., A generalized Jacobi operator in the 4-dimensional Riemannian Geometry, Annuarie de l′ Université de Sofia St. Kliment Ohridski, Faculté de Mathematiques et Informatique, Livre 1-Mathematiques, 85 (1995).

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