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INVERSE THEOREM FOR A NEW SEQUENCE OF LINEAR POSITIVE OPERATORS ON L_p - SPACES

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ABSTRACT

The present paper is a continuation of our work in [2] wherein we had developed a direct theorem in terms of the higher order integral modulus of smoothness for a linear combination of a new sequence of linear positive operators in the L_p −norm. The aim of the present paper is to discuss the corresponding inverse theorem.

1 INTRODUCTION

Following [2], for $f \in L_p[0,\infty)$, $p \geq 1$, the new sequence of linear positive operators M_n is defined as

(1.1) $$M_n(f(u);x) = \int_0^\infty W_n(x,u)\, f(u)\, du$$

where

$$W_n(x,u) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x)\, p_{n,k-1}(u) + (1+x)^{-n}\delta(u),$$

$\delta(u)$ being the Dirac-delta function and $p_{n,k}(u) = \binom{n+k-1}{k} u^k (1+u)^{-(n+k)}$.

The linear combination, due to May [5] and Rathore [7], of the operators M_n is given by

(1.2) $$M_n(f,k,x) = \sum_{j=0}^{k} C(j,k)\; M_{d_j n}(f;x),$$

where

$$C(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, \text{ for } k \neq 0 \text{ and } C(0,0) = 1.$$

Throughout this paper, let $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$, $I_i = [a_i, b_i]$, i =1,2,3 . Also, let $\phi(.)$ denote the characteristic function of I_1.

Agrawal and Thamer [2] established that for smoother functions, the rate of convergence by the linear combination (1.2) of the operators M_n is faster in the L_p −norm. In view of [Theorem, 2], it follows that if $1 \leq p < \infty$, $f \in L_p[0,\infty)$, $0 < \alpha \leq 2k+2$ and

$$\omega_{2k+2}(f,\tau,p,I_1) = O(\tau^{\alpha}) \text{ as } \tau \to 0,$$

then,

$$\|M_n(f,k,.)-f\|_{L_p(I_3)}=O\left(n^{-\alpha/2}\right) \text{ as } n\to\infty.$$

Here, first we obtain a corresponding inverse result, i.e., characterization of the class of functions for which

$$\|M_n(f,k,.)-f\|_{L_p(I_1)}=O(n^{-\alpha/2}) \text{ as } n\to\infty,$$

where $0<\alpha<2k+2$.

The aim of the present paper is to prove the inverse theorem in L_p −norm.

2 PRELIMINARY RESULTS

In order to prove our main result we shall require the following results.

Without any loss of generality we can assume f to have compact support in $[0,\infty)$. To illustrate this, let ψ be the characteristic function of $[a_1-\delta,b_1+\delta],\delta>0$. We can write

$$\|M_n(f\psi,k,.)-f\psi\|_{L_p(I_1)}\le\|M_n(f\psi-f,k,.)\|_{L_p(I_1)}+\|M_n(f,k,.)-f\|_{L_p(I_1)}.$$

Applying Jensen's inequality, we get

$$\|M_n(f(\psi-1),k,.)\|^p_{L_p(I_1)}\le C_0\int_{a_1}^{b_1}\int_0^{\infty}|f(u)|^p\,|\psi(u)-1|^p\,W_n(x,u)\,du\,dx.$$

The presence of $\psi-1$ implies that $|u-x|>\delta$. For sufficiently large u there exist positive constants C_1 and M_0 such that $\dfrac{(u-x)^{2k+2}}{u^{2k+2}+1}>C_1$ for all $u\ge M_0$ and for $u<M_0$ we have $(u-x)^{2k+2}>\delta^{2k+2}$. We break the integration in u in two parts as $u<M_0$ and $u\ge M_0$ and proceeding as in the estimate of J_4 in [Proposition,2] we get

$$\|M_n(f(\psi-1),k,.)\|_{L_p(I_1)}\le C_2\,n^{-l}\|f\|_{L_p[0,\infty)},$$

where $l>0$ is arbitrary but fixed.

LEMMA 2.1. Let $h\in L_1[0,\infty)$ have a compact support. Then, for $r\in N$

$$\left\|\int_0^{\infty}W_n(x,u)\left(\frac{k}{n}-x\right)^r h(u)\,du\right\|_{L_1[0,\infty)}\le C_3\,n^{-r/2}\|h\|_{L_1[0,\infty)},$$

where C_3 is a constant independent of n and h.

PROOF. On an application of Fubini's theorem and Hölder's inequality, we obtain

$$(2.1)\quad (n-1)\int_0^{\infty}\int_0^{\infty}\left(\sum_{k=1}^{\infty}p_{n,k}(x)p_{n,k-1}(u)+(1+x)^{-n}\delta(u)\right)\left|\frac{k}{n}-x\right|^r|h(u)|\,du\,dx$$

$$=\int_0^{\infty}\sum_{k=1}^{\infty}\left\{\int_0^{\infty}(n-1)p_{n,k}(x)\left|\frac{k}{n}-x\right|^r dx\right\}p_{n,k-1}(u)\,|h(u)|\,du$$

$$\leq \int_0^\infty \sum_{k=1}^\infty \left\{ \int_0^\infty (n-1) p_{n,k}(x) \left| \frac{k}{n} - x \right|^{2r} dx \right\}^{1/2} p_{n,k-1}(u) |h(u)| du$$

$$= \int_0^\infty \sum_{k=1}^\infty \left\{ \sum_{j=0}^{2r} \binom{2r}{j} \left(\frac{k}{n}\right)^{2r-j} (-1)^j \int_0^\infty (n-1) p_{n,k}(x) x^j dx \right\}^{1/2} p_{n,k-1}(u) |h(u)| du$$

$$= \int_0^\infty \sum_{k=1}^\infty \left\{ \sum_{j=0}^{2r} \binom{2r}{j} \left(\frac{k}{n}\right)^{2r-j} (-1)^j \frac{\left(\frac{k}{n}+\frac{1}{n}\right)\cdots\left(\frac{k}{n}+\frac{j}{n}\right)}{\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j+1}{n}\right)} \right\}^{1/2} p_{n,k-1}(u) |h(u)| du.$$

Making use of the identities $\prod_{i=1}^{j} \left(\frac{k}{n}+\frac{i}{n}\right) = \left(\frac{k}{n}\right)^j + \left(\frac{k}{n}\right)^{j-1} \frac{1}{n} P_1(j) + \left(\frac{k}{n}\right)^{j-2} \frac{1}{n^2} P_2(j) + \cdots$, $\frac{1}{\prod_{i=2}^{j+1}\left(1-\frac{i}{n}\right)} = 1 + \frac{1}{n} Q_1(j) + \frac{1}{n^2} Q_2(j) + \cdots$, where $P_k(j); Q_k(j)$ are polynomials in j of degree $2k$ and $\sum_{j=0}^{2r} \binom{2r}{j} (-1)^j j^k = 0$, if $k < 2r$, in (2.1), we have

$$\left\| \int_0^\infty W_n(x,u) \left(\frac{k}{n} - x\right)^r h(u)\, du \right\|_{L_1[0,\infty)} \leq C_4 \int_0^\infty \sum_{k=1}^\infty \left\{ \left(\frac{k}{n}\right)^{2r} \frac{1}{n^r} + \frac{1}{n} \left(\frac{k}{n}\right)^{2r-1} \times \frac{1}{n^{r-1}} + \cdots + \frac{1}{n^{2r}} \right\}^{1/2} p_{n,k-1}(u) |h(u)| du.$$

Hölder's inequality for sum and compactness of h imply the result.

LEMMA 2.2. Let $h \in L_p[0,\infty), p \geq 1$ have a compact support; $i, j \in N^0$ (the set of nonnegative integers) and $m > 0$ be fixed. Then for a constant C_5 independent of n and h there holds

$$\left\| \int_0^\infty W_n(x,u) \left(\frac{k}{n} - x\right)^i \int_x^u (u-w)^j h(w) dw\, du \right\|_{L_p(I_3)} \leq C_5 \left\{ n^{-(i+j+1)/2} \|h\|_{L_p(I_1)} + n^{-m} \|h\|_{L_p[0,\infty)} \right\}.$$

PROOF. Let $\psi(u)$ be the characteristic function of I_1. By an application of Jensen's inequality

$$\left| \sum_{k=1}^\infty p_{n,k}(x) \left(\frac{k}{n} - x\right)^i \int_0^\infty (n-1) p_{n,k-1}(u) \int_x^u (u-w)^j h(w)\, dw\, du \right|^p \tag{2.2}$$

$$\leq \sum_{k=1}^\infty p_{n,k}(x) \left| \frac{k}{n} - x \right|^{ip} \int_0^\infty (n-1) p_{n,k-1}(u) |u-w|^s \left| \int_x^u |h(w)|^p dw \right| du, \quad (s = jp + p - 1)$$

$$= \sum_{k=1}^{\infty} p_{n,k}(x)\left|\frac{k}{n}-x\right|^{ip}\int_0^{\infty}(n-1)\psi(u)\,p_{n,k-1}(u)|u-w|^{s}\left|\int_x^u |h(w)|^p\,dw\right|du+\sum_{k=1}^{\infty} p_{n,k}(x)$$

$$\times\left|\frac{k}{n}-x\right|^{p}\int_0^{\infty}(n-1)(1-\psi(u))\,p_{n,k-1}(u)|u-w|^{s}\left|\int_x^u |h(w)|^p\,dw\right|du.$$

In the first term we divide integration in "u" over $[x+ln^{-1/2},x+(l+1)n^{-1/2}]$, $l=0,\pm1,\pm2,\cdots,\pm r$; where $r=r(n)\in N$ satisfies $rn^{-1/2}\le \max(b_1-a_3,b_3-a_1)\le(r+1)n^{-1/2}$. It is similar to [Proposition ,2] and [Theorem3.2, 4]. A typical element of 1st term is now L_p –bounded by

$$\frac{n^2}{l^4}\int_{a_3}^{b_3}\left[\sum_{k=1}^{\infty} p_{n,k}(x)\left|\frac{k}{n}-x\right|^{ip}\int_{x+ln^{-1/2}}^{x+(l+1)n^{-1/2}}(n-1)\,p_{n,k-1}(u)|u-w|^{s+4}\,du\right]$$

$$\times\left(\int_x^{x+(l+1)n^{-1/2}}\psi(w)|h(w)|^p\,dw\right)dx.$$

We, now use Hölder's inequality for infinite sum coupled with moment estimates for a new sequence of linear positive operators [1] and [2] and finally Fubini's theorem to obtain estimate. The presence of factor $(1-\psi(u))$ in second term in (2.2) implies $|u-w|/\delta>1$. This gives arbitrary order $O(n^{-m})$. This completes the proof.

LEMMA 2.3. There exist the polynomials $\phi_{i,j,r}(x)$ independent of n and k such that

$$\frac{d^r}{dx^r}\left[x^k(1+x)^{-(n+k)}\right]=T^{-r}\sum_{\substack{2i+j\le r\\ i,j\ge0}} n^i(k-nx)^j\,\phi_{i,j,r}(x)\,x^k(1+x)^{-(n+k)},$$

where $T=t(1+t)$; and $2i+j\le r, i,j\in N^0$.

LEMMA 2.4. Let $h\in L_p(0,\infty), p\ge1$ and supp $h\subset I_3$. Then

(2.3) $$\left\|M_n^{(2k+2)}(h,.)\right\|_{L_p(I_3)}\le C_6\,n^{k+1}\|h\|_{L_p(I_3)}.$$

Moreover, if $h^{(2k+1)}\in A.C.(I_3)$ and $h^{(2k+2)}\in L_p(I_3)$, then

(2.4) $$\left\|M_n^{(2k+2)}(h,.)\right\|_{L_p(I_3)}\le C_7\left\|h^{(2k+2)}\right\|_{L_p(I_3)},$$

where C_6 and C_7 are constants independent of n and h.

PROOF. Since $\phi_{i,j,2k+2}(x)$ and $T^{-(2r+2)}$ are bounded on I_3, it follows from Lemmas 2.1 and 2.3 that for $h\in L_1[0,\infty)$, $\left\|M_n^{(2k+2)}(h,.)\right\|_{L_1(I_3)}\le C_8\,n^{k+1}\|h\|_{L_1(I_3)}$.

If $h\in L_\infty[0,\infty)$, then by Lemma 2.3 and moment estimates [1], we get

$$\left\|M_n^{(2k+2)}(h,.)\right\|_{L_\infty(I_3)}\le C_9\,n^{k+1}\|h\|_{L_\infty(I_3)}.$$

Now, using Riesz-Thorin interpolation theorem [6], we obtain (2.3).
To prove (2.4), the differentiability properties of h imply that

$$h(u)=\sum_{r=0}^{2k+1}\frac{(u-x)^r}{r!}h^{(r)}(x)+\frac{1}{(2k+1)!}\int_x^u (u-w)^{2k+1}h^{(2k+2)}(w)\,dw.$$

Since $M_n(.,t)$ preserves degree of polynomials, using Lemma 2.3, we have

$$M_n^{(2k+2)}(h,u)=\frac{(n-1)}{(2k+1)!T^{2k+2}}\sum_{k=1}^{\infty}p_{n,k}(x)\left\{\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(k-nx)^j\phi_{i,j,2k+2}(x)\int_0^{\infty}p_{n,k-1}(u)\right.$$
$$\left.\times\int_x^u (u-w)^{2k+1}h^{(2k+2)}(w)\,dw\,du\right\}.$$

Now, applying Lemma 2.2 in the above equality we obtain (2.4).

3 MAIN RESULT

THEOREM. Let $0<\alpha<2k+2,\ f\in L_p[0,\infty), p\ge 1$ and

(3.1) $$\|M_n(f,k,.)-f\|_{L_p(I_1)}=O\left(n^{-\alpha/2}\right),\ n\to\infty,$$

then,

$$\omega_{2k+2}(f,\tau,p,I_3)=O(\tau^{\alpha}),\ \text{ as } \tau\to 0.$$

PROOF. We choose a function $g\in C_0^{2k+2}$ such that $\operatorname{supp} g\subset(t_2,y_2)$, $g(x)=1$ on $[t_3,y_3]$ and $a_1<t_1<t_2<t_3<a_3<b_3<y_3<y_2<y_1<b_1$. Writing $fg=\bar{f}$, for all values of $\gamma\le\tau$, we have

$$\left\|\Delta_\gamma^{2k+2}\bar{f}\right\|_{L_p[t_2,y_2]}\le\left\|\Delta_\gamma^{2k+2}\left(\bar{f}-M_n(\bar{f},k,.)\right)\right\|_{L_p[t_2,y_2]}+\left\|\Delta_\gamma^{2k+2}M_n(\bar{f},k,.)\right\|_{L_p[t_2,y_2]},$$

where Δ_γ^{2k+2} denotes $(2k+2)$th order forward difference. Making use of Jensen's inequality repeatedly and finally Fubini's theorem in second term, we obtain

$$\left\|\Delta_\gamma^{2k+2}\bar{f}\right\|_{L_p[t_2,y_2]}\le\left\|\Delta_\gamma^{2k+2}\left(\bar{f}-M_n(\bar{f},k,.)\right)\right\|_{L_p[t_2,y_2]}$$
$$+\gamma^{2k+2}\left\|M_n^{(2k+2)}(\bar{f},k,.)\right\|_{L_p[t_2,y_2+(2k+2)\gamma]}.$$

Now, we write $\bar{f}=(\bar{f}-\bar{f}_{\eta,2k+2})+\bar{f}_{\eta,2k+2}$ in $\left\|M_n^{2k+2}(\bar{f},k,.)\right\|_{L_p[t_2,y_2+(2k+2)\gamma]}$ where $\bar{f}_{\eta,2k+2}$ is the Steklov mean [2,4,8] of $(2k+2)$th order corresponding to $\bar{f}$ and then recall Lemma 2.4. It follows from properties of Steklov means [2] that for sufficiently small $\eta>0$,

$$\left\|\Delta_\gamma^{2k+2}\bar{f}\right\|_{L_p[t_2,y_2]}\le\left\|\Delta_\gamma^{2k+2}\left(\bar{f}-M_n(\bar{f},k,.)\right)\right\|_{L_p[t_2,y_2]}$$
$$+C_{10}\gamma^{2k+2}\left(n^{k+1}+\eta^{-(2k+2)}\right)\omega_{2k+2}\left(\bar{f},\eta,p,[t_2,y_2]\right).$$

Now, following Berens and Lorentz [3] we can complete the proof, once it is established that

(3.2) $$\left\|\Delta_\gamma^{2k+2}\left(\bar{f}-M_n(\bar{f},k,.)\right)\right\|_{L_p[t_2,y_2]}=O\left(n^{-\alpha/2}\right),\ n\to\infty.$$

Thus,

$$\omega_{2k+2}\left(\bar{f},\tau,p,[t_2,y_2]\right)=O\left(\tau^\alpha\right),\ \tau\to 0.$$

Therefore, as $\bar{f}(x)=f(x)$ for $x\in[t_3,y_3]$

$$\omega_{2k+2}\left(f,\tau,p,I_3\right)=O\left(\tau^\alpha\right),\quad \tau\to 0, \text{ as required.}$$

We prove (3.2) by induction on α. First, assume $\alpha\le 1$.

$$\|M_n(fg,k,.)-fg\|_{L_p[t_2,y_2]}\le\|M_n(g(x)(f(u)-f(x)),k,x)\|_{L_p[t_2,y_2]}$$
$$+\|M_n(f(u)(g(u)-g(x)),k,x)\|_{L_p[t_2,y_2]}.$$

Now, $g(u)-g(x)=(u-x)g'(\xi)$ for some ξ lying between u and x. Using moment estimates and the compactness of f to estimate the second term and statement (3.1) for the first term we have

$$\|M_n(fg,k,.)-fg\|=O\left(n^{-\alpha/2}\right)+O\left(n^{-1/2}\right)=O\left(n^{-\alpha/2}\right).$$

Now, we assume (3.2) to hold true for all values of α satisfying $r-1<\alpha<r$ and prove that the same holds true for $r<\alpha<r+1$. Thus, we have

$$\omega_{2k+2}\left(f,\tau,p,[c,d]\right)=O\left(\tau^{r-1+\beta}\right),\ \tau\to 0,\ 0<\beta<1,$$

for any $[c,d]\subset(a_1,b_1)$. Let $\phi(u)$ denote the characteristic function of $[t_1,y_1]$. The assumed smoothness of f implies [9] that

$$\|M_n(fg,k,.)-fg\|_{L_p[t_2,y_2]}\le\sum_{i=0}^{r-2}\frac{1}{i!}\left\|f^{(i)}(x)M_n\left((u-x)^i(g(u)-g(x)),k,\right)\right\|_{L_p[t_2,y_2]}$$
$$+\frac{1}{(r-2)!}\left\|M_n\left(\phi(u)(g(u)-g(x))\left(\int_x^u(u-w)^{r-2}\left(f^{(r-1)}(w)-f^{(r-1)}(x)\right)dw\right),k,.\right)\right\|_{L_p[t_2,y_2]}$$
$$+\|M_n(F(u,x)(1-\phi(u))(g(u)-g(x)),k,.)\|_{L_p[t_2,y_2]}$$
$$:=J_1+J_2+J_3,$$

Where $F(u,x)=f(u)-\sum_{i=0}^{r-2}\frac{(u-x)^i}{i!}f^{(i)}(x)$; $u\in[0,\infty)$, $x\in[t_2,y_2]$. The direct theorem and moment estimates [1],[2] imply that $J_1,J_3=O\left(n^{-(k+1)}\right),n\to\infty$. Using Jensen's inequality, mean value theorem on g and breaking $[x,u]$ as in Lemma 2.2, we have

$$\int_{t_2}^{y_2}\left|M_n\left(\phi(u)(g(u)-g(x))\left(\int_x^u(u-w)^{r-2}\left(f^{(r-1)}(w)-f^{(r-1)}(x)\right)dw\right),x\right)\right|^p dx$$

$$\leq C_{11}\int_{t_2}^{y_2}\int_{t_1}^{y_1} W_n(x,u)|u-x|^{rp-1}\int_x^u \phi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(x)\right|^p dw\,du\,dx$$

$$\leq C_{11}\sum_{l=1}^{r}\int_{t_2}^{y_2}\left\{\int_{x+ln^{-1/2}}^{x+(l+1)n^{-1/2}} W_n(x,u)\left(n^2 l^{-4}\right)^p |u-x|^{rp+4p-1}\int_x^{x+(l+1)n^{-1/2}} \phi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(x)\right|^p\right.$$

$$\times dwdt+\int_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}} W_n(x,u)(n^2 l^{-4})^p|u-x|^{rp+4p-1}\int_{x-(l+1)n^{-1/2}}^{x} \phi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(x)\right|^p$$

$$\left.\times dw\,dt\right\}dx+\int_{t_2}^{y_2}\int_{t_2-n^{-1/2}}^{y_2+n^{-1/2}} W_n(x,u)|u-x|^{rp-1}\int_{x-n^{-1/2}}^{x+n^{-1/2}} \phi(w)\left|f^{(r-1)}(w)-f^{(r-1)}(x)\right|dw\,du dx$$

$$\leq C_{12}\left\{\sum_{l=1}^{r}\left(n^2 l^{-4}\right)^p n^{-((r+4)p-1)/2}\int_0^{(l+1)n^{-1/2}}\left(\omega\left(f^{(r-1)},w,p,[t_1,y_1]\right)\right)^p dw\right.$$

$$\left.+n^{-(rp-1)/2}\int_0^{n^{-1/2}}\left(\omega\left(f^{(r-1)},w,p,[t_1,y_1]\right)\right)^p dw\right\},$$

on using the moment estimates [1],[2] and then interchanging integration in x and w. Lastly, utilizing $\omega\left(f^{(r-1)},w,p,[t_1,y_1]\right)=O(w^{\beta})$, we find

$$J_2=O(n^{-(r+\beta)/2}),\quad n\to\infty.$$

Combining the estimates of J_1,J_2 and J_3, we obtain(3.2).The proof of (3.2) shows that

(3.3) $\qquad \omega_{2k+2}(f,\tau,p,I_3)=O\left(\tau^{\alpha}\right),\quad \alpha<2k+2,\alpha\neq 2,3,\cdots,2k+1.$

This statement implies that it is true for integer values $2,3,\cdots,2k+1$ also. To prove this, let $\alpha=r$ where r takes any value from $2,3,\cdots,2k+1$. Then, since (3.3) is true for $(r,r+1)$, it follows that

$$\omega_{2k+2}(f,\tau,p,I_3)=O\left(\tau^{r+\theta}\right),\quad 0<\theta<1$$
$$=O\left(\tau^{r}\right).$$

This completes the proof of the inverse theorem.

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