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Riesz and Bessel Potentials, the g^k functions and an Area Function for the Gaussian Measure γ

Liliana Forzani, Roberto Scotto and Wilfredo Urbina *

Abstract

We give proofs of results regarding the boundedness of Littlewood-Paley-Stein functions and Riesz potentials with respect to the Gaussian measure γ. These results were first announced by one of the authors in 1998, [19].

Dedicated to the memory of Eugene Fabes

1 Introduction

Gaussian Harmonic Analysis is the study of operators related to the Ornstein-Uhlenbeck differential operator

$$L := \frac{1}{2}\Delta_x - x \cdot \nabla_x.$$

It is self-adjoint with respect to the Gaussian measure $d\gamma = e^{-|x|^2}dx$ becoming this measure the natural one to study the boundedness of those operators. Gaussian Harmonic Analysis has been under important development for the last twenty years. If $\alpha = (\alpha_1, \ldots \alpha_d) \in \mathbb{N}_0^d$, the Hermite polynomial in $\mathbb{R}^d$ of degree $|\alpha| = \sum_{i=1}^d \alpha_i$ is defined by

$$H_\alpha(x) = \Pi_{i=1}^d H_{\alpha_i}(x_i)$$

where $H_{\alpha_i}(x_i) = (-1)^{\alpha_i} e^{x_i^2} \frac{d^{\alpha_i}}{dx_i^{\alpha_i}}(e^{-x_i^2})$. These polynomials are eigenfunctions of L with eigenvalue $-|\alpha|$.
The semigroups associated to L used through out this paper are

i) The Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$. T_t is defined formally by $T_t = e^{tL}$. Because H_α are eigenfunctions of L we have that

$$T_t H_\alpha = e^{-t|\alpha|} H_\alpha$$

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and extended by linearity to all polynomials in $\mathbb{R}^d$. By using Melher's formula it can be proved that T_t has integral representation whose kernel is

$$O_t(x,y) = \frac{1}{(\pi(1-e^{-2t}))^{\frac{d}{2}}} e^{-\frac{|e^{-t}x-y|^2}{1-e^{-2t}}}.$$

E. Nelson in [13] proved that these operators are hypercontractive, i.e. for $1 < p < +\infty$, $t > 0$, $q(t) = 1+e^{2t}(p-1)$ and $f \in L^p(d\gamma)$ we have $T_t f \in L^{q(t)}(d\gamma)$ and

$$||T_t f||_{q(t),\gamma} \leq ||f||_{p,\gamma}.$$

Moreover, he proved that for $p > 1$,

$$\left|\left|T_t(f - \int f d\gamma)\right|\right|_{p,\gamma} \leq Ce^{-t}||f||_{p,\gamma}.$$

Later, P. Sjögren proved in [16] that $T^* f(x) = \sup_{t>0} |T_t f(x)|$ is weak type $(1,1)$ with respect to the Gaussian measure and since the $L^\infty(d\gamma)$ boundedness is inmediate, the $L^p(d\gamma)$ boundedness follows from Marcinkiewecz interpolation theorem.

ii) The Poisson-Hermite semigroup $\{P_t\}_{t\geq 0}$. P_t is written formally as $P_t = e^{-(-L)^{\frac{1}{2}}t}$. This semigroup can be obtained from the Ornstein-Uhlenbeck semigroup by a subordination formula (see [18]) as

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} T_{\frac{t^2}{4u}} f(x) du.$$

Then it has an integral representation whose kernel is

$$\int_0^1 \frac{t e^{\frac{t^2}{4\log r}}}{2\pi^{\frac{d+1}{2}}(-\log r)^{\frac{3}{2}}} \frac{e^{\frac{|e^{-t}x-y|^2}{1-e^{-2t}}}}{(1-r^2)^{\frac{d}{2}}} \frac{dr}{r}.$$

For every $f \in L^1(d\gamma)$, $P_t f(x)$ turns out to be a smooth function. Taking into account the subordination formula for P_t and that $T^* f(x)$ is strong type (p,p), $1 < p \leq +\infty$, and weak type $(1,1)$, the same is true for $P^* f(x) = \sup_{t>0} |P_t f(x)|$.

In Section 2 we study the Riesz Potentials associated with γ. In [5] it is proved that they need not be weak type $(1,1)$. They are bounded on $L^p(d\gamma)$ for $1 < p < \infty$. We show that although they do not improve integrability, they satisfy an $L^p \log L$ inequality. Also we will consider the Bessel Potentials for γ.
The Littlewood-Paley-Stein theory for the Gaussian measure is a subject still not fully developed. One of the main reasons for studying these Littlewood-Paley functions in the Gaussian context is that in classical Harmonic Analysis they turn out

to be very useful in the proof of the L^p boundedness of singular integral operators, as well as in the characterization of Hardy spaces.
In 1994, C. Gutiérrez introduced in [9] the first order Littlewood-Paley-Stein g function associated with the Gaussian measure as

$$g^1(f)(x) = \left(\int_0^{\infty} t|\nabla P_t f(x)|^2 dt\right)^{1/2} \tag{1}$$

where $\nabla = (\frac{\partial}{\partial t}, \nabla_x)$. He proved the $L^p(d\gamma)$-boundedness of g^1 for $1 < p < \infty$. As for $p = 1$, R. Scotto [15] proved later that it satisfies the weak type $(1,1)$ inequality with respect to γ.
In 1996, C. Gutiérrez, C. Segovia, and J. Torrea introduced in [10] the higher order Littlewood-Paley-Stein functions g^k with respect to a time variable t and with respect to a space variable x, and proved their $L^p(d\gamma)$-boundedness, again for $1 < p < \infty$ as

$$g_T^k f(x) = \left(\int_0^{+\infty} \left|t^k \frac{\partial^k}{\partial t^k} P_t f(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \tag{2}$$

$$g_S^k f(x) = \left(\int_0^{+\infty} \left|t^k \nabla^k P_t f(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \tag{3}$$

where $\nabla^k = \left(\frac{\partial}{\partial x_{\beta_1} \cdots \partial x_{\beta_k}}\right)_{1 \le \beta_j \le d, 1 \le j \le k}$ is the gradient operator of order k and the norm $|\cdot|$ appearing in the integral of g_S^k corresponds to the Euclidean norm in $\mathbb{R}^{d^k}$. They proved the $L^p(d\gamma)$ boundedness of these operators for polynomials and by density on $L^p(d\gamma)$, for $1 < p < \infty$. As it happens with the higher order Riesz transforms associated with the Gaussian measure (see [3], [4]) the case $p = 1$ turns out to be completely different. In Section 3 we prove that the higher order g^k's with respect to t are weak type $(1,1)$ for all k but the ones with respect to x need not be for $k > 2$. After this Journal accepted the paper for publication we knew of an independent proof of part of the results on this section by Sonsoles Pérez.
The definition of an Area function for the Gaussian measure is somewhat problematic, due to the lack of a good definition of a *cone region* in this context. In 1994, L. Forzani and E. Fabes, see [2], gave a definition by choosing a *pencil type zone* as a possible *Gaussian cone.* We discuss this definition and its $L^p(d\gamma)$ continuity in Section 4. We must point out that the possibility of finding more suitable *cones* should not be ruled out, and that further research is needed in this area.

2 Riesz and Bessel Potentials for γ

Riesz Potentials for the Gaussian measure are defined similarly to the ones in Clasical Harmonic Analysis as

$$I_\alpha^\gamma = (-L)^{-\alpha}, \ \alpha > 0;$$

which means that we define $I_\alpha^\gamma H_\beta = \frac{1}{|\beta|^\alpha} H_\beta$ for $|\beta| > 0$, and then extend it by linearity to every polynomial f such that $\int_{\mathbb{R}^d} f\, d\gamma = 0$. In [5] J. García-Cuerva, G. Mauceri, P. Sjögren, and J. Torrea [5] proved that for every $\alpha > 0$, I_α^γ is defined for every polynomial, has integral representation whose kernel is

$$N_\alpha(x,y) = \frac{\pi^{-d/2}}{\Gamma(\alpha)} \int_0^1 (-\log r)^{\alpha-1} \left(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2} \right) \frac{dr}{r} \tag{4}$$

$$= C_{\alpha,d} \int_0^{+\infty} t^{\alpha-1} \left(\frac{e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}}{(1-e^{-2t})^{d/2}} - e^{-|y|^2} \right) dt, \tag{5}$$

thus

$$I_\alpha^\gamma(f) = C_\alpha \int_0^{+\infty} t^{\alpha-1} T_t \left(f - \int f d\gamma \right) dt. \tag{6}$$

They also prove that they are not weak type $(1,1)$ with respect to the Gaussian measure. On the other hand the strong type (p,p) with respect to this measure, for $1 < p < \infty$, follows either by applying a Multiplier Theorem due to P. A. Meyer [12] or directly from the hypercontractivity of the Ornstein-Uhlenbeck semigroup. Moreover we will show that although they do not improve integrability, they satisfy an $L^p \log L$ inequality.

Let us prove first the $L^p(d\gamma)$ boundedness of the Riesz Potentials I_α^γ. Using the integral representation of I_α^γ (4), the hypercontractivity of the semigroup $\{T_t\}_{t>0}$ and Minkowsky's integral inequality we have

$$\begin{aligned} ||I_\alpha^\gamma(f)||_{p,\gamma} &= C \left|\left| \int_0^{+\infty} t^{\alpha-1} T_t \left(f - \int f d\gamma \right) dt \right|\right|_{p,\gamma} \\ &\leq C||f||_{p,\gamma}. \end{aligned}$$

Our next goal is to study if, as in the case of the the Classical Riesz Potentials, they are strong type (p,q) with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. This is not longer true for the Gaussian Riesz Potentials. To see this, for every $a > 0$ let us split kernel (4) into the sum of three kernels

$$\begin{aligned} N_\alpha(x,y) &= C_{\alpha,d} \left(\int_0^{e^{-a}} (-\log r)^{\alpha-1} \left(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2} \right) \frac{dr}{r} \right. \\ &\quad \left. - a^\alpha e^{-|y|^2} + \int_{e^{-a}}^1 (-\log r)^{\alpha-1} \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} \frac{dr}{r} \right) \\ &= N_\alpha^1(x,y) + N_\alpha^2(x,y) + N_\alpha^3(x,y). \end{aligned}$$

Using Minkowski's integral inequality and the hypercontractivity for $\{T_{\frac{t}{2}}\}_{t>0}$ we get that for $q = 1 + (p-1)e^a > p$

$$\begin{aligned}
||I_1 f||_{q,\gamma} &= \left|\left| \int_{\mathbb{R}^d} N^1_\alpha(\cdot, y) f(y) dy \right|\right|_{p,\gamma} \\
&= C \left|\left| \int_a^{+\infty} t^{\alpha-1} T_t \left(f - \int f d\gamma \right) dt \right|\right|_{q,\gamma} \\
&\leq C \int_a^{+\infty} t^{\alpha-1} \left|\left| T_{\frac{t}{2}} T_{\frac{t}{2}} \left(f - \int f d\gamma \right) \right|\right|_{q(\frac{t}{2}),\gamma} dt \\
&\leq C \int_a^{+\infty} t^{\alpha-1} \left|\left| T_{\frac{t}{2}} \left(f - \int f d\gamma \right) \right|\right|_{p,\gamma} dt \\
&\leq C ||f||_{p,\gamma} \int_a^{+\infty} t^{\alpha-1} e^{-\frac{t}{2}} dt \\
&\leq C ||f||_{p,\gamma},
\end{aligned}$$

since $q(\frac{t}{2}) = 1 + (p-1)e^t \geq 1 + (p-1)e^a = q$ for all $t \geq a$ and the gaussian measure is a finite measure.

Since $\left|\int_{\mathbb{R}^d} N^2_\alpha(x,y) f(y) dy\right| \leq C||f||_{p,\gamma}$ and γ is finite measure, we have that if $f \in L^p(d\gamma)$ then $I_2 f \in L^q(d\gamma)$ for any $q \geq 1$.

On the other hand, by taking $f = e^{hy_1^2} \chi_{\{y \in \mathbb{R}^d : y_1 \geq 1\}}$ and applying steps similar to the ones in [3], we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |I_3 f(x)|^q d\gamma(x) &\geq C \int_{\mathbb{R}^d} \left| \int_1^\infty \int_{e^{-a}}^1 (1-r)^{\alpha-1} \frac{e^{-\frac{|rx_1 - y_1|^2}{1-r^2}}}{(1-r^2)^{1/2}} \left(\int_{\mathbb{R}^{d-1}} \frac{e^{-\frac{|rx' - y'|^2}{1-r^2}}}{(1-r^2)^{\frac{d-1}{2}}} dy' \right) dr \right. \\
&\qquad \left. e^{hy_1^2} dy_1 \right|^q d\gamma(x) \\
&\geq C \int_{\{x \in \mathbb{R}^d : x_1 \geq 1\}} \left| e^{x_1^2} \int_{x_1 + 1/x_1}^{2x_1} \int_{1 - r \sim \frac{y_1 - x_1}{y_1}} (1-r)^{\alpha-1} \frac{e^{-\frac{|x_1 - ry_1|^2}{1-r^2}}}{(1-r^2)^{1/2}} dr \right. \\
&\qquad \left. e^{(h-1)y_1^2} dy_1 \right|^q d\gamma(x) \\
&\geq C \int_1^\infty x_1^\beta e^{(hq-1)x_1^2} dx_1.
\end{aligned}$$

Therefore for $p < q$ if we take $h \in (\frac{1}{q}, \frac{1}{p})$, $f \in L^p(d\gamma)$ and $I_3 f \notin L^q(d\gamma)$.

Though the Gaussian Riesz Potentials do not improve integrability, an $L^p \log L(d\gamma)$ inequality can still be obtained.

As E. Fabes suggested, the inequality

$$(7) \quad \int_{\mathbb{R}^d} |I^\gamma_\alpha f(x)|^p \log^+ |I^\gamma_\alpha f(x)| \, d\gamma \leq C \left(\int_{\mathbb{R}^d} |f(x)|^p \, d\gamma + ||f||^p_{p,\gamma} \log^+ ||f||_{p,\gamma} \right).$$

follows from Lemma 3.8 on page 63 proved by L. Gross in [7], [8] which states

Lemma 2.1 *Let (Ω, μ) be a probability measure space. Suppose $1 < p < \infty$, $\epsilon > 0$ and $q > p$. Let $s(t)$ be a real continuously differentiable function on $[0, \epsilon)$ into $(1, \infty)$*

such that $s(0) = p$ *and let* $F(t)$ *be a continuously differentiable function on* $[0, \epsilon)$ *into* $L^q(\mu)$ *with* $F(0) = v \neq 0$. *Then* $||F||_{s(t)}$ *is differentiable at* $t = 0$ *and*

$$\frac{d}{dt}||F(t)||_{s(t)}\bigg|_{t=0} = ||v||_p^{1-p}\left[p^{-1}s'(0)\left\{\int |v|^p \log|v| d\mu - ||v||_p^p \log||v||_p\right\}\right.$$
$$\left. +Re\langle F'(0), |F(0)|^{p-1}\mathrm{sgn}F(0)\rangle\right].$$

Before proving inequality (7) let us introduce the generalized Poisson-Hermite semigroup $Q_t^\alpha = e^{-(-L)^\alpha t}$ for $\alpha > 0$, which is defined on the Hermite polynomials as $Q_t^\alpha H_\beta = e^{-|\beta|^\alpha t}H_\beta$ and then extended to all polynomials by linearity. Let μ_t^α be the probability measure defined on $[0, +\infty)$ such that its Laplace transform satisfies

$$\int_0^\infty e^{-\lambda s}\mu_t^\alpha(ds) = e^{-\lambda^\alpha t}$$

for $0 < \alpha < 1$, see [20]. Therefore $Q_t^\alpha f(x) = \int_0^\infty T_t f(x)\mu_t^\alpha(ds)$. From Minkowski's integral inequality together with the hypercontractivity of the Ornstein-Uhlenbeck semigroup $\{T_t\} = \{Q_t^1\}$, we get the hypercontractivity of the generalized Poisson-Hermite semigroup for $0 < \alpha < 1$. In order to use Lemma 2.1 to prove inequality (7), let us set $\Omega = \mathbb{R}^d$, $\mu = \frac{d\gamma}{\pi^{d/2}}$. Let f be a non-zero polynomial, such that $\int_{\mathbb{R}^d} f\, d\gamma = 0$. Set $s(t) = 1 + (p-1)e^{2t}$, and $F(t) = Q_t^\alpha(I_\alpha^\gamma f)$, then $s(0) = p$ and $F(0) = I_\alpha^\gamma f \neq 0$. From the hypercontractivity of Q_t^α for $0 < \alpha \leq 1$

$$\frac{||F(t)||_{s(t),\gamma} - ||F(0)||_{p,\gamma}}{t} \leq \frac{1-1}{t}||I_\alpha^\gamma f||_{p,\gamma} = 0$$

for every $t > 0$. From this inequality and Lemma 2.1 as $t \to 0^+$, we get

(8)
$$0 \geq \frac{d}{dt}||F(t)||_{s(t),\gamma}\bigg|_{t=0} = ||I_\alpha^\gamma f||_{p,\gamma}^{1-p}[p^{-1}2(p-1)\left(\int_{\mathbb{R}^d} |I_\alpha^\gamma f|^p \log|I_\alpha^\gamma f|\, d\gamma\right)$$
$$- ||I_\alpha^\gamma f||_{p,\gamma}^p \log||I_\alpha^\gamma f||_{p,\gamma}) + Re\langle F'(0), \mathrm{sgn}(I_\alpha^\gamma f)|I_\alpha^\gamma f|^{p-1}\rangle_\gamma].$$

But $F'(0) = -(-L)^\alpha I_\alpha^\gamma f = -f$. Thus,

$$\int_{\mathbb{R}^d} |I_\alpha^\gamma f(x)|^p \log|I_\alpha^\gamma f(x)|\, d\gamma \leq C(||I_\alpha^\gamma f||_{p,\gamma}^p \log||I_\alpha^\gamma f||_{p,\gamma} + \langle |f|, |I_\alpha^\gamma f|^{p-1}\rangle_\gamma).$$

By applying Hölder's inequality to the second term of the sum appearing on the right hand side of the above inequality and using the $L^p(d\gamma)$ continuity of I_α^γ, we get inequality (7) for $0 < \alpha \leq 1$. In order to get inequality (7) for $\alpha > 1$, we write $I_\alpha^\gamma f(x) = I_1^\gamma(I_{\alpha-1}^\gamma f)(x)$, and apply inequality (7) to I_1^γ followed by the $L^p(d\gamma)$ boundedness of $I_{\alpha-1}^\gamma$ together with the fact that $x\log^+ x$ is a non-decreasing function on $(0, +\infty)$.

Also, as in the classical case, Bessel Potentials can be defined formally by

$$J_\alpha^\gamma = (I - L)^{-\alpha}, \ \alpha > 0,$$

and therefore $J_\alpha^\gamma H_\beta = \frac{1}{(1+|\beta|)^\alpha} H_\beta$. Clearly J_α^γ is a positive operator. By the P. A. Meyer's multiplier theorem [12] we have that J_α^γ are bounded on $L^p(d\gamma)$ for $1 < p < \infty$.
On the other hand, again by P. A. Meyer's multiplier theorem,

$$\frac{(-L)^\alpha}{(I-L)^\alpha} \text{ and } \frac{(I-L)^\alpha}{(-L)^\alpha}$$

are bounded operators on every $L^p(d\gamma)$, $1 < p < \infty$. These give the relation between the Riesz and Bessel Potentials, compare with Stein [17] (Lemma 2, page 133).
Finally, Bessel Potentials can be used to define Sobolev spaces with respect to the Gaussian measure, see [20].

3 The higher order g^k functions for γ

The main result of this section is the following

Theorem 3.1 *There exists a constant C, depending on d and k, such that for every $f \in L^1(d\gamma)$ and every $\lambda > 0$,*

$$\gamma\{x \in \mathbb{R}^d : g_T^k f(x) > \lambda\} \leq \frac{C}{\lambda} ||f||_{1,\gamma}, \tag{9}$$

where g_T^k is the higher order g function (see (2)) and $||f||_{1,\gamma} = \int_{\mathbb{R}^d} |f|\, d\gamma$. For $k = 1$ or $k = 2$, g_S^k (see (3)) also satisfies this inequality. If $k > 2$, g_S^k need not satisfy (9).

Before proving this Theorem let us make the following remarks:

1. The operator g_T^k can be viewed as a vector-valued singular integral operator, see [17]. Let $A_1 = \mathbb{R}$ be the set of real numbers and $A_2 = L^2((0,+\infty), dt/t)$ the set of $\mathbb{R}$-valued square integrable functions on $(0,+\infty)$ with respect to the measure dt/t. For $h \in A_2$, let $|h|_2 = (\int_0^{+\infty} |h(t)|^2 dt/t)^{1/2}$. $\mathcal{B}(A_1, A_2)$, the set of bounded linear transformations from A_1 to A_2, can be identified with A_2. Thus

$$g_T^k f(x) = \left| p.v. \int_{\mathbb{R}^d} K_T^k(., x, y) f(y) dy \right|_2,$$

where

$$\begin{aligned} K_T^k(t,x,y) &= t^k \frac{\partial^k}{\partial t^k} P(t,x,y) \\ &= \int_0^1 \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \varphi_k(t,r) \frac{dr}{r} \\ &= \int_0^1 \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{\partial}{\partial r} \left(\int_0^r \varphi_k(t,s) \frac{ds}{s} \right) dr, \end{aligned}$$

with $\varphi_k(t,r) = C_k \frac{t^k H_{k+1}\left(\frac{t}{2(-\log r)^{1/2}}\right)}{(-\log r)^{\frac{k-1}{2}}} \frac{e^{\frac{t^2}{4 \log r}}}{(-\log r)^{\frac{3}{2}}}$.

After an integration by parts,

$$K_T^k(t,x,y) = -\int_0^1 \frac{\partial}{\partial r}\left(\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\right)\tau_k(t,r)dr$$

$$= \int_0^1 \left[\frac{2(rx-y)\cdot(x-ry)}{(1-r^2)^{\frac{d+4}{2}}} - \frac{rd}{(1-r^2)^{\frac{d+2}{2}}}\right] e^{-\frac{|rx-y|^2}{1-r^2}}\tau_k(t,r)dr,$$

where $\tau_k(t,r) = \int_0^r \varphi_k(t,s)\frac{ds}{s}$.

2. i) Clearly, $|\varphi_k(t,r)| \leq Ct\frac{e^{\frac{t^2}{8\log r}}}{(-\log r)^{3/2}}$, for $t > 0$ and $0 < r < 1$.

   ii) $\int_0^1 \varphi_k(t,r)\frac{dr}{r} = 0, \ \ \forall t > 0$, and for all $k \geq 1$.

   This follows from the substitution $u = \frac{t}{2\sqrt{-\log r}}$ and $k-1$ integrations by parts. Indeed,

$$\int_0^1 \varphi_k(t,r)\frac{dr}{r} = C_k\int_0^{+\infty} u^{k-1}H_{k+1}(u)e^{-u^2}du$$

$$= C_k(-1)^{k+1}\int_0^{+\infty} u^{k-1}\frac{d^{k+1}}{du^{k+1}}\left(e^{-u^2}\right)du$$

$$= C_k(k-1)!\int_0^{+\infty}\frac{d^2}{du^2}(e^{-u^2})du = 0.$$

3. $\tau_k(.,r) \in A_2$ and $|\tau_k(.,r)|_2$ is bounded by a constant independent of r. Indeed

$$|\tau_k(.,r)|_2^2 = \int_0^{+\infty}\left(\int_0^r \varphi_k(t,s)\frac{ds}{s}\right)^2\frac{dt}{t}$$

$$= \int_0^{(-\log r)^{1/2}}\left(\int_0^r \varphi_k(t,s)\frac{ds}{s}\right)^2\frac{dt}{t}$$

$$+ \int_{(-\log r)^{1/2}}^{+\infty}\left(-\int_r^1 \varphi_k(t,s)\frac{ds}{s}\right)^2\frac{dt}{t}$$

$$=: (I) + (II)$$

where in the inner integral of (II) we use ii) from remark 2 to replace $\int_0^r \varphi_k(t,s)ds/s$ by $-\int_r^1 \varphi_k(t,s)ds/s$. Then we use remark 2 i) to bound $|\varphi_k(t,s)|$ in the inner integrals of (I) and (II). Once this is done we make the change of variables $-\log s = t^2v$ and (I)+(II) turns out to be bounded by

$$C\left[\int_0^{(-\log r)^{1/2}}\left(\int_{\frac{-\log r}{t^2}}^{+\infty} v^{-3/2}\,dv\right)^2\frac{dt}{t} + \int_{(-\log r)^{1/2}}^{+\infty}\left(\int_0^{\frac{-\log r}{t^2}} dv\right)^2\frac{dt}{t}\right],$$

which is a constant independent of r.

4. Since for $f \in L^1(d\gamma)$, $P_t f(x)$ turns out to be a smooth function we have

$$|\nabla^k P_t f(x)|^2 = \sum_{\substack{1 \leq \beta_j \leq d \\ 1 \leq j \leq k}} \left| \frac{\partial^k}{\partial x_{\beta_1} \cdots \partial x_{\beta_k}} P_t f(x) \right|^2 = C \sum_{|\alpha|=k} |D^\alpha P_t f(x)|^2$$

where $D^\alpha = \frac{\partial^k}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathcal{N}_0^d$. Thus

$$g_S^k f(x) = C \left(\int_0^{+\infty} \sum_{|\alpha|=k} |t^k D^\alpha P_t f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

5. The operator g_S^k can also be viewed as a vector-valued singular integral operator with $A_1 = \mathbb{R}$ and A_2 be the direct sum of $\binom{k+d-1}{d-1}$ copies of $L^2((0, +\infty), dt/t)$.
Let $|h|_2 = (\int_0^{+\infty} \sum_{|\alpha|=k} |h_\alpha(t)|^2 dt/t)^{1/2}$, for $h = (h_\alpha) \in A_2$. Here $\mathcal{B}(A_1, A_2)$ can also be identified with A_2. Thus

$$g_S^k f(x) = \left| p.v. \int_{\mathbb{R}^d} K_S^k(., x, y) f(y) dy \right|_2$$

where $K_S^k(t, x, y) = (K_{S\alpha}^k(t, x, y))_{|\alpha|=k}$,

$$K_{S,\alpha}^k(t, x, y) = D^\alpha P(t, x, y) = C_{d,k} \int_0^1 \eta_k(t, r)\, \omega_k(r)\, H_\alpha \left(\frac{rx - y}{\sqrt{1 - r^2}} \right) \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1 - r^2)^{\frac{d+2}{2}}} dr$$

with $\eta_k(t, r) = \frac{t^{k+1} e^{\frac{t^2}{4 \log r}}}{(-\log r)^{\frac{k+1}{2}}}$, $\omega_k(r) = r^{k-1} \left(\frac{-\log r}{1-r^2} \right)^{\frac{k-2}{2}}$.

6. By direct computation , one can easily see that $|\eta_k(., r)|_{L^2((0,+\infty), dt/t)}$ is bounded by a constant independent of r. Besides $\omega_1(r)$ is a bounded function on $(0, 1)$, and for $k \geq 2$, $\omega_k(r) \leq Cr$ on the same interval.

7. Let $N = \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leq d(1 \wedge \frac{1}{|x|}) \right\}$ and $N_x = \{ y \in \mathbb{R}^d : (x, y) \in N \}$.

Theorem 3.2 *Let $(A_1, |\ \ |_1)$ and $(A_2, |\ \ |_2)$ be two separable Banach spaces. Let T be a bounded linear transformation from $L^p_\gamma(\mathbb{R}^d, A_1)$ to $L^p_\gamma(\mathbb{R}^d, A_2)$ for some p, $1 < p < \infty$ defined as*

$$Tf(x) = p.v. \int_{\mathbb{R}^d} K(x, y) f(y) dy$$

where K defined on the set $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x \neq y\}$ takes its values in $\mathcal{B}(A_1, A_2)$ and satisfies that for every constant $C_1 > 0$ there exists another constant $C_2 > 0$ so that whenever $|x - y| \leq C_1 \left(1 \wedge \frac{1}{|x|}\right)$, *i)* $|K(x,y)|_{\mathcal{B}} \leq \frac{C_2}{|x-y|^d}$,

ii) $\int_{|z-x|\geq 2|x-y|} |K(z,x) - K(z,y)|_{\mathcal{B}}\, dz \leq C_2$.

Let $T_{local}f(x) = T(\chi_{N_x}f)(x)$, *then for every* $f \in L^1_\gamma(\mathbb{R}^d, A_1)$ *and every* $\lambda > 0$, *there exists* $C > 0$ *such that*

$$\gamma\{x \in \mathbb{R}^d : |T_{local}f(x)|_2 > \lambda\} \leq \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|_1 d\gamma(y).$$

Proof Let $\{B_j\}_{j=1}^{\infty}$ be a sequence of hyperbolic balls $B_j = \{x \in \mathbb{R}^d : |x - x_j| \leq d(1 \wedge 1/|x_j|)\}$ such that $\mathbb{R}^d = \bigcup_{j=1}^{\infty} B_j$ and $\sum_{j=1}^{\infty} \chi_{B_j^*}(x) \leq C \;\; \forall\, x \in \mathbb{R}^d$, where $B_j^* = (2d+1)B_j$ is such that $\bigcup_{x\in B_j} N_x \subseteq B_j^*$. Then

$$\begin{aligned}
\gamma\{x \in \mathbb{R}^d : |T_{local}f(x)|_2 > \lambda\} &\leq \sum_{j=1}^{+\infty} \gamma\{x \in B_j : |T(\chi_{N_x}f)(x)|_2 > \lambda\} \\
&\leq \sum_{j=1}^{+\infty} \gamma\{x \in B_j : |T(\chi_{B_j^*}f)(x)|_2 > \lambda/2\} + \\
&\quad + \sum_{j=1}^{+\infty} \gamma(x_j)|\{x \in B_j : |T(\chi_{B_j^*\setminus N_x}f)(x)|_2 > \lambda/2\}| \\
&= (I) + (II)
\end{aligned}$$

The proof that (I) is bounded by $\frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|_1\, d\gamma(y)$ was done in [14], on pages 65-67, for the scalar case. The vector case follows the same steps with minor changes: at the beginning of the Calderón-Zygmund decomposition one has to replace $f \geq 0$ by $|f|_1$ in order to get the sequence of cubes $\{Q_k\}_{k=1}^{\infty}$ such that

(a) $|f(x)|_1 \leq \lambda$ *a.e.* $x \notin \cup Q_k$,

(b) $|\cup Q_k| \leq \frac{1}{\lambda}\int_{B_j^*} |f(x)|_1\, dx$,

(c) $\lambda < \frac{1}{|Q_k|}\int_{Q_k} |f(x)|_1\, dx \leq 2^d\lambda$,

and after that one defines

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \cup Q_k \\ \frac{1}{|Q_k|}\int_{Q_k} f(y)dy & \text{if } x \in Q_k \end{cases}, \text{ and } b(x) = f(x) - g(x),$$

follows the steps written there, and whenever one finds an absolute value this must be changed by $|\cdot|_1$ or $|\cdot|_2$, whichever corresponds. The condition on the gradient of the kernel is replaced by the Hormänder condition *ii)*.

In order to prove that (II) is also bounded by $\frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|_1\, d\gamma(y)$, let us observe that for $x \in B_j$

$$
\begin{aligned}
|T(\chi_{B_j^*\setminus N_x} f)(x)|_2 &\leq \int_{B_j^*\setminus N_x} |K(x,y)|_{\mathcal{B}}\, |f(y)|_1\, dy \\
&\leq C_2 \int_{B_j^*\setminus N_x} \frac{|f(y)|_1}{|x-y|^d}\, dy \\
&\leq C\mathcal{M}(\chi_{B_j^*}|f|_1)(x),
\end{aligned}
$$

where $\mathcal{M}$ is the Hardy-Littlewood Maximal Function. The second inequality comes from condition i) and the last one by taking into account that for $y \in B_j^* \setminus N_x$, we have $d(1 \wedge 1/|x|) < |x-y| \leq (2d+1)d(1 \wedge 1/|x_j|)$, and $|x| \sim |x_j|$. From the fact that $\mathcal{M}$ is weak type $(1,1)$ with respect to Lebesgue measure, we have

$$
\begin{aligned}
(II) &\leq \frac{C}{\lambda} \sum_{j=1}^{+\infty} \gamma(x_j) \int_{B_j^*} |f(y)|_1\, dy \\
&\leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f(y)|_1\, d\gamma(y).
\end{aligned}
$$

where the last inequality was obtained from the fact that the Gaussian density is of constant order of magnitude over each B_j^* and that the sequence $\{B_j^*\}_{j=1}^{\infty}$ has bounded overlap.

8. The following Theorem proved by S. Pérez in [14] will be used in the proof of Theorem 3.1.

Theorem 3.3 *The operators*

$$
T^* f(x) = \int_{\mathbb{R}^d} K^*(x,y)\; f(y)dy, \tag{10}
$$

where

$$
K^*(x,y) = \begin{cases} e^{-|y|^2} & \text{if } x\cdot y \leq 0 \\ \left(\frac{|x+y|}{|x-y|}\right)^2 e^{-\frac{|y|^2-|x|^2}{2} - \frac{|x-y||x+y|}{2}} & \text{if } x \cdot y > 0 \end{cases}
$$

and

$$
T_2^* f(x) = \int_{\mathbb{R}^d} K_2^*(x,y)\; f(y)dy \tag{11}
$$

where

$$
K_2^*(x,y) = \begin{cases} K^*(x,y) & \text{if } x\cdot y \leq 0 \\ \left((|x+y||x-y|)^{\frac{1}{2}} \frac{|x||y|}{|x|^2+|y|^2} + 1\right) K^*(x,y) & \text{if } x\cdot y > 0 \end{cases}
$$

are weak type $(1,1)$ *with respect to the Gaussian measure.*

Proof of Theorem 3.1. In order to prove this Theorem we are going to bound each function $g_{\mathcal{T}}^k f(x)$, $g_{\mathcal{S}}^1 f(x)$, and $g_{\mathcal{S}}^2 f(x)$ by the sum of two which are weak type $(1,1)$ with respect to γ. In the end we will give a counterexample showing that $g_{\mathcal{S}}^k$ need not be weak type $(1,1)$ for $k > 2$.
Clearly for $\mathcal{L} = \mathcal{T}, \mathcal{S}$

$$g_{\mathcal{L}}^k f(x) \leq g_{\mathcal{L},local}^k f(x) + g_{\mathcal{L},global}^k f(x)$$

where

$$g_{\mathcal{L},local}^k f(x) = \left| p.v. \int_{N_x} K_{\mathcal{L}}^k(.,x,y) f(y) dy \right|_2,$$

and

$$g_{\mathcal{L},global}^k f(x) = \int_{\mathbb{R}^d \setminus N_x} |K_{\mathcal{L}}^k(.,x,y)|_2 \ |f(y)| \ dy,$$

The functions $g_{\mathcal{T},global}^k f(x)$ and $g_{\mathcal{S},global}^1 f(x)$ can be bounded by $T^*|f|(x)$. Indeed, by applying Minkowski's integral inequality to $K_{\mathcal{T}}^k$ and remark 3, we get

$$|K_{\mathcal{T}}^k(.,x,y)|_2 \leq C \int_0^1 \left| \frac{\partial}{\partial r} \left(\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \right) \right| dr.$$

and this last term is bounded by $K^*(x,y)$ as it was done in [14], pages 47-49, after the change of variables $t = 1 - r^2$. By applying Minkowski's integral inequality to $K_{\mathcal{S}}^1$ and remark 6, we get

$$|K_{\mathcal{S}}^1(.,x,y)|_2 \leq C \int_0^1 \frac{|rx-y|}{(1-r^2)^{\frac{d+3}{2}}} e^{-\frac{|rx-y|^2}{2(1-r^2)}} dr.$$

The right hand side of this inequality is also bounded by $K^*(x,y)$, see [14], page 39. In the same way, $g_{\mathcal{S},global}^2 f(x)$ turns out to be bounded by $T_2^*|f|(x)$; in fact using Minkowski's integral inequality and remark 6

$$|K_{\mathcal{S}}^2(.,x,y)|_2 \leq C \sum_{|\alpha|=2} \int_0^1 \left| H_\alpha\left(\frac{rx-y}{\sqrt{1-r^2}}\right) \right| \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d+2}{2}}} r dr,$$

and the right hand side of this inequality is bounded by $K_2^*(x,y)$, see [14], page 52. Therefore $g_{\mathcal{T},global}^k$, $g_{\mathcal{S},global}^1$, and $g_{\mathcal{S},global}^2$ turn out to be weak type $(1,1)$ with respect to γ via Theorem 3.3.

It remains to prove that the local parts of these operators are also weak type $(1,1)$ with respect to γ. Since these operators are bounded on $L^p(d\gamma)$, $1 < p < \infty$, see [10], the hypothesis on boundedness of Theorem 3.2 is satisfied. Therefore the result will follow once we check that the kernels of these operators satisfy conditions (i) & (ii) of Theorem 3.2. Before doing this, let us observe that for $y \in N_x$,

a) $|x \cdot (x-y)| \leq |x| \ |x-y| \leq C$ and $|x| \sim |y|$.

b) For $0 < r < 1$, $e^{-\frac{|rx-y|^2}{c(1-r)}} = e^{-\frac{|x-y|^2}{c(1-r)}} e^{-\frac{(1-r)|x|^2}{c}} e^{\frac{2x\cdot(x-y)}{c}} \leq Ce^{-\frac{|x-y|^2}{c(1-r)}} e^{-\frac{(1-r)|x|^2}{c}}$.

c)

$$\begin{aligned}|rx-y|\,|x-ry| &= |x-y-(1-r)x|\,|x-y+(1-r)y| \\ &\le |x-y|^2+(1-r)(|x|+|y|)|x-y|+(1-r)^2|x|\,|y| \\ &\le C(|x-y|^2+(1-r)+(1-r)^2|x|^2)\end{aligned}$$

Condition (i) of Theorem 3.2:
By applying Minkowski's integral inequality to kernel K_T^k and remark 3, we have

$$|K_T^k(.,x,y)|_2 \le C\int_0^1 \left[\frac{|rx-y|\,|x-ry|}{(1-r^2)^{\frac{d+4}{2}}}+\frac{1}{(1-r^2)^{\frac{d+2}{2}}}\right]e^{-\frac{|rx-y|^2}{2(1-r)}}\,dr,$$

which by (c), (b) and the fact that $|t|^m e^{-t^2}\le C$, the right hand side of this inequality can be bounded by

$$\begin{aligned}C\int_0^1 &\left[\frac{|x-y|^2+(1-r)+(1-r)^2|x|^2}{(1-r)^{\frac{d+4}{2}}}\right. \\ &\left.+\frac{1}{(1-r)^{\frac{d+2}{2}}}\right]e^{-\frac{|x-y|^2}{2(1-r)}}e^{-\frac{(1-r)|x|^2}{2}}\,dr \\ &\le C\int_0^1 \frac{e^{-\frac{|x-y|^2}{4(1-r)}}}{(1-r)^{\frac{d+2}{2}}}\,dr \\ &\le \frac{C}{|x-y|^d}.\end{aligned}$$

By applying Minkowski's integral inequality to kernel K_S^k, remark 6, the fact that H_α is a polynomial and that $|t|^m e^{-ct^2}\le C$, we get that

$$\begin{aligned}|K_S^k(.,x,y)|_2 &\le C\sum_{|\alpha|=k}\int_0^1 \left|H_\alpha\left(\frac{rx-y}{\sqrt{1-r^2}}\right)\right|\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d+2}{2}}}\,dr \\ &\le C\int_0^1 \frac{e^{-\frac{|rx-y|^2}{4(1-r)}}}{(1-r)^{\frac{d+2}{2}}}\,dr \\ &\le \frac{C}{|x-y|^d}\end{aligned}$$

for all k.
Condition (ii) of Theorem 3.2:
In order to verify Hörmander's condition ii) of Theorem 3.2, it will be enough to check that $\left|\frac{\partial K_T^k}{\partial y_j}(.,x,y)\right|_2$ and $\left|\frac{\partial K_S^k}{\partial y_j}(.,x,y)\right|_2$ are bounded by $\frac{C}{|x-y|^{d+1}}$. To this end we

use the definition of $K_{\mathcal{T}}^k$ from remark 1 in order to calculate

$$\nabla_y K_{\mathcal{T}}^k(t,x,y) = 2\int_0^1 \left\{ \left[2\frac{(rx-y)\cdot(x-ry)}{(1-r^2)^{\frac{d+4}{2}}} - \frac{rd}{(1-r^2)^{\frac{d+2}{2}}}\right] \frac{rx-y}{1-r^2} - \frac{(x-ry)+r(rx-y)}{(1-r^2)^{\frac{d+4}{2}}} \right\} e^{-\frac{|rx-y|^2}{1-r^2}} \tau_k(t,r)\, dr,$$

and the definition of $K_{\mathcal{S},\alpha}^k$ from remark 5 together with $\nabla H_\alpha = (2\alpha_j H_{\alpha-e_j})_{j=1}^n$ where $e_j = (\delta_{ij})_{i=1}^n$, in order to get

$$\frac{\partial K_{\mathcal{S},\alpha}^k}{\partial y_j}(t,x,y) = C\int_0^1 \eta_k(t,r)\omega(r) \left[\frac{\alpha_j}{\sqrt{1-r^2}} H_{\alpha-e_j}\left(\frac{rx-y}{\sqrt{1-r^2}}\right) + \frac{rx_j-y_j}{1-r^2} H_\alpha\left(\frac{rx-y}{\sqrt{1-r^2}}\right)\right] \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d+2}{2}}}\, dr,$$

for all $j = 1, \ldots, d$.

Thus, by applying Minkowski's integral inequality, remark 3, (b), $|x-ry| \le c|rx-y| + c(1-r)|x|$, (c), and the fact that $|t|^m e^{-ct^2} \le C$ we have

$$\begin{aligned}
\left|\frac{\partial K_{\mathcal{T}}^k}{\partial y_j}(.,x,y)\right|_2 &\le C\int_0^1 \left\{ \left[\frac{|x-ry||x-ry|}{(1-r^2)^{\frac{d+4}{2}}} + \frac{1}{(1-r^2)^{\frac{d+2}{2}}}\right] \frac{|rx-y|}{1-r^2} + \frac{(1-r)|x|}{(1-r^2)^{\frac{d+4}{2}}} \right\} \\
&\qquad e^{-\frac{|rx-y|^2}{4(1-r)}} e^{-\frac{(1-r)|x|^2}{4}} e^{-\frac{|x-y|^2}{4(1-r)}}\, dr \\
&\le C\int_0^1 \left[\frac{|x-y|^2 + (1-r) + (1-r)^2|x|^2}{(1-r^2)^{\frac{d+5}{2}}} + \frac{1}{(1-r^2)^{\frac{d+3}{2}}}\right] e^{-\frac{(1-r)|x|^2}{8}} \\
&\qquad e^{-\frac{|x-y|^2}{4(1-r)}}\, dr \\
&\le C\int_0^1 \frac{e^{-\frac{|x-y|^2}{8(1-r)}}}{(1-r)^{\frac{d+3}{2}}} dr \\
&\le \frac{C}{|x-y|^{d+1}},
\end{aligned}$$

and similarly

$$\left|\frac{\partial K_{\mathcal{S},\alpha}^k}{\partial y_j}(.,x,y)\right|_2 \le C\int_0^1 \frac{e^{-\frac{|x-y|^2}{4(1-r)}}}{(1-r)^{\frac{d+3}{2}}} \le \frac{C}{|x-y|^{d+1}}.$$

According to Theorem 3.2, the operators $g_{\mathcal{T},local}^k$ and $g_{\mathcal{S},local}^k$ turn out to be weak type $(1,1)$ with respect to γ for all k.

To see that $g_{\mathcal{S}}^k$ for $k > 2$ need not satisfy the weak type $(1,1)$ inequality, we refer to [4] where it is shown that the higher order Riesz transforms need not be weak type $(1,1)$ with respect to γ if their order is greater than 2. There they take $y \in \mathbb{R}^d$ such that $|y|$ is large and $y_i \ge \frac{|y|}{2}$, $i = 1, \ldots, d$, and $c > 0$ so that $H_\alpha\left(\frac{y-rx}{\sqrt{1-r^2}}\right) > c|y|^{|\alpha|}$.

Then they define $J = \left\{\xi\frac{y}{|y|} + v: \ \frac{1}{2}|y| < \xi < \frac{3}{4}|y|, v \perp y, \ |v| < 1\right\}$. For $|\alpha| = k$ and $x \in J$

$$\begin{aligned}
K^k_{\mathcal{S},\alpha}(t,x,y) &\geq c\ |y|^k \int_{1/4}^{3/4} \eta_k(t,r)\omega(r)\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d+2}{2}}}\ dr \\
&\geq c\ t^{k+1}e^{-ct^2}|y|^k \int_{1/4}^{3/4} e^{-\frac{|x-ry|^2}{1-r^2}}e^{-|y|^2}e^{|x|^2}\ dr \\
&\geq c\ t^{k+1}e^{-ct^2}|y|^k e^{\xi^2-|y|^2}\int_{1/4}^{3/4} e^{-c(\xi-r|y|)^2}\ dr \\
&\geq c\ t^{k+1}e^{-ct^2}|y|^{k-1}e^{\xi^2-|y|^2}.
\end{aligned}$$

Now, if we take $f = \delta_y e^{|y|^2}$ (by δ_y we mean the delta measure at a mass point y) we get for $x \in J$

$$\begin{aligned}
g^k_{\mathcal{S}}(\delta_y e^{|y|^2})(x) &\geq \left(\int_0^{+\infty} |c\ t^{k+1}e^{-ct^2}|y|^{k-1}e^{\xi^2}|^2\frac{dt}{t}\right)^{1/2} \\
&\geq c\ |y|^{k-1}e^{\xi^2} \\
&\geq c\ |y|^{k-1}e^{\left(\frac{|y|}{2}\right)^2}
\end{aligned}$$

Let us assume that $g^k_{\mathcal{S}}$ is weak type $(1,1)$ with respect to γ. Then

$$\begin{aligned}
\gamma(J) &\leq \gamma\{x \in \mathbb{R}^d : g^k_{\mathcal{S}}(f)(x) > c\ |y|^{k-1}e^{\left(\frac{|y|}{2}\right)^2}\} \\
&\leq C|y|^{-k+1}e^{-\left(\frac{|y|}{2}\right)^2},
\end{aligned}$$

but $\gamma(J) \sim e^{-\left(\frac{|y|}{2}\right)^2}|y|^{-1}$; therefore $k \leq 2$.

4 An Area Function for γ

We define an Area Function $S_\gamma f$ as

$$S^2_\gamma f(x) = \int_{\Gamma_\gamma(x)} |\nabla P_t f(y)|^2\ t\ (t^{-d} \vee |x|^d \vee 1)\ dy\ dt, \tag{12}$$

where $\Gamma_\gamma(x) = \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \ |y-x| < t \wedge \frac{1}{|x|} \wedge 1\right\}$ is a *Gaussian cone.* ($x \wedge y = \min\{x,y\}$ and $x \vee y = \max\{x,y\}$).

The main result of this section is

Theorem 4.1 *Suppose that f is a smooth function. Then* *i) There exists a constant C such that for every $x \in \mathbb{R}^d$,*

(13) $$g_x^1 f(x) \leq C\, S_\gamma f(x).$$

ii) If $1 < p < \infty$, then there exists a constant A_p such that

(14) $$||S_\gamma f||_{p,\gamma} \leq A_p\, ||f||_{p,\gamma},$$

where $||f||_{p,\gamma} = \left(\int_{\mathbb{R}^d} |f|^p d\gamma\right)^{1/p}$.

To prove the above Theorem we basically follow the steps given by E. Stein in [17], pages 86-94. In order to apply those steps we need to use some results about boundedness of an associated maximal operator, certain mean value inequalities which are interesting by themselves, and some inequalities proved by C. Gutiérrez in [9].

In 1994 L. Forzani and E. Fabes [2] defined the non-tangential maximal function for the Poisson-Hermite semigroup as

(15) $$\mathcal{P}^* f(x) = \sup_{(y,t)\in\Gamma_\gamma(x)} |P_t f(y)|,$$

and proved

Theorem 4.2 *i) There exists a constant C, depending only on d, such that for every $f \in L^1(d\gamma)$ and every $\lambda > 0$, the following inequality holds*

$$\gamma\{y \in \mathbb{R}^d : \mathcal{P}^* f(x) > \lambda\} \leq \frac{C}{\lambda}||f||_{1,\gamma}$$

ii) There exists a constant C, depending only on d and p, such that if $f \in L^p(d\gamma)$ for $1 < p \leq \infty$, then

(16) $$||\mathcal{P}^* f||_{p,\gamma} \leq C||f||_{p,\gamma}.$$

Lemma 4.1 *Let us consider the operators*

(17) $$L_1 u = \frac{\partial^2 u}{\partial t^2} + Lu, \text{ and } L_2 u = L_1 u - 2u.$$

If u satisfies $L_1 u = 0$ or $L_2 u = 0$ then

i) (Mean Value inequality)

(18) $$|u(x,t)| \leq \frac{C}{|B((x,t),r)|}\int_{B((x,t),r)} |u(y,s)|dy\,ds,$$

for $r \leq t \wedge \frac{1}{|x|} \wedge 1$.

ii) If $u \geq 0$ in $B((x,t),2r)$ then

(19)
$$u(z,l) \approx \frac{1}{|B((x,t),r)|}\int_{B((x,t),r)} u(y,s)dy\,ds,$$

for any $(z,l) \in B((x,t),r)$, with $r \leq t \wedge \frac{1}{|x|} \wedge 1$.

iii) (Harnack inequality) There exists a constant $C > 0$ such that if $u \geq 0$ on $B((x,t),2r)$

(20)
$$\sup_{B((x,t),r)} u \leq C \inf_{B((x,t),r)} u$$

for $r \leq t \wedge \frac{1}{|x|} \wedge 1$.

Suppose we have proved these propositions. Then we have

Proof of Theorem 4.1.
i) The components of $\nabla u(y,t) = \nabla P_t f(y)$ (for f smooth enough) are solutions of $L_2 u = 0$. Then, applying the Mean Value Inequality (Lemma 4.1, (18)) in the definition of $g_S^1(f)$ we get, after using Schwarz inequality,

$$(g_S^1 f)^2(x) \leq C\int_0^\infty t(t^{-d-1} \vee |x|^{d+1} \vee 1)\int_{B((x,t),t\wedge\frac{1}{|x|}\wedge 1)} |\nabla u(y,s)|^2 dydsdt.$$

Since $B((x,t),t\wedge\frac{1}{|x|}\wedge 1) \subset \Gamma_\gamma(x)$, and if $(y,s) \in B((x,t),t\wedge\frac{1}{|x|}\wedge 1)$, $|s-t| < t\wedge\frac{1}{|x|}\wedge 1$, we have

$$\begin{aligned}(g_S^1 f)^2(x) &\leq C\int_{\Gamma_\gamma(x)} |\nabla u(y,s)|^2 \int_{|s-t|<s\wedge\frac{1}{|x|}\wedge 1} t(t^{-d-1} \vee |x|^{d+1} \vee 1)dtdyds\\ &\leq C\int_{\Gamma_\gamma(x)} |\nabla u(y,s)|^2 s(s^{-d} \vee |x|^d \vee 1)dsdy\\ &= CS_\gamma^2(f)(x),\end{aligned}$$

as we wanted to prove.

ii) To prove the L^p inequality we consider two cases
1) $p \geq 2$. Let ψ be a positive function in $\mathbb{R}^d$. Interchanging integrals in the definition of the area function and taking into account that $\gamma(B(x,t\wedge\frac{1}{|x|}\wedge 1)) \sim (t^d \wedge |x|^{-d} \wedge 1)e^{-|x|^2}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} S_\gamma(f)^2(x)\psi(x)d\gamma(x)\\ &\leq C\int_{\mathbb{R}^d}\int_0^\infty t|\nabla u(y,t)|^2 \frac{1}{\gamma(B(y,t\wedge\frac{1}{|y|}\wedge 1))}\int_{B(y,t\wedge\frac{1}{|y|}\wedge 1)} \psi(x)d\gamma(x)dtd\gamma(y)\\ &\leq C\int_{\mathbb{R}^d} (g_S^1(f)(y))^2 M_T\psi(y)d\gamma(y),\end{aligned}$$

where M_T is the truncated maximal function

$$M_T f(y) = \sup_{r\in\left(0,\frac{1}{|y|}\wedge 1\right]} \frac{1}{\gamma(B(y,r))}\int_{B(y,r)} |f(z)|d\gamma(z), \tag{21}$$

which is known to be strong type (p,p) for $p>1$ and weak type $(1,1)$ with respect to the Gaussian measure γ (see [11]). If we apply the $L^p(d\gamma)$-boundedness of $g_S^1 f$ (see [9]) in the last inequality we get (14) for $p \geq 2$.

2) $1<p<2$. If f is smooth enough and positive we can use that $L_1(P_t f)^p = C_p(P_t f)^{p-2}|\nabla u|^2$ to have

$$S_\gamma^2 f(x) \leq C_p(\mathcal{P}^* f(x))^{2-p} I^*(x), \tag{22}$$

where $\mathcal{P}^* f(x)$ is the non-tangential maximal function asociated with the Poisson-Hermite semigroup (15), and

$$I^*(x) = \int_{\Gamma_\gamma(x)} tL_1(P_t f(y))^p (t^{-d}\vee |x|^d \vee 1)dydt.$$

C. Gutiérrez proved in [9] that

$$\int_0^\infty \int_{\mathbb{R}^d} tL_1(P_t f(y))^p dt d\gamma \leq \int_{\mathbb{R}^d} |f|^p d\gamma$$

therefore

$$\int_{\mathbb{R}^d} I^*(x)d\gamma(x) \leq C\int_0^\infty \int_{\mathbb{R}^d} t\, L_1(P_t f)^p \int_{B(y,c(t\wedge\frac{1}{|y|}\wedge 1))} (t^{-d}\vee|x|^d\vee 1)dx\, dtdy \tag{23}$$

$$\leq C\int_{\mathbb{R}^d} |f|^p\, d\gamma. \tag{24}$$

Now, let us prove (14) for this case. From inequality (22), Hölder's inequality with exponents $r=\frac{2}{p}$ and its conjugate, inequality (23), and Theorem 4.2 (ii), we get

$$\begin{aligned}\int_{\mathbb{R}^d} |S_\gamma(f)(x)|^p d\gamma(x) &\leq C\int_{\mathbb{R}^d} (\mathcal{P}^* f(x))^{\frac{(2-p)p}{2}} I^*(x)^{\frac{p}{2}} d\gamma(x)\\ &\leq C\left(\int_{\mathbb{R}^d}(\mathcal{P}^* f(x))^p d\gamma(x)\right)^{\frac{2-p}{2}} \left(\int_{\mathbb{R}^d} I^*(x)d\gamma(x)\right)^{p/2}\\ &\leq C\int_{\mathbb{R}^d} |f(x)|^p d\gamma(x).\end{aligned}$$

Proof of Theorem 4.2.

It is enough to prove the weak type $(1,1)$ of $\mathcal{P}^*$ since the L^∞ boundedness is immediate, and by applying Marcinkiewicz Interpolation Theorem we get (16) for $1<p<\infty$. To prove the weak type $(1,1)$, it suffices to consider $f\geq 0$. Let us note that for all $(y,t)\in\Gamma_\gamma(x)$, $(y,t)\in B((x,t), t\wedge\frac{1}{|x|}\wedge 1)$; thus from (20) in Lemma 4.1 we have

$$u(y,t)\leq Cu(x,t).$$

Therefore, $\mathcal{P}^* f(x) \leq C \sup_{t>0} P_t f(x) = P^* f(x)$, and this last maximal function is weak type $(1,1)$ with respect to the Gaussian measure (see Section 1).

Proof of Lemma 4.1.

Let us prove it for u solution of $L_1 u = 0$; the case for $L_2 u = 0$ is analogous. For each $(x_0, t_0) \in \mathbb{R}^{n+1}_+$, $|x_0| > 1$, set $B = B((x_0, t_0), \frac{1}{|x_0|})$. Let us define on B the transformation

$$x = x_0 + \frac{1}{|x_0|} x',$$

$$t = t_0 + \frac{1}{|x_0|} t'.$$

Then $(x, t) \in B$ if and only if $(x', t') \in B((0,0), 1)$. Now, let us define the function

$$U(x', t') = u\left(x_0 + \frac{1}{|x_0|} x', t_0 + \frac{1}{|x_0|} t'\right) \quad \text{on } B((0,0),1).$$

The function U satisfies the equation

$$\Delta_{x',t'} U - 2\frac{1}{|x_0|}\left(x_0 + \frac{1}{|x_0|} x'\right) \nabla_{x'} U = 0.$$

Since $(x', t') \in B((0,0), 1)$ then $\frac{1}{|x_0|}\left(x_0 + \frac{1}{|x_0|} x'\right)$ is bounded by a constant. From the Classical Mean Value Inequality ([6] page 244) for elliptic differential operators with bounded first order coefficients, we have

$$|U(0,0)| \leq \frac{C}{s^{n+1}} \int_{B((0,0),s)} |U(x', t')| dx' dt'$$

for all $s \leq 1$.

Now, by the definition of U, the last inequality can be rewritten as

$$\begin{aligned} |u(x_0, t_0)| &\leq \frac{C}{s^{d+1}} \int_{B((0,0),s)} \left| u\left(x_0 + \frac{1}{|x_0|} x', t_0 + \frac{1}{|x_0|} t'\right)\right| dx' dt' \\ &= \frac{C|x_0|^{d+1}}{s^{d+1}} \int_{B((x_0,y_0),\frac{s}{|x_0|})} |u(x,t)| dx dt. \end{aligned}$$

Hence, in order to obtain inequality (18), if $t_0 < \frac{1}{|x_0|}$, take $s = |x_0| t_0$ and if $t_0 > \frac{1}{|x_0|}$, $s = 1$. If $|x_0| \leq 1$ we can apply to u the Classical Mean Value Inequality in the ball $B((x_0, t_0), 1)$.

To prove (19) and (20) we use, as before, the results known for classical positive solutions, see [6] pages 244-250.

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Liliana Forzani - Departamento de Matemáticas, Universidad Nacional del Litoral and IMAL-CONICET. Santa Fe 3000 Argentina. e-mail:lforzani@intec.unl.edu.ar

Roberto Scotto - Departamento de Matemáticas, Universidad Nacional del Litoral. Santa Fe 3000 Argentina. e-mail:scotto@math.unl.edu.ar

Wilfredo Urbina - Escuela de Matemáticas, Facultad de Ciencias UCV. Apt. 47195 Los Chaguaramos, Caracas 1041-A Venezuela. e-mail:wurbina@euler.ciens.ucv.ve

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