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Free $(n+1)$–valued C–algebras

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Abstract

A method to determine the number of elements of the $(n+1)$–valued C–algebra with a finite set of free generators is described. Applying this method for $n=1$ and $n=2$, we find again the results obtained by L. Iturrioz and A. Monteiro ([7]) and by L. Iturrioz and O. Rueda ([8]) respectively.

1 Preliminaries

C–algebras [9] represent the algebraic countpart of the implicational fragment of the infinite–valued Lukasiewicz logic and they have been studied by several authors under different names.

Recall that a C–algebra is an algebra $\langle A, \rightarrow, 1\rangle$ of type $(2,0)$ fulfilling the following identities:

(C1) $1 \rightarrow x = x$,

(C2) $x \rightarrow (y \rightarrow x) = 1$,

(C3) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,

(C4) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(C5) $((x \rightarrow y) \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow x) = 1$.

It is well known that the variety of the algebras $\langle A, \rightarrow, 1\rangle$ of type $(2,0)$ verifying (C1),..., (C4) are the dual of commutative BCK–algebras [15, 18] in the sense that $y * x$ and 0 are replaced by $x \rightarrow y$ and 1 respectively.

It is simple to see that the relation $\leq$ defined by $x \leq y$ iff $x \rightarrow y = 1$ is a partial order on A and $x \leq 1$ for all $x \in A$. Furthermore, $(A, \leq)$ is a join semilattice where $x \vee y = (x \rightarrow y) \rightarrow y$ is the supremum of the elements x and y.

A C^0–algebra is an algebra $\langle A, \rightarrow, 0, 1\rangle$ of type $(2,0,0)$ such that $\langle A, \rightarrow, 1\rangle$ is a C–algebra and 0 is the first element for $\leq$.

We denote by $\mathbf{C}$ and $\mathbf{C^0}$ the varieties of C–algebras and C^0–algebras respectively. In [13], it is shown that $\mathbf{C^0}$ coincides with the variety $\mathbf{W}$ of Wasjberg algebras. On the other hand, as is well known, the variety of Wasjberg algebras is equivalents to Chang's MV–algebras [2].

If $\mathbf{K}$ is one of the varieties $\mathbf{C}$ or $\mathbf{C^0}$, we are going to denote by $Con_{\mathbf{K}}(A)$, $Hom_{\mathbf{K}}(A, B)$ and $Epi_{\mathbf{K}}(A, B)$ the sets of $\mathbf{K}$–congruences, $\mathbf{K}$–homomorphisms from A into B and $\mathbf{K}$–epimorphisms from A onto B, respectively. Besides, if $S \subseteq A$ is a $\mathbf{K}$–subalgebra of A we write $S \lhd_{\mathbf{K}} A$. We denote by $[G]_{\mathbf{K}}$ the $\mathbf{K}$–subalgebra of A generated by G. The subindex $\mathbf{K}$ will be omitted when there is no doubt about it.

Let $A \in \mathbf{K}$. $D \subseteq A$ is a deductive system of A if $1 \in D$ and if $x, x \rightarrow y \in D$, then $y \in D$. If $\mathcal{D}(A)$ is the set of all deductive system of A, then $Con_{\mathbf{K}}(A) = \{R(D) : D \in \mathcal{D}(A)\}$ where $R(D) = \{(x, y) \in A^2 : x \rightarrow y, y \rightarrow x \in D\}$ ([3, 9, 13]). If $R = R(D)$, we denote by A/D the quotient algebra.

Let $h \in Hom_{\mathbf{K}}(A, B)$. The set $Ker(h) = \{x \in A : h(x) = 1\}$ is called the kernel of h. It is easy to see that if $h \in Hom_{\mathbf{K}}(A, B)$, then $Ker(h) \in \mathcal{D}(A)$.

Let n be an integer, $n \geq 1$. A C_{n+1}–algebra (or C^0_{n+1}–algebra) is a C–algebra (or C^0 –algebra) which satisfies the identity:

(C6) $(x^n \rightarrow y) \vee x = 1$,

where $x^1 \rightarrow y = x \rightarrow y$ and $x^{n+1} \rightarrow y = x \rightarrow (x^n \rightarrow y)$, for $n = 1, 2, \ldots$.

We denote by $\mathbf{C_{n+1}}$ and $\mathbf{C^0_{n+1}}$ the varieties of C_{n+1}–algebras and C^0_{n+1}–algebras respectively.

Next, we summarize some properties of C_{n+1}–algebras and for their proofs we refer the readers to the list of references.

(T1) Let $C_{n+1} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ be the set of the rational fractions. If we define $x \rightarrow y = min\ \{1, 1 - x + y\}$ for all $x, y \in C_{n+1}$, then $\langle C_{n+1}, \rightarrow, 1\rangle \in \mathbf{C_{n+1}}$.

(T2) If $S \lhd C_{n+1}$ and $|S| > 1$, then $S \simeq_{\mathbf{C^0_{t+1}}} C_{t+1}$ where $t \leq n$ ([3]).

(T3) Let $[a, b] = \{x \in A : a \leq x \leq b\}$, then $[\frac{n-t}{n}, 1] \lhd C_{n+1}$ for all t, $0 \leq t \leq n$. Besides, taking into account that over a finite chain the operation $\rightarrow$ is determined by the order, we can state that $[\frac{n-t}{n}, 1] \simeq C_{t+1}$ ([3, 13, 16]).

(T4) If $S \lhd C_{n+1}$ such that $\frac{n-1}{n} \in S$, then there is $\frac{u}{n} \in C_{n+1}$ such that $S = [\frac{u}{n}, 1]$ ([3]).

(T5) If $A \in \mathbf{C}_{n+1}$ is non-trivial, then A is isomorphic to a subalgebra of $P = \prod_{M \in \mathcal{M}(A)} A/M$ where $\mathcal{M}(A)$ is the set of maximal deductive systems of A. Furthermore, if $A \in \mathbf{C}^0_{n+1}$ is finite, then $A \simeq P$ ([3, 13]).

(T6) If $A \in \mathbf{C}_{n+1}$ and $S \lhd A$, then for any $D \in \mathcal{M}(S)$ there is a unique $M \in \mathcal{M}(A)$ such that $D = M \cap S$ ([3]).

(T7) If $A \in \mathbf{C}_{n+1}$, then the following conditions are equivalent ([3]):

(i) $M \in \mathcal{M}(A)$,

(ii) there is $h \in Hom(A, C_{n+1})$ such that $Ker(h) = M$,

(iii) $A/M \simeq C_{j+1}$ for some j, $1 \leq j \leq n$.

2 Free $C_{n+1}-$Algebras

Throughout this section $\mathcal{L}(c) \in \mathbf{C}_{n+1}$ denotes the C_{n+1}–algebra with a set G of free generators, $|G| = c$, where c is a cardinal number. If we wish to emphasize that $\mathcal{L}(c) \in \mathbf{C}_{n+1}$, we write $\mathcal{L}(n, c)$. Estas algebras libres fueron descriptas por A. Rose en [14]

Let $X \subseteq A$ be an ordered set. We denote by $\mu(X)$ the set of minimal elements of X.

Lemma 2.1 *Let $A \in \mathbf{C}$ and $X \subseteq A$. If $[X]_{\mathbf{C}} = A$, then $\mu(X) = \mu(A)$.*

Proof. If $|A| = 1$, the lemma holds. Assume now that $|A| > 1$. If $z \in \mu(A)$, then $B = A \setminus \{z\}$ is a subalgebra of A. If $z \notin X$, then $X \subseteq B$ and $[X]_{\mathbf{C}} \neq A$ which is a contradiction. Hence $\mu(A) \subseteq X$ and so $\mu(A) \subseteq \mu(X)$. Conversely, if $t \in \mu(X)$, then $S = \{x \in A : x \not< t\}$ is a subalgebra of A and $x \not< t$ for all $x \in X$. Thus $X \subseteq S$ and so $A = S$. Therefore, $t \in \mu(A)$ and $\mu(X) \subseteq \mu(A)$. □

Corollary 2.1 *If $A \in \mathbf{C}$ and $[X]_{\mathbf{C}} = A$, then the following conditions are equivalent:*

(i) $X = \mu(A)$,

(ii) *X is an antichain.*

Lemma 2.2 *If G is a set of free generators of $\mathcal{L}(c)$ such that $|G| = c$, then $G = \mu(\mathcal{L}(c))$.*

Proof. If $|G| = 1$, then $G = \mu(G)$ and by Lemma 2.1 we have that $G = \mu(\mathcal{L}(1))$. Assume now that $|G| > 1$ and let $g, g' \in G$ be such that $g \neq g'$. If $g < g'$ we consider the application $f : G \longrightarrow C_{n+1}$ defined by $f(t) = 1$ if $t = g$ and $f(t) = 0$ otherwise. Therefore, there exists $h \in Hom(\mathcal{L}(c), C_{n+1})$ which extends f. Then $h(g) = 1$ and $h(g') = 0$, which is a contradiction because h is isotone. If $g' < g$ the proof is similar. Hence g and g' are incomparable and by Corollary 2.1 we get $G = \mu(G) = \mu(A)$. □

From Lemma 2.2 we immediately have

Lemma 2.3 $\mathcal{L}(c) = \bigcup\limits_{g \in G} [g, 1]$

The proof of Lemma 2.4 is a consequence of McNaugton's representation theorem ??. An explicit proof of finiteness of finitely generated free $(n+1)-$valued $MV-$algebra follows from results obtained by R. Grigolia in 1972 (see [5]).

Lemma 2.4 *$\mathcal{L}(n, m)$ is finite.*

Method for the determination of $|\mathcal{L}(n, m)|$

In what follows m is an integer, $m \geq 1$ and $G = \{g_1, g_2, \ldots, g_m\}$ is a set of free generators of $\mathcal{L}(n, m)$. By Lemma 2.3 we can write

$$|\mathcal{L}(n, m)| = \sum_{i=1}^{m} (-1)^{k+1} a_k. \tag{1}$$

where

$$a_k = \sum_{1 \leq i_1 < \ldots < i_k \leq m} |\bigcap_{t=1}^{k} [g_{i_t}, 1]|. \tag{2}$$

By the symmetry of the problem it is sufficient to compute $|\bigcap\limits_{i=1}^{k} [g_i, 1]|$.

Let $G_k = \{g_1, g_2, \ldots, g_k\}$, $G_{m-k} = G \setminus G_k$ and $g_k^* = \bigvee\limits_{i=1}^{k} g_i$. Hence, it is easy to see that $B_k = \bigcap\limits_{i=1}^{k} [g_i, 1] = [g_k^*, 1]$. Then,

$$|\mathcal{L}(n,m)| = \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}|B_k|. \tag{3}$$

It is not difficult to show that B_k is a finite subalgebra of $\mathcal{L}(n,m)$ with first element g_k^*. Then by T5

$$B_k = \prod_{D\in\mathcal{M}(B_k)} B_k/D. \tag{4}$$

Let $\mathcal{M}_i(B_k) = \{D \in \mathcal{M}(B_k) : B_k/D \simeq C_{i+1}\}$, $1 \le i \le n$, $1 \le k \le m$, then

$$\beta_{i,k}^n = |\mathcal{M}_i(B_k)|. \tag{5}$$

By (4), (5) and T7 we have

$$|B_k| = \prod_{i=1}^{n}(i+1)^{\beta_{i,k}^n}. \tag{6}$$

From (3) and (6)

$$|\mathcal{L}(n,m)| = \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}\prod_{i=1}^{n}(i+1)^{\beta_{i,k}^n}. \tag{7}$$

From T6 we have that for any $D \in M_i(B_k)$ there is a unique $M \in \mathcal{M}(\mathcal{L}(n,m))$ such that $D = M \cap B_k$. Then to compute $\beta_{i,k}^n$ we must determine the number of maximal deductive systems M of $\mathcal{L}(n,m)$ which satisfy

(M1) $B_k \not\subseteq M$,

(M2) If $D = M \cap B_k$ then $B_k/D \simeq C_{i+1}$.

Let $\mathcal{M}_{i,k}^n = \{M \in \mathcal{M}(\mathcal{L}(n,m)) : M$ verifies M1 and M2$\}$. Then,

Lemma 2.5 *For any $h \in Hom(\mathcal{L}(n,m), C_{n+1})$ such that $|h(\mathcal{L}(n,m)| > 1$ there is $\overline{h} \in Hom(\mathcal{L}(n,m), C_{n+1})$ which verifies:*

(i) $Ker(\overline{h}) = Ker(h)$,

(ii) $\overline{h}(\mathcal{L}(n,m)) = [\frac{n-t}{n}, 1] \simeq h(\mathcal{L}(n,m))$.

Proof. Since $h(\mathcal{L}(n,m)) = S$ is a subalgebra of C_{n+1}, by T2 and T3 there are isomorphisms $h_1 : S \longrightarrow C_{t+1}$, $1 \leq t \leq n$, and $h_2 : C_{t+1} \longrightarrow [\frac{n-t}{n}, 1]$. Then $\overline{h} = h_2 \circ h_1 \circ h$ satisfies (i) and (ii). □

By T6, T7 and Lemma 2.5 we have that for any $M \in \mathcal{M}^n_{i,k}$ there is a unique $h \in Hom(\mathcal{L}(n,m), C_{i+1})$ which satisfies

(H0) $M = Ker(h)$,

(H1) $B_k \not\subseteq Ker(h)$,

(H2) $h(B_k) = [\frac{n-i}{n}, 1]$.

As $h(B_k)$ is a subalgebra of $\mathcal{L}(n,m))$, from H2 and T4 it follows

(H3) $h(\mathcal{L}(n,m)) = [\frac{n-j}{n}, 1] \supseteq [\frac{n-i}{n}, 1]$.

Let $\mathcal{H}^n_{i,k} = \{h \in Hom(\mathcal{L}(n,m), C_{i+1}) : h$ verifies H1 and H2$\}$, then

$$\beta^n_{i,k} = |\mathcal{H}^n_{i,k}|. \tag{8}$$

For each $h \in \mathcal{H}^n_{i,k}$, the restriction $f = h|G$, verifies

(F1) $f(G_k) \subseteq [0, \frac{n-i}{n}]$,

(F2) there exists $g \in G_k$ such that $f(g) = \frac{n-i}{n}$,

(F3) $[f(G)]_C \supseteq [\frac{n-i}{n}, 1]$.

Indeed:

(F1): By H2 we have $f(g_k^*) = \frac{n-i}{n}$. Since $g \leq g_k^*$ for all $g \in G_k$ and h is isotone, we have $h(g) \leq h(g_k^*)$. Then $f(g) \leq \frac{n-i}{n}$ for all $g \in G_k$.

(F2): If $f(g) < \frac{n-i}{n}$ for all $g \in G_k$, then we have $h(g_k^*) < \frac{n-i}{n}$ which contradicts H2.

(F3): From $B_k \lhd \mathcal{L}(n,m)$ it results $h(B_k) \lhd h(\mathcal{L}(n,m))$. Moreover, $h(\mathcal{L}(n,m)) = h([G]_C) = [f(G)]_C$. By H2 $[\frac{n-i}{n}, 1] \lhd [f(G)]_C$ and by T4 we have $[f(G)]_C = [\frac{n-j}{n}, 1] \supseteq [\frac{n-i}{n}, 1]$, where $\frac{n-j}{n} = \min f(G)$.

Let C^G_{n+1} be the set of the functions from G into C_{n+1} and

$\mathcal{F}^n_{i,k} = \{f \in C^G_{n+1} : f$ satisfies F1, F2 and F3$\}$.

Let $f \in \mathcal{F}^n_{i,k}$ and $h \in Hom(\mathcal{L}(n,m), C_{i+1})$ be the unique extension of f. Then $h \in \mathcal{H}^n_{i,k}$. Indeed:

(H1): $h(g_k^*) = h(g_1 \vee g_2 \vee \ldots \vee g_k) = f(g_1) \vee f(g_2) \vee \ldots \vee f(g_k) = \frac{n-i}{n} < i$, hence $B_k \not\subseteq Ker(h)$.

(H2): Let $z \in h(B_k)$ and $x \in B_k$ be such that $h(x) = z$. From $g_k^* \leq x$ and h isotone, we obtain $h(g_k^*) \leq h(x)$. Hence $\frac{n-i}{n} \leq z$ and therefore $h(B_k) \subseteq [\frac{n-i}{n}, 1]$. On the other hand if $z' \in [\frac{n-i}{n}, 1]$, then by F3 we have $z' \in [\frac{u}{n}, 1] = [f(G)]_C = h(\mathcal{L}(n,m))$ and so there is $a \in \mathcal{L}(n,m)$ such that $h(a) = z'$. Hence $g_k^* \vee a \in B_k$ and $h(g_k^* \vee a) = z'$.

(H3): It is a consequence from the fact that h extends f.

The application $f \mapsto h$ is a bijection. Then by (8)

$$\beta^n_{i,k} = |\mathcal{F}^n_{i,k}| \tag{9}$$

In order to determine $|\mathcal{L}(n+1,m)|$ in function of n and m in some particular cases, we compute $|\beta^n_{i,k}|$ for some integers i,k.

Let $U_t = \{u_1, u_2, \ldots, u_t\}$ be a set of free generetors of C^0_{n+1}–algebra $\mathcal{L}^0(n+1,t)$. In [13, pp. 135] it is proved that

$$v_t(n+1) = |Epi_{C^0}(\mathcal{L}^0(n+1,t), C_{n+1})| = (n+1)^t - \sum_{j/n,\ j\neq n} v_t(j+1). \tag{10}$$

Lemma 2.6 $|\beta^n_{n,k}| = v_{m-k}(n+1)$.

Proof. Let $\beta : G_{m-k} \longrightarrow U_{m-k}$ be the mapping defined by $\beta(g_{k+j}) = u_j$, $1 \leq j \leq m-k$, and for each $h \in Epi_{C^0}(\mathcal{L}^0(n+1,m-k), C_{n+1})$ we consider $f_1 = h_{1|U_{m-k}}$, $f_2 = f_1 \circ \beta$ and $f_h : G \longrightarrow C_{n+1}$ defined by

$$f_h(g) = \begin{cases} 0, & \text{if } g \in G_k, \\ f_2(g), & \text{otherwise} \end{cases}.$$

Then, the mapping $\psi : Epi_{C^0}(\mathcal{L}^0(n+1,m-k), C_{n+1}) \longrightarrow \mathcal{F}^n_{n,k}$ defined by $\psi(h) = f_h$ is a bijection. We only check that $f_h \in \mathcal{F}^n_{n,k}$. It is clear that f_h verifies F1 and F2, on the other hand $[f_h(G)]_C = [f_1(U_{m-k})]_C \subseteq [f_1(U_{m-k})]_{C^0} = C_{n+1}$. Since $f_h(G_k) = \{0\}$ we have that $0 \in [f_1(G)]_C$, then $[f_1(U_{m-k})]_C = C_{n+1}$ and so $[f_h(G)]_C = C_{n+1}$ and it verifies F3. Then, from (10) we have that $|\mathcal{F}^n_{n,k}| = |Epi_{C^0}(\mathcal{L}^0(n+1,m-k), C_{n+1})| = v_{m-k}(n+1)$. □

Lemma 2.7 $|\beta^n_{(n-1),k}| = (2^k - 1)\cdot(n+1)^{m-k} - n^{m-k} + \upsilon_{m-k}((n-1)+1)$.

Proof. Let

$$A = \{f \in C^G_{n+1} : f(G_k) = \{0, \tfrac{1}{n}\}\},$$

$$B = \{f \in C^G_{n+1} : f(G_k) = \{\tfrac{1}{n}\}, 0 \in f(G_{m-k})\},$$

$$C = \{f \in C^G_{n+1} : f(G_k) = \{\tfrac{1}{n}\}, f(G_{m-k}) \subseteq [\tfrac{1}{n}, 1], [f(G)]_{\mathbf{C}} = [\tfrac{1}{n}, 1]\}.$$

It is easy to see that $\{A, B, C\}$ is a partition of $\mathcal{F}^n_{(n-1),k}$ and it holds:

(i) $|A| = (2^k - 2)\cdot(n+1)^{m-k}$,

(ii) $|B| = (n+1)^{m-k} - n^{m-k}$.

If $\beta : G_{m-k} \longrightarrow U_{m-k}$ is defined as in Lemma 2.6, $\alpha : C_n \longrightarrow C_{n+1}$ is defined by $\alpha(\frac{j}{n-1}) = \frac{j+1}{n}$, and for each $h \in Epi_{\mathbf{C}^0}(\mathcal{L}^0(n+1, m-k), C_n)$ we consider $f_1 = h_{|U_{m-k}}$, $t_h = \alpha \circ f_1 \circ \beta$ and $\bar{f}_h : G \longrightarrow C_{n+1}$ defined by

$$\bar{f}_h(g) = \begin{cases} \frac{1}{n}, & \text{if } g \in G_k, \\ t_h(g), & \text{otherwise} \end{cases},$$

then, the mapping $\psi : Epi(\mathcal{L}^0(n+1, m-k), C_{n+1}) \longrightarrow C$, $\psi(h) = \bar{f}_h$ is a bijection. We only check that $\bar{f}_h \in \mathcal{F}^n_{(n-1),k}$. It is easy to see that $\bar{f}_h$ verifies F1 and F2. Since $[\bar{f}_h(G)]_{\mathbf{C}} \subseteq [\frac{1}{n}, 1]$ and $[\bar{f}_h(G_{m-k})]_{\mathbf{C}}| = [\alpha(f_1(U_{m-k})]_{\mathbf{C}} = \alpha(C_n)$, we get $[f_h(G)]_{\mathbf{C}} = [\frac{1}{n}, 1]$. Then

(iii) $|C| = \upsilon_{m-k}(n)$.

From (i), (ii) and (iii) it follows Lemma 2.7. □

Lemma 2.8 $|\beta^n_{1,k}| = (n^k - (n-1)^k)\cdot(n+1)^{m-k}$.

Proof. Let

$$A = \{f \in C^{G_k}_{n+1} : f(G_k) \subseteq [0, \tfrac{n-1}{n}], \tfrac{n-1}{n} \in f(G_k)\},$$

$$B = C^{G_{m-k}}_{n+1}.$$

We define $\psi : F_{1+1}(k) \rightarrow A \times B$ by $\psi(f) = (f_A, f_B)$ where $f_A = f_{|G_k}$, $f_B = f_{|G_{m-k}}$. It is easy to see that ψ is injective. Let $(f_1, f_2) \in A \times B$ and $f \in C^G_{n+1}$ be defined by $f(g) = f_1(g)$ if $g \in G_k$ and $f(g) = f_2(g)$ if $g \in G_{m-k}$. Then, $f \in \mathcal{F}^n_{1,k}$. Indeed, it is clear that f verifies F1 and F2. Since $\frac{n-1}{n} \in f(G)$, then by T4 $[f(G)] = [\frac{n}{n}, 1]$ and so f verifies F3. Furthermore $\psi(f) = (f_1, f_2)$, then ψ is onto.

It holds:

(i) $|A| = n^{m-k} - (n-1)^{m-k}$,

(ii) $|B| = (n+1)^{m-k}$.

From (i) and (ii) it follows the Lemma 2.8. □

Examples 2.1 Now, we apply the above results to those values of n for which no additional calculus must be done.

From (10) it follows:

$$
\begin{aligned}
v_{m-k}(1+1) &= 2^{m-k}, &(11)\\
v_{m-k}(2+1) &= 3^{m-k} - 2^{m-k}, &(12)\\
v_{m-k}(3+1) &= 4^{m-k} - 2^{m-k}. &(13)
\end{aligned}
$$

Taking into account (11), (12) and (13), we obtain

(E1) $n = 1$:

$$
\begin{aligned}
|\beta^1_{1,k}| &= (1^k - (1-1)^k) \cdot (1+1)^{m-k} = 2^{m-k},\\
|\mathcal{L}(2,m)| &= \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \cdot 2^{2^{m-k}}. \qquad (14)
\end{aligned}
$$

The formula 14 has been obtained by A. Monteiro and L. Iturrioz in [7].

(E2) $n = 2$:

$$
\begin{aligned}
|\beta^2_{1,k}| &= (2^k - (2-1)^k) \cdot (2+1)^{m-k} = (2^k - 1) \cdot 3^{m-k},\\
|\beta^2_{2,k}| &= v_{m-k}(2+1) = 3^{m-k} - 2^{m-k},\\
|\mathcal{L}(3,m)| &= \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \cdot 2^{(2^k-1)\cdot 3^{m-k}} \cdot 3^{3^{m-k} - 2^{m-k}}. \qquad (15)
\end{aligned}
$$

The formula 15 has been obtained by L. Iturrioz and O. Rueda in [8].

(E3) $n = 3$:

$$\begin{aligned} |\beta^3_{1,k}| &= (3^k - 2^k)\, 4^{m-k}, \\ |\beta^3_{2,k}| &= (2^k - 1)\cdot 4^{m-k} - 3^{m-k} + 3^{m-k} - 2^{m-k} = (2^k - 1)\cdot 4^{m-k} - 2^{m-k}, \\ |\beta^3_{3,k}| &= v_{m-k}(3+1) = 4^{m-k} - 2^{m-k}, \\ |\mathcal{L}(4,m)| &= \sum_{k=1}^{m} (-1)^{k+1}\binom{m}{k}\cdot 2^{(3^k-2^k)\cdot 4^{m-k}}\cdot 3^{(2^k-1)\cdot 4^{m-k}}\cdot 4^{4^{m-k}-2^{m-k}}. \end{aligned}$$

References

[1] B. Bosbach, *Residuation groupoids*, Bull. Acad. Pol. Sci., 22(1974), 103–104.

[2] C. C. Chang, *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc., 88(1958), 467–490.

[3] A. V. Figallo, *I_{n+1}–álgebras con operaciones adicionales*, Doctoral Thesis, Univ. Nac. del Sur, 1989, Bahia Blanca, Argentina.

[4] J. M. Font, A. Rodriguez and A. Torrens, *Wajsberg algebras*, Stochastica, 8, 1(1984), 5–31.

[5] R. Grigolia, *Algebraic analysis of Lukasiewicz - Tarski n–valued systems*, in R. Wojcicki and G. Malinowski (Eds.).

[6] K. Iseki and S. Tanaka, *An introduction to the theory of* BCK*–algebra*, Mat. Japonica, 23(1978), 1–26.

[7] L. Iturrioz and A. Monteiro, *Cálculo Proposicional implicativo clásico con n variables proposicionales*, Rev. de la Unión Mat. Argentina, 22(1966), 146.

[8] L. Iturrioz and O. Rueda, *Algèbres implicatives trivalentes de Lukasiewicz libres*, Discrete Mathematics, 18(1977), 35–44.

[9] Y. Komori, *The separation theorem of the $\aleph_0$–valued Lukasiewicz propositional logic*, Reports of the Faculty of Sciences, Shizuoka University, 12(1978), 1–5.

[10] R. McNaugton, *A theorem about infinite - valued sentential logic*, Jour. Symb. Logic, 16(1951), 1–13.

[11] R. K. Meyer, *Pure denumerable Lukasiewiczian implication*, Jour. Symb. Logic, 31(1966), 575–580.

[12] A. Monteiro, *Algebras implicativas trivalentes de Lukasiewicz*, Lectures given at the Univ. Nac. del Sur, Bahia Blanca, Argentina, 1968.

[13] A. J. Rodriguez Salas, *Un estudio algebraico de los cálculos proposicionales de Lukasiewicz*, Doctoral Thesis, Univ. de Barcelona (1980).

[14] A. Rose, *Extensions de quelques theoremes de Mc Naughton*, C. R. Acad. Sci. Paris, 292(1981), 979–981.

[15] S. Tanaka, *On $\wedge$-commutative algebras*, Math. Seminar Notes, 3(1975), 59–64.

[16] T. Traczyk, *On the variety of bounded commutative* BCK-*algebras*, Math. Japonica, 24, 3(1979), 283–292.

[17] B. Wozniakowska, *Algebraic proof of the separation theorem, for the infinite – valued logic of Lukasiewicz*, Rep. on Math. logic., 9(1978), 129–137.

[18] H. Yutani, *Quasi-commutative* BCK-*algebras and conguence relations*, Math. Seminar Notes, 5(1977), 469–480.

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