Revista de la
Unión Matemática Argentina
Volumen 41, Nro. 4, 2000, 45-46

Counterexample to a conjecture of Mujica

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Abstract

Let U be an open subset of a Banach space E. In [2] Mujica shows there is a unique Banach space $G^\infty(U)$ and a bounded holomorphic mapping δ_U from U into $G^\infty(U)$ with the property that given any Banach space F every bounded holomorphic function from U into F factors through $G^\infty(U)$ as a continuous linear mappings. The properties the Banach space $G^\infty(U)$ are similar to those of E. In [4] Mujica asks if $G^\infty(U)$ is weakly sequentially complete when E is weakly sequentially complete. In this paper we provide a counterexample to this conjecture.

In [2] Mujica proves the following result:

Theorem 1. *(Mujica) Let U be an open subset of a Banach space E then there is a Banach space $G^\infty(U)$ and $\delta_U \in \mathcal{H}^\infty(U; G^\infty(U))$ such that the following universal property holds: Given any Banach space F and any $f \in \mathcal{H}^\infty(U; F)$ there is a unique continuous linear operator $T_f\colon G^\infty(U) \to F$ such that $f = T_f \circ \delta_U$.*

The pair $G^\infty(U)$ and δ_U are characterized up to isometric isomorphism by this property. The Banach space $G^\infty(U)$ can be realised as the space of all linear functionals on $\mathcal{H}^\infty(U)$ whose restriction to each multiple of the unit ball of $\mathcal{H}^\infty(U)$ is continuous for the compact open topology. The holomorphic function δ_U is then defined by $\delta_U(x) = \delta_x$ where $\delta_x(f) = f(x)$. Furthermore $\mathcal{H}^\infty(U)$ is isometrically isomorphic to $G^\infty(U)'_b$, the strong dual of $G^\infty(U)$.
The vector space properties of $G^\infty(U)$ are closely related to those of E. Indeed in [2], Mujica shows that if U is balanced open in E then $G^\infty(U)$ has the approximation property if and only if E has the approximation property, while if B_E is the open unit ball of E then $G^\infty(B_E)$ has the metric approximation property if and only if E has the metric approximation property.
Continuing the study of $G^\infty(U)$ in [4] Mujica poses the following question:

Problem. *Let U be a bounded open subset of a weakly sequentially complete Banach space E. Is $G^\infty(U)$ weakly sequentially complete?*

We will give an example to show that the answer to this question is no. We begin with the observation that $L^1(\partial\Delta)/H_0^1(\Delta)$ is the unique isometric predual of $\mathcal{H}^\infty(\Delta)$ (see [1]) and that this space is weakly sequentially complete and has cotype 2. We shall need the following result of Pisier [5].

Theorem 2. *(Pisier) There is a weakly sequentially complete Banach space Z with cotype 2 such that $\left(L^1(\partial\Delta)/H_0^1(\Delta)\right) \widehat{\bigotimes}_\pi Z$ contains a copy of c_0.*

In particular this will mean that $\left(L^1(\partial\Delta)/H_0^1(\Delta)\right) \widehat{\bigotimes}_\pi Z$ is not weakly sequentially complete and does not have cotype 2.
Clearly we have that $\mathbf{C} \bigoplus_\infty Z$ is weakly sequentially complete.
By Proposition 2.3 of [2] we see that Z is isomorphic to a complemented subspace of $G^\infty(B_Z)$. By Theorem 6.1 of [3]

$$\begin{aligned} G^\infty(\Delta \times B_Z) &\simeq G^\infty(\Delta) \widehat{\bigotimes}_\pi G^\infty(B_Z) \\ &\simeq \left(L^1(\partial\Delta)/H_0^1(\Delta)\right) \widehat{\bigotimes}_\pi G^\infty(B_Z) \end{aligned}$$

which contains $\left(L^1(\partial\Delta)/H_0^1(\Delta)\right) \widehat{\bigotimes}_\pi Z$ as a complemented subspace. Therefore we see that $G^\infty(\Delta \times B_Z)$ cannot be weakly sequentially complete. The space $\mathbf{C} \bigoplus_\infty Z$ is also an example of a Banach space with cotype 2 with open unit ball $\Delta \times B_Z$ such that $G^\infty(\Delta \times B_Z)$ does not have cotype 2.

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Recibido : 11 de Febrero de 2000
Aceptado : 22 de Marzo de 2000