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AN ITERATION PROCEDURE FOR NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT

We study the general semilinear second order ordinary differential equation (1) with a nonlinear boundary condition (H). By Schauder's Theorem we obtain solutions of the problem (1-H) under growth conditions on f and h. Moreover, we show that a solution may be constructed using an iterative procedure.

INTRODUCTION

We consider the general second order semilinear equation

$$u''(t) = f(t, u, u') \tag{1}$$

with nonlinear boundary conditions

$$u(0) = h_0(u(0), u'(0)), \qquad u(T) = h_T(u(T), u'(T)) \tag{H}$$

where $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and $h = (h_0, h_T) : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous functions.

In a previous paper [AMS], we have studied the particular case of the two-point boundary value problem (i.e. the Dirichlet problem), which corresponds to constant h, proving that at least one solution exists if f grows linearly on (u, u'), namely

$$|f(t, u, x)| \leq c|(u, x)| + d \qquad \text{for any } u, x \in \mathbb{R} \text{ and } t \in [0, T]$$

with slope c small enough. Furthermore, we have shown that a solution of (1-Dir) can be obtained constructively by a continuation-type method. Related cases and also periodic type and Sturm-Liouville conditions are studied in [AS], [B], [Br], [FM], [M], among other authors. Topological methods have been applied from the pioneering work of Severini [S].

Nonlinear boundary conditions are not so widely treated in the literature, although they appear frequently in different models. For example, in 1995 Rebelo and Sanchez considered the problem

$$u'' + g(t,u) = 0, \qquad u(0) = f_1(-u'(0)), \quad u(\pi) = f_1(u'(\pi))$$

with $f_1^{-1} \in C(\mathbb{R},\mathbb{R})$ continuous and strictly increasing, which may be regarded as a mathematical model for the axial deformation of a nonlinear elastic beam [RS].

In this work, we show that some topological techniques may be applied also in this case. More precisely, we extend the results of [AMS] for the nonlinear case, showing that (1-H) can be solved under linear growth conditions for f and h. Moreover, we find solutions by a recursive method.

1. A FIXED POINT OPERATOR AND SOLVABILITY OF (1-H)

In order to find solutions of (1-H) we will define a fixed point operator on $C^1[0,T]$: recalling the Green function G associated to the second derivative ∂^2, i.e.

$$G(t,s) = \begin{cases} \frac{t(s-T)}{T} & \text{if } s \geq t \\ \frac{(t-T)s}{T} & \text{if } s \leq t \end{cases}$$

we define $K : C^1[0,T] \to C^1[0,T]$ by

$$Ku(t) = \varphi_u(t) + \int_0^T G(t,s) f(s,u,u')ds$$

where

$$\varphi_u(t) = \frac{h_T(u(T),u'(T)) - h_0(u(0),u'(0))}{T} t + h_0(u(0),u'(0))$$

It is immediate to see that K is compact, and that any solution of (1-H) may be regarded as a fixed point of K. Moreover,

$$\|Ku\|_\infty \leq \|\varphi_u\|_\infty + \sup_{0\leq t\leq T} \{\|G(t,\cdot)\|_p\} \|f(\cdot,u,u')\|_{p'}$$

and

$$\|(Ku)'\|_\infty \le \|\varphi_u'\|_\infty + \sup_{0\le t\le T}\{\|\frac{\partial G}{\partial t}(t,\cdot)\|_p\}\|f(\cdot,u,u')\|_{p'}$$

with $\frac{1}{p}+\frac{1}{p'}=1$. As

$$\|G(t,\cdot)\|_p^p = \left(\frac{t}{T}\right)^p \frac{(T-t)^p}{p+1}T,$$

we conclude that

$$\sup_{0\le t\le T}\{\|G(t,\cdot)\|_p\} = \frac{T}{4}\left(\frac{T}{p+1}\right)^{1/p}$$

In the same way,

$$\sup_{0\le t\le T}\|\{\frac{\partial G}{\partial t}(t,\cdot)\|_p\} = \left(\frac{T}{p+1}\right)^{1/p}$$

We will state an existence result for (1-H), assuming that f and h satisfy the linear-growth conditions

$$|h(x,y)| \le k_h|(x,y)| + l_h, \qquad |f(t,x,y)| \le k_f|(x,y)| + l_f$$

for some positive constants k_h, k_f, l_h, l_f. In this case, we will also see that $p=1$ is the best choice for the previous apriori bounds for G.

THEOREM 1

(1-H) admits a solution in any of the following cases:

i) $T\ge 4$ *and* $k_h+\frac{T^2}{8}k_f<1$.

ii) $T\le 2$ *and* $\frac{2}{T}k_h+\frac{T}{2}k_f<1$.

iii) $2<T<4$ *and* $(k_f,k_h)\in C^\circ$, *where* $C\subset \mathbb{R}^2$ *is the convex hull of the points* $(0,0)$, $(0,1)$, $(\frac{2}{T},0)$, $(\frac{4(T-2)}{T^2},\frac{4-T}{2})$.

Proof

From the growth conditions we obtain:

$$\|f(t,u,u')\|_{p'} \le T^{1/p'}(k_f\|u\|_{1,\infty}+l_f)$$

and

$$\|\varphi_u\|_\infty = \max\{|h_0(u(0),u'(0))|,|h_T(u(T),u'(T))|\} \le k_h\|u\|_{1,\infty}+l_h$$

and

$$\|\varphi_u'\|_\infty = \frac{|h_T(u(T),u'(T)) - h_0(u(0),u'(0))|}{T} \leq \frac{2}{T}(k_h\|u\|_{1,\infty} + l_h)$$

Then, using the previous computations, we have:

$$\|Ku\|_\infty \leq (k_h + k_f\frac{T^2}{4(p+1)^{1/p}})\|u\|_{1,\infty} + c$$

and

$$\|(Ku)'\|_\infty \leq (\frac{2}{T}k_h + k_f\frac{T}{(p+1)^{1/p}})\|u\|_{1,\infty} + c$$

for some constant c. Let us suppose that

$$(*_p) \qquad k_h + k_f\frac{T^2}{4(p+1)^{1/p}} < 1, \qquad \frac{2}{T}k_h + k_f\frac{T}{(p+1)^{1/p}} < 1$$

holds. Then, for large R we conclude that $K(B_R(0)) \subset B_R(0)$ and by Schauder Theorem K has a fixed point. As $(p+1)^{1/p}$ is nonincreasing, its maximum value is achieved for $p = 1$. Hence, it will suffice to prove that $(*_p)$ holds for $p = 1$, namely

$$(*_1) \qquad k_h + k_f\frac{T^2}{8} < 1, \qquad \frac{2}{T}k_h + k_f\frac{T}{2} < 1$$

If $T \geq 4$ and $k_h + \frac{T^2}{8}k_f < 1$, then $k_h + \frac{T^2}{4}k_f < 1 + \frac{T^2}{8}k_f < 2 - k_h < \frac{T}{2}$, and $(*_1)$ holds.
If $T \leq 2$ and $k_h + \frac{T^2}{4}k_f < \frac{T}{2}$, then $k_h + \frac{T^2}{8}k_f < \frac{T}{2} \leq 1$ and $(*_1)$ holds.
Finally, if $2 < T < 4$ we have that $(*_1)$ holds if and only if (k_f, k_h) lies in the first quadrant below the lines

$$k_h + k_f\frac{T^2}{8} = 1, \qquad k_h + k_f\frac{T^2}{4} = \frac{T}{2}$$

$\square$

2. AN ITERATIVE PROCEDURE FOR (1-H)

In this section we will embed the problem (1-H) in a family of problems

$$(1\text{-H})_\lambda \begin{cases} u''(t) = \lambda f(t,u,u') \\ u(0) = h_0(u(0),u'(0)), \quad u(T) = h_T(u(T),u'(T)) \end{cases}$$

The aim of the method is, starting at a solution u_0 for λ_0, to define recursively a sequence which will converge in $C^1([0,T])$ to a solution of (1-H) $_{\lambda_0+\varepsilon}$ for some step ε. We recall the following result from the theory of linear operators (see [AMS]):

LEMMA 2

Let L be the linear operator given by $Lu = u'' + r(t)u' + s(t)u$, with $r, s \in L^\infty(0,T)$, $s \le 0$. Then there exists a constant c depending only on $\|r\|_\infty$ and $\|s\|_\infty$ such that

$$\|u\|_{2,2} \le c\|Lu\|_2$$

for any $u \in H^2 \cap H_0^1(0,T)$. Moreover, the problem

$$\begin{cases} Lu = \psi \\ u|_{\partial I} = \varphi \end{cases}$$

is uniquely solvable in $H^2(0,T)$ for any $\psi \in L^2(0,T)$ and any boundary Dirichlet data φ.

Assuming that u_0 is a solution of (1-H) $_{\lambda_0}$ we define the sequence $\{u_n\} \subset H^2(0,T)$ by the problems:

$$\begin{cases} \frac{u''_{n+1}}{\lambda_0+\varepsilon} = \frac{\partial f}{\partial u'}(t,u_n,u'_n)(u_{n+1}-u_n)' + \frac{\partial f}{\partial u}(t,u_n,u'_n)(u_{n+1}-u_n) + f(t,u_n,u'_n) \\ u_{n+1}(0) = h_0(u_n(0),u'_n(0)), \quad u_{n+1}(T) = h_T(u_n(T),u'_n(T)) \end{cases}$$

As a basic assumption, we will suppose that f is C^2 with respect to u, u', and $\frac{\partial f}{\partial u}(t,u_0(t),u'_0(t)) \ge 0$.

We remark that if $\{u_n\}$ is well defined and $u_n \to u$ for the C^1-norm, then u is a solution of (1-H) $_{\lambda_0+\varepsilon}$. Moreover, if $\frac{\partial f}{\partial u}(t,u_n(t),u'_n(t)) \ge 0$, from lemma 2 we conclude that u_{n+1} is well defined. In this case, for $z_n = u_{n+1} - u_n$ we have:

$$L_n z_n := z''_n - (\lambda_0+\varepsilon)[r_n(t)z'_n + s_n(t)z_n] = (\lambda_0+\varepsilon)R_n$$

with $r_n(t) = \frac{\partial f}{\partial u'}(t,u_n,u'_n)$, $s_n(t) = \frac{\partial f}{\partial u}(t,u_n,u'_n)$ and R_n the Taylor remainder

$$R_n(t) = \frac{1}{2}\left[\frac{\partial^2 f}{\partial u^2}(t,\xi)z_{n-1}^2 + 2\frac{\partial^2 f}{\partial u \partial u'}(t,\xi)z_{n-1}z'_{n-1} + \frac{\partial^2 f}{\partial u'^2}(t,\xi)(z'_{n-1})^2\right]$$

for some mean value $\xi \in L^\infty([0,T],\mathbb{R}^2)$. Writing

$$\varphi_n(t) = m_n t + h_0(u_n(0),u'_n(0)) - h_0(u_{n-1}(0),u'_{n-1}(0))$$

where the slope m_n is given by

$$m_n = \frac{h_T\Big|_{(u_{n-1}(T),u'_{n-1}(T))}^{(u_n(T),u'_n(T))} - h_0\Big|_{(u_{n-1}(0),u'_{n-1}(0))}^{(u_n(0),u'_n(0))}}{T}$$

we obtain from lemma 2 and the imbedding $H^2(0,T) \hookrightarrow C^1([0,T])$:

$$\|z_n - \varphi_n\|_{1,\infty} \leq c_n \|L_n(z_n - \varphi_n)\|_2 \leq c_n(\lambda_0 + \varepsilon)\left(\|R_n\|_2 + \|r_n\varphi'_n + s_n\varphi_n\|_2\right)$$

for some constant c_n depending only on $\|r_n\|_\infty$ and $\|s_n\|_\infty$.
Thus, if $u_{n-1}, u_n \in B_R(u_0)$ we have that

$$\|z_n\|_{1,\infty} \leq (1+\overline{c})\|\varphi_n\|_{1,\infty} + c\|z_{n-1}\|_{1,\infty}^2$$

where the constants $\overline{c}$, c can be choosen depending only on R. Furthermore, if h is Lipschitz with constant k_h then $\|\varphi_n\|_{1,\infty} \leq k_h \max\{1, \frac{2}{T}\}\|z_{n-1}\|_{1,\infty}$.
Hence,

$$\|z_n\|_{1,\infty} \leq b\|z_{n-1}\|_{1,\infty} + c\|z_{n-1}\|_{1,\infty}^2$$

for $b = (1+\overline{c})k_h \max\{1, \frac{2}{T}\}$. Moreover,

$$z''_0 - (\lambda_0 + \varepsilon)\left[\frac{\partial f}{\partial u'}(t, u_0, u'_0)z'_0 + \frac{\partial f}{\partial u}(t, u_0, u'_0)z_0\right] = \varepsilon f(t, u_0, u'_0)$$

and as $z_0|_{\partial I} = 0$, we obtain:

$$\|z_0\|_{1,\infty} \leq \varepsilon c_0 \|f(\cdot, u_0, u'_0)\|_2$$

Thus we have:

THEOREM 3

With the previous notations, let us assume that u_0 *is a solution of (1-H)*$_{\lambda_0}$, *and*

i) $\frac{\partial f}{\partial u}(t,x,y) \geq 0$ *for* $(x,y) \in K_R = B_R(u_0([0,T]) \times u'_0([0,T]))$

ii) h *is Lipschitz on* K_R *with constant* $k_h < \frac{\min\{1,\frac{T}{2}\}}{(1+\overline{c})}$.

Then the sequence $\{u_n\}$ *is well defined and converges in* $B_R(u_0) \subset C^1([0,T])$ *for any step* ε *such that*

$$(*) \qquad \varepsilon c_0 \|f(\cdot, u_0, u'_0)\|_2 < \frac{R(1-b)}{(1+cR)}$$

Proof

From the previous computations and ($*$), it follows that $\|z_0\|_{1,\infty} \leq R$ or, equivalently, $u_1 \in B_R(u_0)$. This proves that u_2 is well defined. Moreover, as

$$\varepsilon c_0 \|f(\cdot, u_0, u_0')\|_2 < R[1 - (b + c\varepsilon c_0 \|f(\cdot, u_0, u_0')\|_2)]$$

we conclude that $b + c\|z_0\|_{1,\infty} < 1$ and $\|z_1\|_{1,\infty} \leq (b + c\|z_0\|_{1,\infty})\|z_0\|_{1,\infty} < R$. Inductively, we see that the sequence $\{u_n\}$ is well defined, and

$$\|z_n\|_{1,\infty} \leq (b + c\|z_0\|_{1,\infty})^n \|z_0\|_{1,\infty}$$

Hence

$$\sum_{n=0}^{\infty} \|z_n\|_{1,\infty} \leq \frac{\|z_0\|_{1,\infty}}{1 - (b + c\|z_0\|_{1,\infty})} \leq R,$$

which implies that $\{u_n\}$ is a Cauchy sequence with $\lim_{n\to\infty} u_n = u \in B_R(u_0)$.
□

Furthermore, if f and its first and second order derivatives with respect to u, u' are bounded in $[0,T] \times \mathrm{IR}^2$, with $\frac{\partial f}{\partial u} \geq 0$ the step ε can be choosen independently of u_0. Hence we have:

COROLLARY 4

Let us assume that u_0 is a solution of (1-H)$_{\lambda_0}$, and

i) f, ∂f and $\partial^2 f$ are bounded, and $\frac{\partial f}{\partial u} \geq 0$ in $[0,T] \times \mathrm{IR}^2$.

ii) h is Lipschitz with constant k_h small enough.

Then there exists a sequence $\lambda_0 < \lambda_1 < ... < \lambda_n = 1$ such that a solution of (1-H)$_{\lambda_j}$ can be constructed recursively.

REMARK:

By Theorem 1, as $|h(x,y)| \leq k_h|(x,y)| + |h(0,0)|$ we conclude that (1-H)$_0$ is solvable for $k_h < \min\{1, \frac{T}{2}\}$.

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