Revista de la
Unión Matemática Argentina
Volumen 42, Nro. 1, 2000, 51-73

$T1$ theorems on generalized Besov and Triebel-Lizorkin spaces over spaces of homogeneous type

Silvia I. Hartzstein*and Beatriz E. Viviani†

Presentado por Carlos Segovia Fernández

Abstract

In this paper we extend the definition of the Besov and the Triebel-Lizorkin spaces in the context of spaces of homogeneous-type given by Han and Sawyer in [HS]. We consider, as a control of the 'local regularity', functions $\psi(t)$ more general than the potentials t^α used in their case. We also state $T1$-type theorems in these spaces. Our approach yields some new results for kernels satisfying integral regularity conditions.

1 Introduction

In the context of spaces of homogeneous type, G. David, J.L. Journèe and S. Semmes, in [DJS], showed how to construct an appropiate family of operators $\{D_k\}_{k\in\mathbb{Z}}$ whose kernels satisfy certain size, smoothness and moment conditions and the nondegeneracy condition $\sum_{k\in Z} D_k = I$ on L^2. In [HS], Han and E. Sawyer introduced a class of distributions on spaces of homogeneous type and then established a Calderón-type reproducing formula associated to that family of operators for this class. This formula allowed them to define the Besov spaces $\dot{B}_p^{\alpha,q}$, $1 \le p,q < \infty$ and the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$, $1 < p,q < \infty$ and to show that those spaces are independent of the family of operators $\{D_k\}_{k\in\mathbb{Z}}$ involved in their definition and, in this way, to develop Littlewood-Paley characterizations of them.

By considering more general functions $\psi(t)$ than the potential functions t^α as a measure of the local regularity, in this paper we define the Besov spaces $\dot{B}_p^{\psi,q}$, $1 \le p,q < \infty$ and Triebel-Lizorkin spaces $\dot{F}_p^{\psi,q}$, $1 < p,q < \infty$ on spaces of homogeneous-type. We also state $T1$-theorems of boundedness of generalized Calderón-Zygmund operators on these spaces for kernels satisfying integral conditions of size and smoothness .

In the context of $\mathbb{R}^n$, Y. Han and S. Hofmann in [HH] prove $T1$- theorems on the Besov spaces $\dot{B}_p^{\alpha,q}$, $1 \le p,q \le \infty$ and the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(w)$, $1 < p,q <$

*Supported by UNL and IMAL-CONICET
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∞. In the case of the Besov spaces they consider the following smoothness conditions

$$\text{(L'1)} \quad \sup_{\substack{R>0\\|u|+|v|\le R}} \left(\int_{2^jR\le|x-y|} |K(x+u,y+v)-K(x,y)|dx + \int_{2^jR\le|x-y|} |K(x+u,y+v)-K(x,y)|dy\right) \equiv \gamma_1(2^{-j}),$$

where the 'modulus of continuity' γ_1 satisfies $\sum_{j=1}^{\infty}\gamma_1((2A)^{-j})<\infty$, for $\alpha=0$ and $\gamma_1(t)=t^\epsilon$ for $0<\alpha<\epsilon$.
In the case of the Triebel-Lizorkin spaces they consider the conditions

$$\text{(Lr2)} \quad \sup_{\substack{R>0\\|u|+|v|\le R}} (2^kR)^{n/r'}\left(\left(\int_{2^kR\le|x-y|\le 2^{k+1}R} |K(x+u,y+v)-K(x,y)|^r dx\right)^{1/r} + \left(\int_{2^kR\le|x-y|\le 2^{k+1}R} |K(x+u,y+v)-K(x,y)|^r dy\right)^{1/r}\right) \equiv \delta_r(2^{-k}),$$

where $\int_0^1 \delta_r(t)\log\frac{1}{t}\frac{dt}{t}<\infty$ for $\alpha=0$ and $\delta_r(t)=t^\epsilon$ for $0<\alpha<\epsilon$.

On the other hand, in the context of homogeneous-type spaces, Han and Sawyer in [HS] prove $T1$- theorems on the Besov and Triebel-LIzorkin spaces for kernels satisfying standard conditions of size and smoothness. These are

$$\text{(P1)} \quad |K(x,y)|\le A\delta(x,y)^{-1}$$

$$\text{(P2)} \quad |K(x,y)-K(x',y)|+|K(y,x)-K(y,x')|\le\left(\frac{\delta(x,x')}{\delta(x,y)}\right)^{\epsilon}\delta(x,y)^{-1},$$

for $\delta(x,y)\ge 2\delta(x,x')$.

In the same context, we consider integral kernel estimates, slightly stronger than the ones established in [HH], when they are compared in $\mathbb{R}^n$ for convolution kernels, although our assumptions concerning to the 'modulus of continuity' and on the local regularity control of the spaces are weaker than theirs. Our results are a refinement of those obtained in [HS] since we recover these last ones for standard kernels and local regularity controlled by $\psi(t)=t^\alpha$.

In the next we establish the general settings of this work.

Given a set X we shall say that a real valued function $\delta(x,y)$ defined on $X\times X$ is a quasi-distance on X if there exists a constant $A>0$ such that for all $x,y,z\in X$ it verifies:

a) $\delta(x,y)\ge 0$ and $\delta(x,y)=0$ if and only if $x=y$

b) $\delta(x,y)=\delta(y,x)$

c) $\delta(x,y)\le A[\delta(x,z)+\delta(z,y)]$.

In a set X endowed with a quasi-distance $\delta(x,y)$, the balls $B_\delta(x,r)=\{y:\delta(x,y)<r\}$ form a basis of neighborhoods of x for the topology induced by the uniform structure on X. Let μ be a positive measure on a σ- algebra of subsets of X which contains the

open set and the balls $B_\delta(x,r)$. We say that $X := (X,\delta,\mu)$ is a *space of homogeneous type* if there exists a finite constant A' such that

$$\mu(B_\delta(x,2r)) \leq A'\mu(B_\delta(x,r)) \tag{1.1}$$

holds for all $x \in X$ and $r > 0$. Macías and Segovia ([MS]) showed that it is always possible to find a quasi-distance $d(x,y)$ equivalent to $\delta(x,y)$ and $0 < \theta \leq 1$, such that

$$|d(x,y) - d(x',y)| \leq Cr^{1-\theta}d(x,x')^\theta \tag{1.2}$$

holds whenever $d(x,y) < r$ and $d(x',y) < r$.
We also say that (X,δ,μ) is of order θ if δ satisfies (1.2). (X,δ,μ) is a *normal space* if there exist constants A_1 y A_2 such that

$$A_1 r \leq \mu(B_\delta(x,r)) \leq A_2 r \tag{1.3}$$

hols for every $x \in X$ and $r > 0$.

In this paper $X := (X,\delta,\mu)$ will mean a normal space of homogeneous type of order θ.

Given a ball B in X and a number η, $0 < \eta \leq \theta$, we denote by $\Lambda^\eta(B)$ the set of all the complex- valued functions f with support in B such that

$$|f(x) - f(y)| \leq C\delta(x,y)^\eta, \quad x,y \in X.$$

We denote $|f|_\eta$ the infimum of the constants appearing in (1.4) and $\|f\|_\eta = \|f\|_\infty + |f|_\eta$. We say that a function f belongs to Λ_0^η if $f \in \Lambda^\eta(B)$ for some ball B. The space Λ_0^η is the inductive limit of the Banach spaces $\Lambda^\eta(B)$. The space of all continous linear functionals on Λ_0^η will be denoted $(\Lambda_0^\eta)'$.

A nonegative real function ϕ defined on the positive numbers is said to be of *lower type* $\alpha \geq 0$ if there exists a constant $C_1 > 0$ such that

$$\phi(st) \leq C_1 s^\alpha \phi(t) \text{ for } 0 < s \leq 1 \text{ and } t > 0.$$

Similarly, ϕ is said to be of *upper type* β if there exists a constant $C_2 > 0$ such that

$$\phi(st) \geq C_2 s^\beta \phi(t) \text{ for } 0 < s \leq 1 \text{ and } t > 0.$$

In the next we state the properties of an approximation to the identity as defined in [HS]. In [DJS], it is shown how to build such approximation to the identity. Let A be the constant of the triangular inequality associated to δ

Definition 1.1 *A sequence $(S_k)_{k\in Z}$ of integral operators is called an* approximation to the identity, *if the kernels $S_k(x,y)$ associated to S_k are functions from $X \times X$ in $\mathbb{C}$ and there exist $0 < \epsilon \leq \theta$ and a finite constant C such that for all $k \in Z$ and $x,x',y,y' \in X$ they satisfy*

$$S_k(x,y) = 0 \quad \textit{if} \quad \delta(x,y) \geq (2A)^{-k} \quad \textit{and} \ \|S_k\|_\infty \leq C(2A)^k, \tag{1.4}$$

$$|S_k(x,y) - S_k(x',y)| \leq C(2A)^{k(1+\epsilon)}\delta(x,x')^\epsilon, \tag{1.5}$$

$$|S_k(x,y) - S_k(x,y')| \le C(2A)^{k(1+\epsilon)}\delta(y,y')^\epsilon, \tag{1.6}$$
$$|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \le C(2A)^{k(1+2\epsilon)}\delta(x,x')^\epsilon\delta(y,y')^\epsilon, \tag{1.7}$$
$$\int_X S_k(x,y)d\mu(y) = \int_X S_k(x,y)d\mu(x) = 1. \tag{1.8}$$

In all this paper the constant ϵ, $0 < \epsilon \le \theta$, will denote that associated to an approximation to the identity satisfying (1.5), (1.6) and (1.7) of Definition (1.1).

The operators $D_k = S_k - S_{k-1}$ satisfy $\sum_{k \in Z} D_k = I$ in L^2 since $\lim_{k\to\infty} S_k f = f$ and $\lim_{k\to -\infty} S_k f = 0$ in L^2. Moreover, their associated kernels $D_k(x,y)$ satisfy properties (1.4) to (1.7) of Definition (1.1) and

$$\int_X D_k(x,y)d\mu(y) = \int_X D_k(x,y)d\mu(x) = 0. \tag{1.9}$$

In [HS] was introduced a suitable class of test functions defined on X, the set $M^{(\beta,\gamma)}$ and its dual space $(M^{(\beta,\gamma)})'$.

DEFINITION 1.2 *Given $0 < \beta \le 1$, $\gamma > 0$ and $x_0 \in X$ fix. A function f defined on X is a smooth molecule of type (β,γ) of width d centered in x_0, if there exists a constant $C > 0$ such that*

$$|f(x)| \le C\frac{d}{(d+\delta(x,x_0))^{1+\gamma}},$$
$$|f(x) - f(x')| \le C\delta(x,x')^\beta\left(\frac{d}{(d+\delta(x,x_0))^{1+\gamma}} + \frac{d}{(d+\delta(x',x_0))^{1+\gamma}}\right),$$
$$\int f(x)d\mu(x) = 0,$$

hold for every $x \in X$.

We denote by $\|f\|_{(\beta,\gamma)}$, the infimum of the constants appearing in (1.10) and (1.10). With this norm $M^{(\beta,\gamma)}$ is a Banach space and the space $(M^{(\beta,\gamma)})'$ is the set of all continous and linear functional on $M^{(\beta,\gamma)}$. We denote by $< h, f >$ the natural application of $h \in (M^{(\beta,\gamma)})'$ to $f \in M^{(\beta,\gamma)}$.

In [HS], the authors prove Calderón-type reproduction formulas for both spaces. These formulas are stated in the following theorems:

THEOREM 1.1 *Let $(S_k)_{k\in Z}$ be an approximation to the identity and set $D_k = S_k - S_{k-1}$. There exist families of operators $(\tilde{D}_k)_{k\in Z}$ and $(\hat{D}_k)_{k\in Z}$ such that for all $f \in M^{(\beta,\gamma)}$*

$$f = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k f = \sum_{k=-\infty}^{\infty} D_k \hat{D}_k f,$$

where the series converges in $M^{(\beta',\gamma')}$, for $\beta' < \beta$ and $\gamma' < \gamma$.

If $(\tilde{D}_k)_{k\in Z}$ y $(\hat{D}_k)_{k\in Z}$ are like in Theorem (1.1) then their associated kernels $\tilde{D}_k(x,y)$ and $\hat{D}_k(x,y)$ are (ϵ',ϵ')-smooth molecules of width $(2A)^{-k}$, as functions of the first and second variable respectively. Therefore, $\tilde{D}_k^* f$ and $\hat{D}_k^* f \in M^{(\beta,\gamma)}$, whenever $f \in M^{(\beta,\gamma)}$, $0 < \beta,\gamma < \epsilon$. This allows to define $\tilde{D}_k h$ and $\hat{D}_k h$ as elements of $\left(M^{(\beta,\gamma)}\right)'$ for $h \in \left(M^{(\beta,\gamma)}\right)'$, by $< \tilde{D}_k h, f >=< h, \tilde{D}_k^* f >$ and $< \hat{D}_k h, f >=< h, \hat{D}_k^* f >$. It is then proved in [HS] that the formulas in Theorem (1.1) are also valid in the sense of distributions. More precisely

THEOREM 1.2 *Let $(D_k)_{k\in Z}$, $(\tilde{D}_k)_{k\in Z}$ and $(\hat{D}_k)_{k\in Z}$ be like in Theorem (1.1). Then for all $f \in (M^{(\beta,\gamma)})'$, we have that*

$$f = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k f = \sum_{k=-\infty}^{\infty} D_k \hat{D}_k f,$$

in the sense of

$$\langle f, g\rangle = \lim_{M\to\infty} \langle \sum_{|k|\le M} \tilde{D}_k D_k f, g\rangle = \lim_{M\to\infty} \langle \sum_{|k|\le M} D_k \hat{D}_k f, g\rangle$$

for all $g \in M^{(\beta',\gamma')}$, with $\beta' > \beta$ and $\gamma' > \gamma$.

2 Generalized Besov and Triebel–Lizorkin spaces

In the context of spaces of homogeneous type, Han and Sawyer ([HS]) define the Besov spaces $\dot{B}_p^{\alpha,q}$ and Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$, of distributions whose 'local regularity' is controlled by the function t^α, with $-\epsilon < \alpha < \epsilon$, and its integrability by p and q. Replacing the potentials t^α by more general functions $\psi(t)$, we define the spaces $\dot{B}_p^{\psi,q}$ and $\dot{F}_p^{\psi,q}$.

In the sequel we denote by ψ the function $\psi = \phi_1/\phi_2$, where $\phi_1(t)$ and $\phi_2(t)$ are quasi increasing functions of upper type $s_1 < \epsilon$ and $s_2 < \epsilon$, respectively and $\{D_k\}_{k\in Z}$ the family of operators defined in Theorem (1.1).

DEFINITION 2.1 *For $f \in (M^{(\beta,\gamma)})'$, with $0 < \beta,\gamma < \epsilon$, we define*

$$\|f\|_{\dot{B}_p^{\psi,q}} = \left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}\|D_k f\|_p\right)^q\right)^{\frac{1}{q}} \quad if\ 1 \le p \le \infty, 1 \le q \le \infty,$$

with the obvious change for the case $q = \infty$ Interchanging the order of the norms in L^p and l^q we have

$$\|f\|_{\dot{F}_p^{\psi,q}} = \| \left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}|D_k f|\right)^q\right)^{\frac{1}{q}} \|_{L^p}, \quad if\ 1 < p,q < \infty.$$

Also, if w is a nonnegative locally integrable function, we denote

$$\|f\|_{\dot{F}_p^{\psi,q}(w)} = \| \left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}|D_k f|\right)^q\right)^{\frac{1}{q}} \|_{L^p(w)}, \quad if\ 1 < p,q < \infty.$$

In a similar way to the case $\psi(t) = t^\alpha$, (see [HS]), it can be proved that if $(P_k)_{k\in Z}$ is another approximation to the identity of order ϵ and $E_k = P_k - P_{k-1}$ then the norms obtained replacing D_k by E_k are equivalent to the defined in (2.1), (2.1) and (2.1). The same result is true replacing the operators D_k by $\tilde{D}_k^*$ or $\hat{D}_k$.

The Besov space $\dot{B}_p^{\psi,q}$, $1 \le p, q \le \infty$, is the set of all $f \in \left(M^{(\beta,\gamma)}\right)'$, with $\beta > s_1$ and $\gamma > s_2$, such that

$$\|f\|_{\dot{B}_p^{\psi,q}} < \infty \text{ and } |\langle f, h\rangle| \le C\|f\|_{\dot{B}_p^{\psi,q}}\|h\|_{(\beta,\gamma)},$$

for all $h \in M^{(\beta,\gamma)}$.
Analogously, The Triebel-Lizorkin space $\dot{F}_p^{\psi,q}(w)$, with $1 < p, q < \infty$, is the set of all $f \in \left(M^{(\beta,\gamma)}\right)'$, with $\beta > s_1$ and $\gamma > s_2$, such that

$$\|f\|_{\dot{F}_p^{\psi,q}(w)} < \infty, \text{ and } |\langle f, h\rangle| \le \|f\|_{\dot{F}_p^{\psi,q}(w)}\|h\|_{(\beta,\gamma)},$$

for all $h \in M^{(\beta,\gamma)}$.

When $\psi(t) = t^\alpha$ we have the usual Besov space $\dot{B}_p^{\alpha,q}$ and the Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}(w)$.
In the following, we state the main properties of the generalized Besov and Triebel-Lizorkin spaces, $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\psi,q}(w)$, $1 < p, q < \infty$, without including their proof in order of not extending this work. Both classes are Banach spaces and the corresponding dual spaces are $\dot{B}_{p'}^{1/\psi,q'}$ and $\dot{F}_{p'}^{1/\psi,q'}(w^{-p'/p})$ respectively, with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. The molecular space $M^{(\beta,\gamma)}$ is continously embbeded in both of them if $s_1 < \beta$ and $s_2 < \gamma$. Moreover, $M^{(\epsilon',\epsilon')}$ is dense in $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\psi,q}$,$1 < p, q < \infty$, for all ϵ', such that $\max(s_1, s_2) < \epsilon' < \epsilon$.

In the case of $X = \mathbb{R}^n$, we give some examples of classical distributions spaces that can be characterized as special cases of the Besov and Triebel-Lizorkin spaces.

For $1 < p < \infty$, $\dot{F}_p^{0,2} = L^p$. (See [Tr2] and [FJW]).

If ϕ is of positive lower type and upper type lower than 1, $\dot{B}_p^{\phi,q} = \dot{\Lambda}_\phi^{p,q}$, (see [Tr2], [S], [J], [B], [I]) where $\dot{\Lambda}_\phi^{p,q}$ is the set of all the functions (modulus constants) such that

$$\left[\int_{\mathbb{R}^n}\left(\frac{\|f(x+y)-f(x)\|_p}{\phi(|y|)}\right)^q \frac{dy}{|y|^n}\right]^{1/q} < \infty, \quad \text{for} \quad ,1 \le p \le \infty, 1 \le q < \infty$$

and

$$\sup_{y\in\mathbb{R}^n, y\ne 0} \frac{\|f(x+y)-f(x)\|_p}{\phi(|y|)} < \infty, \quad \text{for} \quad ,1 \le p \le \infty \text{ and } \; q = \infty.$$

The homogeneous Sobolev space $\dot{L}_p^k$, with $1 \le p \le \infty$ and k a nonegative integer, consists of all tempered distributions f such that $D^\gamma f \in L^p(\mathbb{R}^n)$ for $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $|\gamma| = k$. Endowed with the norm $\|f\|_{\dot{L}_p^k} = \sum_{|\gamma|=k}\|D^\gamma f\|_p$ we have that, $\dot{L}_p^k \simeq \dot{F}_p^{k,2}$.(See [Tr2] and [FJW]).

Let consider the fractional derivative operator D_α defined by $\widehat{D_\alpha h}(\xi) = |\xi|^\alpha \hat{h}(\xi)$, $0 < \alpha < n$, for $h \in S_o = \{f \in S : sop\hat{f} \subset \mathbb{R}^n - 0\}$. Given $f \in S_o'$, $D_\alpha f$ is defined by $< D_\alpha f, h > = < f, D_\alpha h >$, for $h \in S_o$. The homogeneous fractional Sobolev space $\dot{L}^\alpha_p$, $\alpha > 0$, $1 < p < \infty$, is the set of all $f \in S'_o$ such that $D_\alpha f \in L^p$ endowed with the norm $\|f\|_{\dot{L}^p_\alpha} = \|D_\alpha f\|_{L^p}$. Then that $\dot{L}^p_\alpha = \dot{F}^{\alpha,2}_p$ with equivalences of norms. In the setting of homogeneous-type spaces this result is obtained by Gatto and Vàgi in [GV]

3 Definition of the Calderón-Zygmund generalized operators and main theorems

Let be $\Delta = \{(x,x)/x \in X\}$ and consider the continous linear mapping $T: \Lambda^\beta_o \to (\Lambda^\beta_o)'$ for every $0 < \beta \leq \theta$, associated to a kernel $K(x,y)$, defined on $X \times X - \Delta$ and locally integrable outside Δ such that

$$\langle Tf, g \rangle = \int \int g(x) K(x,y) f(y) d\mu(x) d\mu(y) \tag{3.10}$$

for all $f, g \in \Lambda^\beta_o$ with disjoint supports.

We say that T has the weak boundary property of order β, $0 < \beta \leq \theta$, if T verifies

$$\text{(WBP)} \qquad |\langle Tf, g \rangle| \leq C\mu(B)^{1+2\beta} \|f\|_\beta \|g\|_\beta,$$

for f and g in $\Lambda^\beta(B)$ and every ball $B \subset X$. Note that (WBP) is also true for every $\epsilon \geq \beta$.

To obtain the continuity of T on the generalized Besov spaces we require the following size and smoothness conditions on K:

$$\text{(S0)} \quad \sup_{R>0} \int_{R \leq \delta(x,y) \leq 2AR} (|K(x,y)| + |K(y,x)|) d\mu(x) \leq C, \text{ for every } y \in X;$$

$$\text{(S1)} \quad \int_{\delta(x,y) \geq (2A)(2A)^j R} \left(\sup_{0<s\leq R} \frac{1}{s} \int_{\delta(z,x)<s} |K(w,z) - K(y,x)| d\mu(z) \right) d\mu(x) \leq \gamma_1((2A)^{-j}),$$

$$\text{(S1')} \quad \int_{\delta(x,y) \geq (2A)(2A)^j R} \left(\sup_{0<s\leq R} \frac{1}{s} \int_{\delta(z,x)<s} |K(z,w) - K(x,y)| d\mu(z) \right) d\mu(x) \leq \gamma_1((2A)^{-j}),$$

for every $w, y \in X$ and $R > 0$ such that $\delta(w,y) < R$, for $j = 1,2,3,...$ and where the *modulus of continuity* γ_1 is a quasi-increasing function defined in $t > 0$ such that $\lim_{t \to 0} \gamma_1(t) = 0$ and which satisfies

$$\sum_{j=1}^{\infty} (2A)^{j\alpha} \gamma_1((2A)^{-j}) < \infty \tag{3.11}$$

for some $\alpha \geq 0$, (or the equivalent condition $\int_0^1 \gamma_1(t) \frac{1}{t^{\alpha+1}} dt < \infty$).

If K satisfies the punctual smoothness condition

$$\text{(P)} \quad |K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \omega_\infty \left(\frac{\delta(x,x')}{\delta(x,y)} \right) \delta(x,y)^{-1}$$

for $\delta(x,y) \geq 2A\delta(x,x')$, where ω_∞ is a quasi-increasing function such that $\sum_{l=1}^\infty l\omega_\infty((2A)^{-l}) < \infty$, then K verifies $(S1)$ and $(S1')$ and γ_1 verifies $\sum_{l=0}^\infty \gamma_1((2A)^{-l}) < \infty$.

If K verifies $(S1)$ then it satisfies the following Hörmander-type condition:

$$\text{(H1)} \quad \int_{\delta(x,y)\geq(2A)^j R} |K(w,x) - K(y,x)| d\mu(x) \leq \gamma_1((2A)^{-j})$$

for every $w, y \in X$ and $R > 0$ such that $\delta(w,y) < R$, $j \in \mathbb{N}$, where γ_1 is as in (S_1). Similarly, $(S1')$ implies

$$(H1') \quad \int_{\delta(x,y)\geq(2A)^j R} |K(x,w) - K(x,y)| d\mu(x) \leq \gamma_1((2A)^{-j})$$

for every $w, y \in X$ and $R > 0$ such that $\delta(w,y) < R$, $j \in \mathbb{N}$.

In order to establish continuity results on the generalized weighted Triebel-Lizorkin spaces we need the following conditions on the kernel $K(x,y)$ associated to the operator T:

Let $1 < r < \infty$ and r' such that $1/r + 1/r' = 1$, then we set

$$(S^r0) \quad \sup_{R>0} R^{1/r'} \left(\int_{R\leq\delta(x,y)\leq 2AR} (|K(x,y)|^r + |K(y,x)|^r) d\mu(x) \right)^{1/r} \leq C, \quad \text{for every } y \in X;$$

$$(S^r1) \quad \left[\int_{\substack{(2A)^j R\leq\delta(x,y)\\ \leq(2A)^{j+1}R}} \left(\sup_{0<s\leq R} \frac{1}{s} \int_{\delta(z,x)<s} |K(w,z) - K(y,x)|^r d\mu(z) \right) d\mu(x) \right]^{1/r} \leq ((2A)^j R)^{-1/r'} \gamma_r((2A)^{-j}), \text{ and}$$

$$(S^r1') \quad \left[\int_{\substack{(2A)^j R\leq\delta(x,y)\\ \leq(2A)^{j+1}R}} \left(\sup_{0<s\leq R} \frac{1}{s} \int_{\delta(z,x)<s} |K(z,w) - K(x,y)|^r d\mu(z) \right) d\mu(x) \right]^{1/r} \leq ((2A)^j R)^{-1/r'} \gamma_r((2A)^{-j}).$$

for every $w, y \in X$ and $R > 0$ such that $\delta(w,y) < R$, $j = 2, 3, \ldots$ and γ_r is a quasi-increasing function such that $\lim_{t\to 0} \gamma_r(t) = 0$ satisfying either $\sum_{l=1}^{\infty} l\gamma_r((2A)^{-l}) < \infty$ or (3.11).

If K satisfies the punctual estimate (P), then K also satisfies (S^r1) and (S^r1') and $\gamma_r = \omega_\infty$.

If K verifies (S^r1) then it also satisfies:

$$(H^r1) \quad \left(\int_{(2A)^j R\leq\delta(x,y)\leq(2A)^{j+1}R} |K(w,x) - K(y,x)|^r d\mu(x) \right)^{1/r} \leq ((2A)^j R)^{-1/r'} \gamma_r((2A)^{-j})$$

for every $w, y \in X$ and $R > 0$ such that $\delta(w,y) < R$.

Analogously, from (S^r1') we obtain

$$(H^r1') \quad \left(\int_{(2A)^j R\leq\delta(x,y)\leq(2A)^{j+1}R} |K(x,w) - K(x,y)|^r d\mu(x) \right)^{1/r} \leq ((2A)^j R)^{-1/r'} \gamma_r((2A)^{-j})$$

whenever $w, y \in X$ and $R > 0$ is such that $\delta(w,y) < R$.

We now state the main theorems of this work:

THEOREM 3.1 *Let $T: \Lambda_o^\beta \to (\Lambda_o^\beta)'$ be a linear continous operator, with $0 < \beta < \epsilon$, weakely bounded of order ϵ associated to a kernel K which verifies $(S0)$, $(S1)$ and $(S1')$.*
Let ϕ_1 and ϕ_2 be functions of lower types, i_1 e i_2 and of upper types $s_1 < \epsilon$ and s_2, ϵ, respectively. Suppose that γ_1 verifies $\sum_{j=0}^\infty (2A)^{j\alpha}\gamma_1((2A)^{-j}) < \infty$ for some α, such that $0 \le \alpha < \epsilon$.
If $T1 = 0$ then T is a bounded operator on $\dot{B}_p^{\phi_1/\phi_2,q}$, for $0 < i_1 - s_2 \le s_1 - i_2 \le \alpha$ with $0 < \alpha < \epsilon$ and $1 \le p, q < \infty$.
If $T1 = T^*1 = 0$ then T is bounded on $\dot{B}_p^{\phi_1/\phi_2,q}$, for $-\alpha \le i_1 - s_2 \le s_1 - i_2 \le \alpha$ and $1 \le p, q < \infty$.

THEOREM 3.2 *Let $1 < p < \infty$, $1 < q < \infty$, $1 < r' < min\{p,q\}$, r such that $1/r + 1/r' = 1$ and $w \in A_{p/r'}$.*
Let $T: \Lambda_o^\beta \to (\Lambda_o^\beta)'$ be a linear continous operator with $0 < \beta < \epsilon$, weakely bounded of order ϵ, associated to a kernel K satisfying (S^r0), (S^r1) and (H^r1') with modulus of continuity *γ_r, a quasi-increasing function such that* $\lim_{t\to 0}\gamma_r(t) = 0$.

1. *Let suppose that $\sum_{l=1}^\infty l\gamma_r((2A)^{-l}) < \infty$. If $T1 = T^*1 = 0$ then T is bounded in $\dot{F}_p^{0,q}(w)$.*

2. *Let ϕ_1 and ϕ_2 be of lower types i_1 and i_2, and of upper types s_1 and s_2 lower than ϵ, respectively.*
Suppose that $\sum_{l=1}^\infty (2A)^{l\alpha}\gamma_r((2A)^{-l}) < \infty$, for some $0 < \alpha < \epsilon$.
If $T1 = 0$ then T is bounded in $\dot{F}_p^{\phi_1/\phi_2,q}(w)$ for $0 < i_1 - s_2 \le s_1 - i_2 \le \alpha$.
If $T1 = T^*1 = 0$ then T is bounded in $\dot{F}_p^{\phi_1/\phi_2,q}(w)$ for $-\alpha \le i_1 - s_2 \le s_1 - i_2 \le \alpha$.

4 Proof of the theorems

Note that if the kernel K satisfies $(S0)$ or (S^r0) then T can be extended to a continous linear operator, $T: M^{(\beta,\gamma)} \to (\Lambda_o^\beta)'$, for every $\gamma > 0$.
In fact, for $f \in M^{(\beta,\gamma)}$ and $g \in \Lambda_o^\beta$ we consider $x_0 \in X$, like in the definition of $M^{(\beta,\gamma)}$ and $R > 0$ such that $sopg \in B(x_0, R)$. We choose $\xi \in \Lambda_o^\theta$ such that $\xi \equiv 1$ in $B(x_0, 2AR)$ and $\xi \equiv 0$ in $B(x_0, 4A^2R)$, and consider the following extension

$$\langle Tf, g\rangle := \langle T(f\xi), g\rangle + \langle Tf(1-\xi), g\rangle, \tag{4.12}$$

where the first term in (4.12) is well defined since $f\xi \in \Lambda_o^\beta$ and the second term must be understood as the integral

$$I = \int\int K(x,y)f(y)(1-\xi(y))g(x)d\mu(y)d\mu(x) \tag{4.13}$$

which is absolutely convergent for K satisfying $(S0)$ if f and g are molecules. It is not hard to see that this extension is independent of the choice of ξ and coincides with the original operator when $f \in \Lambda_o^\beta$. In order to prove the boundedness of this operator on the Besov and Triebel-Lizorkin spaces, in view of Theorem (1.1) and since $D_k^* g \in \Lambda_o^\beta$ for every $k \in Z$, we have that

$$\begin{aligned} < D_k T f, g > &= < Tf, D_k^* g > \\ &= \lim_{N\to\infty} \sum_{|j|\le N} < TD_j(\hat{D}_j f), D_k^* g > \\ &= \lim_{N\to\infty} \sum_{|j|\le N} < D_k T D_j(\hat{D}_j f), g >, \end{aligned} \tag{4.14}$$

for every $f \in M^{(\beta,\gamma)}$ and $g \in \Lambda_0^\beta$. Setting $T_{k,j} = D_k T D_j$, the application

$$K_{k,j}(x,y) = \langle TD_j(.,y), D_k(x,.)\rangle,$$

is the associated kernel to $T_{k,j}$ since for $f \in M^{(\beta,\gamma)}$ and $g \in \Lambda_o^\beta$, we have that

$$\begin{aligned} < T_{k,j} f, g > &= < TD_j f, D_k^* g > \\ &= < T\int D_j(.,y) f(y) d\mu(y), \int D_k(x,.) g(x) d\mu(x) > \\ &= \int\int < TD_j(.,y), D_k(x,.) > f(y) g(x) d\mu(x) d\mu(y), \end{aligned} \tag{4.15}$$

where (4.15) follows from the point of view of the theory of Bochner's integral. To prove Theorem (3.1) we need the following technical lemma:

LEMMA 4.1 *Let T be a linear continous operator from Λ_o^β to $(\Lambda_o^\beta)'$, for some $0 < \beta < \epsilon$, which is weakely bounded of order ϵ and such that $T1 = 0$. Suppose that T is associated to a kernel K satisfying $(S0)$, $(S1)$ and $(S1')$.*
Then, for $k \ge j$, we have

$$\int_X |K_{k,j}(x,y)| d\mu(y) + \int_X |K_{k,j}(x,y)| d\mu(x) \le \omega((2A)^{-|k-j|}) \tag{4.16}$$

where ω satisfies $\sum_{l=1}^\infty \omega((2A)^{-l})(2A)^{l\alpha} < \infty$, whenever $\sum_{l=1}^\infty \gamma_1((2A)^{-l})(2A)^{l\alpha} < \infty$, for some α, with $0 \le \alpha < \epsilon$. For $k < j$, the left-hand side of (4.16) is bounded by a constant.

PROOF:

Let us first consider the case $k \ge j$ and suppose that $\delta(x,y) \ge 4A^2(2A)^{-j}$. Since $sopD_k(x,.)$ and $sopD_j(.,y)$ are disjoint sets and $\int_X D_k(x,z) d\mu(z) = 0$ then $K_{k,j}$ is well defined in the form
$K_{k,j}(x,y) = \int_X \int_X D_k(x,z)[K(z,u) - K(x,u)] D_j(u,y) d\mu(u) d\mu(z)$.
As $\int |D_j(.,y)| d\mu(y) \le C$ and $\delta(u,y) \le (2A)^{-j}$ for $u \in sop\ D_j(.,y)$, we get that $\delta(x,u) \ge (2A)^{-j+1}$ and then,

$$\begin{aligned} &\int_{\delta(x,y)\ge 4A^2(2A)^{-j}} |K_{k,j}(x,y)| d\mu(y) \\ &\le \int_{\delta(x,z)\le(2A)^{-k}} |D_k(x,z)| \int_{\delta(x,u)\ge 2A(2A)^{-j}} |K(z,u) - K(x,u)| \\ &\qquad \times \left(\int |D_j(u,y)| d\mu(y)\right) d\mu(u) d\mu(z) \\ &\le C \int_{\delta(x,z)\le(2A)^{-k}} |D_k(x,z)| \\ &\qquad \times \left(\int_{\delta(x,u)\ge 2A(2A)^{k-j}(2A)^{-k}} |K(z,u) - K(x,u)| d\mu(u)\right) d\mu(z). \end{aligned} \tag{4.17}$$

Applying $(H1)$, which follows from $(S1)$, the inner integral in (4.17) is then bounded by $\gamma_1((2A)^{-(k-j)})$ and, as $\|D_k(x,.)\|_1$ is uniformly bounded in k and x, we obtain that

$$\int_{\delta(x,y)\geq 4A^2(2A)^{-j}} |K_{k,j}(x,y)|d\mu(y) \leq C\gamma_1((2A)^{-(k-j)}), \tag{4.18}$$

To handle the integral in $d\mu(x)$ on the set $\delta(x,y) \geq 4A^2(2A)^{-j}$, we apply $(S1')$ and the property $\|D_k\|_\infty \leq C(2A)^k$ to get

$$\begin{aligned}
&\int_{\delta(x,y)\geq 4A^2(2A)^{-j}} |K_{k,j}(x,y)|d\mu(x) \\
&\leq C\int |D_j(u,y)| \int_{\delta(x,u)\geq 2A(2A)^{k-j}(2A)^{-k}} \\
&\qquad \times \left((2A)^k \int_{\delta(x,z)\leq(2A)^{-k}} |K(z,u)-K(x,u)|d\mu(z)\right) d\mu(x)d\mu(u) \\
&\leq C\gamma_1((2A)^{-(k-j)}).
\end{aligned}$$

We now consider the case $\delta(x,y) \leq 4A^2(2A)^{-j}$. Choosing $\xi \in C_0^\infty(-3A,3A)$ such that $\xi \equiv 1$ in $[-2A,2A]$ we define $h_k(z) = \xi\left((2A)^k\delta(x,z)\right)$. Since $T1 = 0$, we can split $K_{k,j}$ as

$$\begin{aligned}
K_{k,j}(x,y) &= < D_k(x,.), T(D_j(.,y)h_k) > \\
&\quad + < D_k(x,.), T(D_j(.,y)(1-h_k)) > \\
&= < D_k(x,.), T((D_j(.,y)-D_j(x,y))h_k) > \\
&\quad + < D_k(x,.), T((D_j(.,y)-D_j(x,y))(1-h_k)) > \\
&= D + B
\end{aligned} \tag{4.19}$$

But, since $\|D_k(x,.)\|_\epsilon \leq C(2A)^{k(1+\epsilon)}$, $\|[D_j(.,y) - D_j(x,y)]h_k\|_\epsilon \leq C(2A)^{j(1+\epsilon)}$ and their supports are both contained in the ball $B(x,(2A)^{-k})$ then, applying the weak boundary property, we have that $|D| \leq C(2A)^j(2A)^{-(k-j)\epsilon}$, where the constant C is independent of k and j and $\gamma_2((2A)^{-(k-j)}) := (2A)^{-(k-j)\epsilon}$ satisfies (3.11) when $\alpha < \epsilon$.
On the other side, since $\delta(z,u) \geq (2A)^{-k}$ and $\int_X D_k(x,z)d\mu(z) = 0$, the second term in (4.19) can be written as

$$\begin{aligned}
B &= \int\int D_k(x,z)(K(z,u)-K(x,u))(D_j(u,y)-D_j(x,y)) \\
&\qquad \times(1-h_k(u))d\mu(u)d\mu(z).
\end{aligned} \tag{4.20}$$

Next we split $|B|$ as

$$\begin{aligned}
|B| &\leq \left(\int\int_{(2A)(2A)^{-k}\leq\delta(x,u)\leq(2A)(2A)^{-j}} + \int\int_{\delta(x,u)\geq(2A)(2A)^{-j}}\right) \\
&\qquad |D_k(x,z)||K(z,u)-K(x,u)||D_j(u,y)-D_j(x,y)|d\mu(u)d\mu(z) \\
&= B_1 + B_2.
\end{aligned} \tag{4.21}$$

Since there is a positive contant C, independent of j, such that $|D_j(u,y) - D_j(x,y)| \leq C\min\left((2A)^{j(1+\epsilon)}\delta(x,u)^\epsilon, (2A)^j\right)$, we first get that

$$B_2 \leq C(2A)^j \int |D_k(x,z)|\left(\int_{\delta(x,u)\geq(2A)(2A)^{-j}} |K(z,u)-K(x,u)|d\mu(u)\right)d\mu(z). \tag{4.22}$$

Spliting $(2A)^{-j} = (2A)^{k-j}(2A)^{-k}$, $(k \geq j)$, and applying $(H1)$, we obtain that $B_2 \leq C(2A)^j\gamma_1((2A)^{-(k-j)})$. We also get that

$$B_1 \leq C(2A)^{j(1+\epsilon)}$$
$$\times \int |D_k(x,z)| \left(\int_{(2A)^{-k+1} \leq \delta(x,u) \leq (2A)^{-j+1}} \delta(x,u)^{\epsilon} |K(z,u) - K(x,u)| d\mu(u) \right) d\mu(z). \tag{4.23}$$

Applying $(H1)$, the inner integral in (4.23) is dominated by

$$\sum_{m=1}^{k-j} \int_{\substack{(2A)^{-k+m} \leq \delta(x,u) \\ \leq (2A)^{-k+m+1}}} \delta(x,u)^{\epsilon} |K(z,u) - K(x,u)| d\mu(u)$$
$$\leq C(2A)^{-k\epsilon} \sum_{m=1}^{k-j} (2A)^{m\epsilon} \int_{\delta(x,u) \geq (2A)^m (2A)^{-k}} |K(z,u) - K(x,u)| d\mu(u)$$
$$\leq C(2A)^{-k\epsilon} \sum_{m=1}^{k-j} (2A)^{m\epsilon} \gamma_1((2A)^{-m}),$$

and then it follows that $B_1 \leq C(2A)^j \gamma_3((2A)^{-(k-j)})$, where $\gamma_3((2A)^{-l}) = (2A)^{-l\epsilon} \sum_{m=1}^{l} (2A)^{m\epsilon} \gamma_1((2A)^{-m})$ verifies (3.11) for $\alpha < \epsilon$. Denoting $\omega = \gamma_1 + \gamma_2 + \gamma_3$, from the above results, for $k \geq j$ we have that

$$\int_{\delta(x,y) \leq (4A^2)(2A)^{-j}} |K_{k,j}(x,y)| \{d\mu(x) + d\mu(y)\}$$
$$\leq \int_{\delta(x,y) \leq (4A^2)(2A)^{-j}} D + B\{d\mu(x) + d\mu(y)\} \leq C\omega((2A)^{-(k-j)}). \tag{4.24}$$

Let now consider the case $k < j$. As $\int D_j d\mu(u) = 0$, for $\delta(x,y) \geq 4A^2(2A)^{-k}$, we have that
$K_{k,j}(x,y) = \int_X \int_X D_j(u,y)(K(z,u) - K(z,y)) D_k(x,z) d\mu(u) d\mu(z)$.
Since in this case we get that $\delta(z,u) \geq (2A)^{-k}$, from $(H1')$, we deduce that

$$\int_{\delta(x,y) \geq 4A^2(2A)^{-k}} |K_{k,j}(x,y)| d\mu(x)$$
$$\leq C \int |D_j(u,y)| \left(\int_{\delta(z,u) \geq (2A)^{-k}} |K(z,u) - K(z,y)| d\mu(z) \right) d\mu(u)$$
$$\leq C\gamma_1(1) \int |D_j(u,y)| d\mu(u) \leq C. \tag{4.25}$$

Similarly, from the null average of $D_k(x,.)$, we write
$K_{k,j}(x,y) = \int_X \int_X D_k(x,z)(K(z,u) - K(x,u)) D_j(u,y)$
and, by $(H1)$, we get

$$\int_{\delta(x,y) \geq 4A^2(2A)^{-k}} |K_{k,j}(x,y)| d\mu(y)$$
$$\leq C \int |D_k(x,z)| \left(\int_{\delta(x,u) \geq (2A)^{-k}} |K(z,u) - K(x,u)| d\mu(u) \right) d\mu(z)$$
$$\leq C. \tag{4.26}$$

For $\delta(x,y) \le 4A^2(2A)^{-k}$ we procceed as in the case $k \ge j$. In fact, denoting $l_j(z) = \xi\left((2A)^j\delta(y,z)\right); z \in X$, where ξ is defined like in that case, we display $K_{k,j}$ as

$$\begin{aligned} K_{k,j}(x,y) &= \langle D_k(x,.)l_j, T(D_j(.,y))\rangle \\ &\quad + \int\int D_k(x,z)K(z,u)D_j(u,y)(1-l_j(z))d\mu(u)d\mu(z) \\ &= \tilde{D}+\tilde{B}. \end{aligned} \tag{4.27}$$

From the (WBP), the first term $\tilde{D}$, which must be understood in the sense of distributions, satisfies $|\tilde{D}| \le C(2A)^k$, because $|D_k l_j|_\epsilon \le C(2A)^k(2A)^{j\epsilon}$, $|D_j|_\epsilon \le C(2A)^{j(1+\epsilon)}$ and their supports are both contained in $B(y, 3A(2A)^{-j})$.
From the null average of $D_j(.,y)$ and the property $\|D_k(x,.)\|_\infty \le C(2A)^k$, applying $(H1')$, we also get that

$$\begin{aligned} &|\tilde{B}| \\ &\le C(2A)^k \int_{\delta(y,u)\le(2A)^{-j}} |D_j(u,y)|\Big(\int_{\substack{\delta(y,z)\ge(2A)(2A)^{-j} \\ \delta(x,z)<(2A)^{-k}}} |K(z,u)-K(z,y)|d\mu(z)\Big)d\mu(u) \\ &\le C(2A)^k. \end{aligned}$$

By integrating $|\tilde{D}|+|\tilde{B}|$ over the set $\{\delta(x,y) \le 4A^2(2A)^{-k}\}$ in $d\mu(x)$ and in $d\mu(y)$ we obtain the desired estimate and this ends the proof of Lemma (4.1).◇

REMARKS **4.2** *Note that if in addition we have $T^*1 = 0$, then we also obtain (4.16) for $k < j$ since conditions on T and T^* are symmetric and*

$$\begin{aligned} K_{k,j}(x,y) &= < D_k(x,.), TD_j(.,y) > = < T^*D_k(x,.), D_j(.,y) > \\ &= < T^*D_k(.,x), D_j(y,.) > = K^*_{j,k}(y,x). \end{aligned} \tag{4.28}$$

PROOF: OF THEOREM (3.1)

Let denote $\Omega = \dot{B}_p^{\psi,q}$ and $\beta = \max(s_1, s_2)$, where $\psi = \phi_1/\phi_2$.
Since $M^{(\epsilon',\epsilon')}$ is dense in Ω, $1 \le p, q < \infty$, for all ϵ' such that $\beta < \epsilon' < \epsilon$ it is enough to show that there exists a constant $C > 0$ such that $\|Tf\|_\Omega \le C\|f\|_\Omega$ for all $f \in M^{(\epsilon',\epsilon')}$.
By Lemma (4.1), $T_{k,j}$ is an integral operator defined by

$$T_{k,j}h(x) = \int K_{k,j}(x,y)h(y)d\mu(y), \quad x \in X.$$

and for $k \ge j$ and $1 \le p < \infty$, it satisfies

$$\|T_{k,j}h\|_p \le C\omega((2A)^{-(k-j)})\|h\|_p, \tag{4.29}$$

In fact, applying Hölder's inequality, for $1 < p < \infty$ we have

$$\begin{aligned} \|T_{k,j}h\|_p &\le \left(\int\left(\int |K_{k,j}(x,y)||h(y)|d\mu(y)\right)^p d\mu(x)\right)^{1/p} \\ &\le \left(\int\left(\int |K_{k,j}(x,y)|d\mu(y)\right)^{p/p'}\left(\int |K_{k,j}(x,y)||h(y)|^p d\mu(y)\right)d\mu(x)\right)^{1/p} \\ &\le C\omega((2A)^{-(k-j)})\|h\|_p \end{aligned} \tag{4.30}$$

and, for $p=1$, we have

$$\|T_{k,j}h\|_1 \;\le\; \int\int |K_{k,j}(x,y)||h(y)|d\mu(y)d\mu(x) \le C\omega((2A)^{-(k-j)})\|h\|_1. \quad (4.31)$$

For $k<j$ and also from Lemma (4.1) we obtain

$$\|T_{k,j}(\hat{D}_j f)\|_p \le C\|\hat{D}_j f\|_p. \quad (4.32)$$

On the other hand, from (4.14) we have

$$\begin{aligned}
\|Tf\|_{\dot{B}_p^{\psi,q}} &= \left(\sum_{k\in Z}\left(\frac{\|D_k(Tf)\|_p}{\psi((2A)^{-k})}\right)^q\right)^{1/q}\\
&\le \left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}\sum_{j\in Z}\|D_kTD_j(\hat{D}_j f)\|_p\right)^q\right)^{1/q}\\
&= \left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}\sum_{j\in Z}\|T_{k,j}(\hat{D}_j f)\|_p\right)^q\right)^{1/q}\\
&\le \left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}\sum_{j\le k}\|T_{k,j}(\hat{D}_j f)\|_p\right)^q\right)^{1/q}\\
&\quad + \left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}\sum_{j>k}\|T_{k,j}(\hat{D}_j f)\|_p\right)^q\right)^{1/q} = S_1+S_2. \quad (4.33)
\end{aligned}$$

Nevertheless, from the definitions of lower and upper type, we obtain

$$\begin{aligned}
\frac{1}{\psi((2A)^{-k})} &\le \frac{\phi_2((2A)^{-k})}{\phi_1((2A)^{-k})} \le C(2A)^{(k-j)(s_1-i_2)}\frac{\phi_2((2A)^{-j})}{\phi_1((2A)^{-j})}\\
&= C(2A)^{(k-j)(s_1-i_2)}\frac{1}{\psi((2A)^{-j})} \text{ for } k\ge j, \quad (4.34)
\end{aligned}$$

and

$$\frac{1}{\psi((2A)^{-k})} \le C(2A)^{(j-k)(s_2-i_1)}\frac{1}{\psi((2A)^{-j})} \text{ for } k<j. \quad (4.35)$$

Therefore, applying (4.34) and (4.29) we get

$$\begin{aligned}
S_1 &\le C\left(\sum_{k\in Z}\left(\sum_{j\le k}(2A)^{(k-j)(s_1-i_2)}\omega((2A)^{-(k-j)})\frac{1}{\psi((2A)^{-j})}\|\hat{D}_j f\|_p\right)^q\right)^{1/q}\\
&= C\left(\sum_{k\in Z}\left(\sum_{j\ge 0}(2A)^{j(s_1-i_2)}\omega((2A)^{-j})\frac{1}{\psi((2A)^{-(k-j)})}\|\hat{D}_{k-j} f\|_p\right)^q\right)^{1/q}\\
&\le C\sum_{j\ge 0}(2A)^{j(s_1-i_2)}\omega((2A)^{-j})\left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-(k-j)})}\|\hat{D}_{k-j} f\|_p\right)^q\right)^{1/q}\\
&\le C\|f\|_{\dot{B}_p^{\psi,q}}, \quad (4.36)
\end{aligned}$$

since by hypothesis,

$$\sum_{j\geq 0}(2A)^{j(s_1-i_2)}\omega((2A)^{-j}) \leq \sum_{j\geq 0}(2A)^{j\alpha}\omega((2A)^{-j}) < \infty.$$

On the other side, applying (4.35) and (4.32) we have

$$\begin{aligned} S_2 &\leq C\left(\sum_{k\in Z}\left(\sum_{j>k}(2A)^{(j-k)(s_2-i_1)}\frac{1}{\psi((2A)^{-j})}\|\hat{D}_j f\|_p\right)^q\right)^{1/q} \\ &= C\left(\sum_{k\in Z}\left(\sum_{j<0}(2A)^{j(i_1-s_2)}\frac{1}{\psi((2A)^{-(k-j)})}\|\hat{D}_{k-j} f\|_p\right)^q\right)^{1/q} \\ &\leq C\sum_{j<0}(2A)^{j(i_1-s_2)}\left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-(k-j)})}\|\hat{D}_{k-j} f\|_p\right)^q\right)^{1/q} \qquad (4.37)\\ &\leq C\|f\|_{\dot{B}_p^{\psi,q}}, \qquad (4.38)\end{aligned}$$

whenever $i_1 - s_2 > 0$. Finally, by Remark (4.2), if $T1 = T^*1 = 0$ then (4.16) is valid, and also (4.29), for all k and $j \in \mathbb{Z}$. Therefore, instead of (4.37) the bound for S_2 is

$$\begin{aligned} S_2 &\leq C\left(\sum_{k\in Z}\left(\sum_{j>k}(2A)^{(j-k)(s_2-i_1)}\omega((2A)^{-(j-k)})\frac{1}{\psi((2A)^{-j})}\|\hat{D}_j f\|_p\right)^q\right)^{1/q} \\ &\leq C\sum_{j>0}(2A)^{j(s_2-i_1)}\omega((2A)^{-j})\left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-(k-j)})}\|\hat{D}_{k-j} f\|_p\right)^q\right)^{1/q} \qquad (4.39)\\ &\leq C\|f\|_{\dot{B}_p^{\psi,q}}, \qquad (4.40)\end{aligned}$$

whenever $s_2 - i_1 \leq \alpha$. In this way, the proof of this theorem is complete. ◊

To prove Theorem (3.2) we need the following two technical lemmas:

LEMMA 4.3 *Let T be associated to a kernel K satisfying (S^r1) with* modulus of continuity *γ_r, $1 < r < \infty$ and $1/r + 1/r' = 1$. Then, for $k \geq j$, we have*

$$\begin{aligned} &\left(\int_{(2A)^i(2A)^{-j}\leq\delta(x,y)\leq(2A)^{i+1}(2A)^{-j}} |K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}} \\ &\leq C((2A)^i(2A)^{-j})^{-\frac{1}{r'}}\sum_{l=-1}^{1}\gamma_r((2A)^{-(i+l)}(2A)^{-|k-j|}),\ i=2,3,\ldots \qquad (4.41)\end{aligned}$$

For $k < j$, we have

$$\begin{aligned} &\left(\int_{(2A)^i(2A)^{-k}\leq\delta(x,y)\leq(2A)^{i+1}(2A)^{-k}} |K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}} \\ &\leq C((2A)^i(2A)^{-k})^{-\frac{1}{r'}}\sum_{l=-1}^{1}\gamma_r((2A)^{-(i+l)}(2A)^{-|k-j|}),\ i=2,3,\ldots \qquad (4.42)\end{aligned}$$

PROOF:

Let first consider the case $k \geq j$. Denote $t = (2A)^{-j} ands = (2A)^{-k}$. Let also define the set $Q_i = \{y : (2A)^i t \leq \delta(x,y) \leq (2A)^{i+1}t\}$, $i = 2,3,\dots$. For $y \in Q_i$, $D_k(x,z) \neq 0$ and $D_j(u,y) \neq 0$, we get that $\delta(z,u) \geq (2A)^{i-1}t$ and then the kernel $K_{k,j}(x,y)$ is well defined as

$$\begin{aligned} K_{k,j}(x,y) &= \int_X\int_X D_k(x,z)K(z,u)D_j(u,y)d\mu(u)d\mu(z) \\ &= \int_X\int_X D_k(x,z)\left(K(z,u)-K(x,u)\right)D_j(u,y)d\mu(u)d\mu(z), \quad (4.43) \end{aligned}$$

as $\int D_k(x,z)d\mu(z) = 0$. Since $sopD_j(.,y) \in B(y,t)$ and $\|D_j\|_\infty \leq C1/t$ we have that

$$\begin{aligned} &\left(\int_{Q_i} |K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}} \\ &\leq C\left(\int_{Q_i}\left(\int |D_k(x,z)|(\frac{1}{t}\int_{\delta(u,y)<t} |K(z,u)-K(x,u)|d\mu(u))d\mu(z)\right)^r d\mu(y)\right)^{\frac{1}{r}} \quad (4.44) \end{aligned}$$

Applying Hölder's inequality to the inner integral in (4.44), we obtain that

$$\begin{aligned} &\left(\int_{Q_i} |K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}} \\ &\leq C\left(\int_{Q_i}\left(\int |D_k(x,z)|(\frac{1}{t}\int_{\delta(u,y)<t} |K(z,u)-K(x,u)|^r d\mu(u))^{\frac{1}{r}} d\mu(z)\right)^r d\mu(y)\right)^{\frac{1}{r}} \quad (4.45) \end{aligned}$$

Then applying Minkowski's inequality, we get

$$\begin{aligned} &\left(\int_{Q_i} |K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}} \\ &\leq C\int_{\delta(x,z)\leq s} |D_k(x,z)|\left(\int_{Q_i}\frac{1}{t}\int_{\delta(u,y)<t} |K(z,u)-K(x,u)|^r d\mu(u)d\mu(y)\right)^{\frac{1}{r}} d\mu(z). \end{aligned} \quad (4.46)$$

Moreover, if $y \in Q_i$ and $\delta(u,y) < t$, then $(2A)^{i-1}t \leq \delta(x,u) \leq (2A)^{i+2}t$. Therefore, writing $t = (2A)^{k-j}s$ and applying Tonelli's theorem to the integrals in $d\mu(u)$ and $d\mu(y)$ we obtain the bound

$$\begin{aligned} &\left(\int_{Q_i} |K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}} \\ &\leq C \sup_{\substack{(x,z):\delta(x,z)<s \\ s>0}} \left(\int_{(2A)^{i+k-j-1}s\leq\delta(x,u)<(2A)^{i+k-j+2}s} |K(z,u)-K(x,u)|^r d\mu(u)\right)^{\frac{1}{r}} \quad (4.47) \end{aligned}$$

Since $i+k-j \geq 1$, we apply the weaker condition (H^r1) to prove that

$$\left(\int_{Q_i} |K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}} \leq C((2A)^{i-j})^{-\frac{1}{r'}} \sum_{l=-1}^{1} \gamma_r((2A)^{-(i+k-j+l)}) \quad (4.48)$$

and then we get (4.41).

Let now consider the case $k<j$ and denote $\hat{Q}_i = \{y : (2A)^i s \leq \delta(x,y) \leq (2A)^{i+1}s\}$,with $i=2,3,\ldots$.
Substructing $K(z,y)$ instead of $K(x,u)$ in (4.43) and procceeding as in (4.44), (4.45) and (4.46), we get that

$$\left(\int_{\hat{Q}_i} |K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}}$$

$$\leq C\int |D_k(x,z)| \left(\int_{\hat{Q}_i}\left(\frac{1}{t}\int_{\delta(u,y)<t} |K(z,u)-K(z,y)|^r d\mu(u)\right) d\mu(y)\right)^{\frac{1}{r}} d\mu(z). \quad (4.49)$$

But, if $y \in \hat{Q}_i$, $\delta(x,z)<s$ and $\delta(u,y)<t$ then $(2A)^{i-1}s \leq \delta(z,y) < (2A)^{i+2}s$. Moreover, writing $s=(2A)^{j-k}t$ and applying condition (S^r1), we obtain

$$\left(\int_{\hat{Q}_i} |K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}}$$

$$\leq C \sup_{\delta(v,w)<t}\left(\int_{\substack{(2A)^{i+j-k-1}t\leq\delta(w,y)\\<(2A)^{i+j-k+2}t}} \sup_{0<\tau\leq t}\left(\frac{1}{\tau}\int_{\delta(u,y)<\tau} |K(v,u)-K(w,y)|^r d\mu(u)\right) d\mu(y)\right)^{\frac{1}{r}}$$

$$\leq C((2A)^{i-k})^{-\frac{1}{r'}} \sum_{l=-1}^{1} \gamma_r((2A)^{-(i+j-k+l)}).\Diamond \quad (4.50)$$

Lemma 4.4 *Let r, r' and γ_r be like in Lemma (4.3). Let $T\colon \Lambda_o^\beta \to (\Lambda_o^\beta)'$ be a linear continous operator, $0<\beta\leq\epsilon$ which is weakely bounded of order ϵ with $0<\epsilon\leq\theta$ and such that $T1=0$. Let also K, its associated kernel, verify (S^r0), (S^r1) and (H^r1'). Then,*

(a) For $k \geq j$, we have

$$\int |K_{k,j}(x,y)||h(y)|d\mu(y) \leq \omega\left((2A)^{-|k-j|}\right)\left(M(|h|^{r'})(x)\right)^{\frac{1}{r'}} \quad (4.51)$$

where M is the Hardy-Littlewood maximal operator. Moreover, ω satisfies $\sum_{l=0}^{\infty}\omega((2A)^{-l})<\infty$; whenever γ_r satisfies $\sum_{l=0}^{\infty} l\gamma_r((2A)^{-l})<\infty$, and ω satisfies (3.11) with $0<\alpha<\epsilon$, whenever γ_r satisfies the same condition.

(b) For $k<j$ there is a constant C, not depending of k and j, such that if γ_r verifies
$\sum_{l=0}^{\infty}\gamma_r((2A)^{-l})<\infty$ then

$$\int |K_{k,j}(x,y)||h(y)|d\mu(y) \leq C\left(M(|h|^{r'})(x)\right)^{\frac{1}{r'}}. \quad (4.52)$$

PROOF:
We first consider the case $k \geq j$. Denote, as in the previous lemma, $t = (2A)^{-j}$, $s = (2A)^{-k}$ and $Q_i = \{(2A)^i t \leq \delta(x,y) \leq (2A)^{i+1}t\}$ with $i = 2,3,\ldots$. Then, we have

$$\begin{aligned}\int |K_{k,j}(x,y)||h(y)|d\mu(y) &= \left(\int_{\delta(x,y)\leq 4A^2t} + \sum_{i=2}^{\infty}\int_{Q_i}\right) |K_{k,j}(x,y)||h(y)|d\mu(y)\\ &= I_1 + I_2. \end{aligned} \tag{4.53}$$

To estimate I_1 we use the bounds obtained in the proof of Lemma (4.1) for the case $\delta(x,y) \leq 4A^2(2A)^{-j}$ and $k \geq j$.
Using the hypothesis $T1 = 0$, in (4.19) we have $K_{k,j}(x,y) = D + B$, with

$$|D| \leq C(2A)^j(2A)^{-(k-j)\epsilon} := C(2A)^j\delta_1((2A)^{-(k-j)})$$

and $|B| \leq B_1 + B_2$, with

$$\begin{aligned} B_1 &\leq \int\int_{(2A)(2A)^{-k}\leq\delta(x,u)\leq(2A)(2A)^{-j}} |D_k(x,z)||K(z,u) - K(x,u)|\\ &\quad \times|D_j(u,y) - D_j(x,y)|d\mu(u)d\mu(z), \end{aligned} \tag{4.54}$$

$$\begin{aligned} B_2 &\leq \int\int_{\delta(x,u)\geq(2A)(2A)^{-j}} |D_k(x,z)||K(z,u) - K(x,u)|\\ &\quad \times|D_j(u,y) - D_j(x,y)|d\mu(u)d\mu(z). \end{aligned} \tag{4.55}$$

By the fact that $\|D_j\|_\infty \leq C(2A)^j$, spliting the inner integral in (4.55) as the series of the integrals over the sets $(2A)^i t \leq \delta(x,u) \leq (2A)^{i+1}t$ and applying Hölder's inequality, we get that

$$\begin{aligned} B_2 &\leq C(2A)^j \int |D_k(x,z)|\\ &\quad \times \sum_{i=1}^{\infty}\left((2A)^i t\right)^{\frac{1}{r'}} C\left(\int_{(2A)^i t\leq\delta(x,u)\leq(2A)^{i+1}t} |K(z,u) - K(x,u)|^r d\mu(u)\right)^{\frac{1}{r}} d\mu(z). \end{aligned} \tag{4.56}$$

As $i - j = (i + k - j) - k$ and $i + k - j > 1$, it is enough to apply the weaker condition $(H^r 1)$ to conclude that

$$B_2 \leq C(2A)^j\delta_2((2A)^{-(k-j)}), \tag{4.57}$$

with $\delta_2((2A)^{-l}) := \sum_{i=1}^{\infty}\gamma_r((2A)^{-i}(2A)^{-l})$. On the other side, like in (4.23), we have that

$$\begin{aligned} B_1 &\leq C(2A)^{j(1+\epsilon)}\\ &\quad \times \int |D_k(x,z)| \left(\int_{(2A)s\leq\delta(x,u)\leq(2A)t} \delta(x,u)^\epsilon |K(z,u) - K(x,u)|d\mu(u)\right) d\mu(z) \end{aligned} \tag{4.58}$$

Spliting the inner integral as the sum of $k - j$ integrals over the sets $\{(2A)^m s \leq \delta(x,u) \leq (2A)^{m+1}s\}$, applying Hölder's inequality and, once more, condition $(H^r 1)$, we obtain

$$B_1 \leq C_\epsilon(2A)^j\delta_3((2A)^{-(k-j)}), \tag{4.59}$$

with $\delta_3((2A)^{-l}) := (2A)^{-l\epsilon}\sum_{m=1}^{l}(2A)^{m\epsilon}\gamma_r((2A)^{-m})$.
It is easy to check that $\omega_1 = \delta_1+\delta_2+\delta_3$, satisfies the summability properties enunciated in this Lemma and also, the first term in (4.53) satisfies

$$\begin{aligned} I_1 &\leq C\omega_1((2A)^{-(k-j)})(2A)^j \int_{\delta(x,y)\leq 4A^2(2A)^{-j}} |h(y)|d\mu(y) \\ &\leq C\omega_1((2A)^{-(k-j)})\left((2A)^j \int_{\delta(x,y)\leq 4A^2(2A)^{-j}} |h(y)|^{r'}d\mu(y)\right)^{\frac{1}{r'}} \\ &\leq C\omega_1((2A)^{-|k-j|})[M(|h|^{r'})(x)]^{\frac{1}{r'}}. \end{aligned} \tag{4.60}$$

On the other side, from Hölder's inequality and inequality (4.41) obtained in Lemma (4.3), it follows that

$$\begin{aligned} I_2 &\leq \sum_{i=2}^{\infty}\left(\int_{Q_i}|K_{k,j}(x,y)|^r d\mu(y)\right)^{\frac{1}{r}}\left(\int_{\delta(x,y)<(2A)^{i+1}t}|h(y)|^{r'}d\mu(y)\right)^{\frac{1}{r'}} \\ &\leq \sum_{i=2}^{\infty}\sum_{l=-1}^{1}\gamma_r((2A)^{-(i+l)}(2A)^{-(k-j)})[(2A)^i t]^{-\frac{1}{r'}}\left(\int_{\delta(x,y)<(2A)^{i+1}t}|h(y)|^{r'}d\mu(y)\right)^{\frac{1}{r'}} \\ &\leq C\omega_2((2A)^{-(k-j)})\left[M(|h|^{r'})(x)\right]^{\frac{1}{r'}}, \end{aligned} \tag{4.61}$$

with $\omega_2((2A)^{-l}) := \sum_{i=1}^{\infty}\gamma_r((2A)^{-i}(2A)^{-l})$. As ω_2 satisfies the required summability properties, taking $\omega = \omega_1+\omega_2$ we completed the proof of this lemma for the case $k \geq j$.
We now consider the case $k < j$. In a similar fashion to the previous case we have

$$\begin{aligned} \int |K_{k,j}(x,y)||h(y)|d\mu(y) &= \left(\int_{\delta(x,y)\leq 4A^2 s} + \sum_{i=2}^{\infty}\int_{\tilde{Q}_i}\right)|K_{k,j}(x,y)||h(y)|d\mu(y) \\ &= \tilde{I}_1+\tilde{I}_2, \end{aligned} \tag{4.62}$$

where $\tilde{Q}_i = \{(2A)^i s \leq \delta(x,y) \leq (2A)^{i+1}s\}$.

Procceeding as in (4.27) of Lemma (4.1), for $\delta(x,y)\leq 4A^2 s$ we write

$$\begin{aligned} K_{k,j}(x,y) &= \langle D_k(x,.)l_j, T(D_j(.,y))\rangle \\ &+ \int D_j(u,y)\int D_k(x,z)|K(z,u)-K(z,y)|(1-l_j(z))d\mu(u)d\mu(z) \\ &= \tilde{D}+\tilde{B}, \end{aligned}$$

where $l_j(z) = \xi((2A)^j\delta(y,z))$ and ξ is defined as in that lemma.
By the weak boundary property (WBP), we have that $\tilde{D}\leq C(2A)^k$.
Taking in account that $\|D_k\|_\infty \leq C(2A)^k$, applying Hölder's inequality, then the hypothesis (H^r1') and, finally, the weaker property $\sum_{i=1}^{\infty}\gamma_r((2A)^{-i})\leq C$, we get

$$|\tilde{B}| \leq C(2A)^k\int|D_j(u,y)|$$

$$\times \sum_{i=2}^{\infty}((2A)^i t)^{1/r'} \left(\int_{\delta(z,y)\le(2A)^{i+1}t} |K(z,u)-K(z,y)|^r d\mu(z)\right)^{1/r} d\mu(u)$$
$$\le C(2A)^k \sum_{i=2}^{\infty} \gamma_r((2A)^{-i}) \le C(2A)^k \tag{4.63}$$

Then it follows that

$$\tilde{I}_1 \le C(2A)^k \int_{\delta(x,y)\le 4A^2(2A)^{-k}} |h(y)|d\mu(y) \le C\left[M(|h|^{r'})(x)\right]^{\frac{1}{r'}}. \tag{4.64}$$

To estimate $\tilde{I}_2$ we first apply Hölder's inequality, then inequality (4.42) obtained in Lemma (4.3) and, as γ_r is quasi increasing we get that

$$\begin{aligned}\tilde{I}_2 &\le \sum_{i=2}^{\infty}\sum_{l=-1}^{1} \gamma_r((2A)^{-(i+l)}(2A)^{-(j-k)}) \\ &\quad\times \left((2A)^i(2A)^{-k}\right)^{\frac{-1}{r'}} \left(\int_{\delta(x,y)\le(2A)^{i+1}(2A)^{-k}} |h(y)|^{r'} d\mu(y)\right)^{\frac{1}{r'}} \\ &\le \sum_{i=2}^{\infty}\sum_{l=-1}^{1} \gamma_r((2A)^{-(i+l)}) \left[M(|h|^{r'})(x)\right]^{\frac{1}{r'}} \le C\left[M(|h|^{r'})(x)\right]^{\frac{1}{r'}}. \end{aligned} \tag{4.65}$$

In this way the case $k < j$ is also proved.◇

REMARKS 4.5 *Note that if we have $T^*1 = 0$ in addition of the hypothesis of Lemma (4.4), then we also obtain (4.51) for the case $k < j$. In fact, we proceed in a similar way to that of the case $k \ge j$ but, for the case $\delta(x,y) \le 4A^2(2A)^{-k}$ we apply (H^r1'), and for the case $\delta(x,y) > 4A^2(2A)^{-k}$, we use (4.42).*

PROOF: OF THEOREM (3.2)

Let denote $\Omega = \dot{F}_p^{\psi,q}(w)$ and, $\beta = 0$ for $\psi(t) = 1$ or $\beta = \max(s_1, s_2)$ for $\psi = \phi_1/\phi_2$. Since the space $M^{(\epsilon',\epsilon')}$ is dense in Ω for all ϵ' such that $\beta < \epsilon' < \epsilon$, it is enough to show that there is a constant $C > 0$ such that $\|Tf\|_\Omega \le C\|f\|_\Omega$ for all $f \in M^{(\epsilon',\epsilon')}$. But,

$$\begin{aligned}\|Tf\|_\Omega &= \left\|\left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}|D_k(Tf)(x)|\right)^q\right)^{1/q}\right\|_{L^p(w)} \\ &= \left\|\left(\sum_{k\in Z}\left(\frac{1}{\psi((2A)^{-k})}|< Tf, D_k(x,.) >|\right)^q\right)^{1/q}\right\|_{L^p(w)} \\ &\le \left\|\left(\sum_{k\in Z}\frac{1}{\psi((2A)^{-k})}\left(\sum_{j\in Z}|D_kTD_j(\hat{D}_jf)(x)|\right)^q\right)^{1/q}\right\|_{L^p(w)} \\ &= \left\|\left(\sum_{k\in Z}\frac{1}{\psi((2A)^{-k})}\left(\sum_{j\in Z}|T_{k,j}(\hat{D}_jf)(x)|\right)^q\right)^{1/q}\right\|_{L^p(w)},\end{aligned}$$

$$
\begin{aligned}
&\leq \ \| \left(\sum_{k\in Z} \frac{1}{\psi((2A)^{-k})} \left(\sum_{j\leq k} |T_{k,j}(\hat{D}_j f)(x)| \right)^q \right)^{1/q} \|_{L^p(w)} \\
&\quad + \| \left(\sum_{k\in Z} \frac{1}{\psi((2A)^{-k})} \left(\sum_{j>k} |T_{k,j}(\hat{D}_j f)(x)| \right)^q \right)^{1/q} \|_{L^p(w)} \\
&= \ \|S_1(x)\|_{L^p(w)} + \|S_2(x)\|_{L^p(w)}, \qquad (4.66)
\end{aligned}
$$

where $T_{k,j}(\hat{D}_j f)(x) = \int K_{k,j}(x,y)(\hat{D}_j f)(y)d\mu(y)$.
To estimate S_1 we apply (4.51) of Lemma (4.4) to obtain that $T_{k,j}$ satisfies

$$
|T_{k,j}(\hat{D}_j f)(x)| \leq C\omega((2A)^{-(k-j)})(M|\hat{D}_j f|^{r'}(x))^{\frac{1}{r'}},
$$

for $k \geq j$. From inequality (4.34) obtained in the proof of Theorem 3.1 (which is obviously true in the case $\psi(t) = 1$), and Minkowski's inequality it follows that

$$
\begin{aligned}
&S_1(x) \\
&\leq \left(\sum_{k\in Z} \left(\sum_{j\leq k} (2A)^{(k-j)(s_1-i_2)} \omega((2A)^{-(k-j)}) \left(M(\frac{|\hat{D}_j f|}{\psi((2A)^{-j})})^{r'}(x) \right)^{\frac{1}{r'}} \right)^q \right)^{1/q} \\
&\leq \sum_{j\geq 0} (2A)^{j(s_1-i_2)} \omega((2A)^{-j}) \left(\sum_{k\in Z} \left(M(\frac{|\hat{D}_{k-j} f|}{\psi((2A)^{-(k-j)})})^{r'}(x) \right)^{\frac{q}{r'}} \right)^{1/q} \\
&= \sum_{j\geq 0} (2A)^{j(s_1-i_2)} \omega((2A)^{-j}) \left(\sum_{k\in Z} \left(M(\frac{|\hat{D}_k f|}{\psi((2A)^{-k})})^{r'}(x) \right)^{\frac{q}{r'}} \right)^{1/q}
\end{aligned} \qquad (4.67)
$$

Nevertheless, by Lemma (4.4), the first factor in the last inequality is a finite constant since, from the hypothesis $\sum_{j\geq 0} j\gamma_r((2A)^{-j}) < \infty$, in the case $\psi(t) = 1$ it is equal to $\sum_{j\geq 0} \omega((2A)^{-j}) < \infty$, and, from the hypothesis $\sum_{j\geq 0}(2A)^{j\alpha}\gamma_r((2A)^{-j}) < \infty$, in the case $\psi(t) = \phi_1(t)/\phi_2(t)$ and $s_1 - i_2 \leq \alpha$, it is lower than or equal to $\sum_{j\geq 0}(2A)^{j\alpha}\omega((2A)^{-j}) < \infty$.
Therefore, we have proved that

$$
S_1(x) \ \leq \ C \left(\sum_{k\in Z} \left(M(\frac{|\hat{D}_k f|}{\psi((2A)^{-k})})^{r'}(x) \right)^{\frac{q}{r'}} \right)^{1/q}. \qquad (4.68)
$$

Since $1 < p/r', q/r' < \infty$, we are able to apply the weighted version of the Fefferman-Stein vector valued maximal inequality to obtain that

$$
\begin{aligned}
\|S_1\|_{L^p(w)} &\leq \ C\| \left(\sum_{k\in Z} \left(M(\frac{|\hat{D}_k f|}{\psi((2A)^{-k})})^{r'}(x) \right)^{\frac{q}{r'}} \right)^{1/q} \|_{L^p(w)} \\
&\leq \ \| \left(\sum_{k\in Z} (\frac{|\hat{D}_k f|}{\psi((2A)^{-k})})^q(x) \right)^{1/q} \|_{L^p(w)} = C\|f\|_{\dot{F}_p^{\psi,q}(w)} \qquad (4.69)
\end{aligned}
$$

Let now estimate S_2. From Remark (4.5), when $T^*1 = 0$ we also have $|T_{k,j}(\hat{D}_j f)(x)| \leq C\omega((2A)^{-(j-k)})(M|\hat{D}_j f|^{r'}(x))^{\frac{1}{r'}}$ for $k < j$. Then using inequality (4.35) and proceeding like in the previous case we obtain that

$$S_2(x)$$

$$\leq C\left(\sum_{k\in Z}\left(\sum_{j>k}(2A)^{(j-k)(s_2-i_1)}\omega((2A)^{-(j-k)})\left(M\left(\frac{|\hat{D}_j f|}{\psi((2A)^{-j})}\right)^{r'}(x)\right)^{\frac{1}{r'}}\right)^q\right)^{1/q}$$

$$\leq C\sum_{j>0}(2A)^{j(s_2-i_1)}\omega((2A)^{-j})\left(\sum_{k\in Z}\left(M\left(\frac{|\hat{D}_k f|}{\psi((2A)^{-k})}\right)^{r'}(x)\right)^{\frac{q}{r'}}\right)^{1/q} \tag{4.70}$$

$$\leq C\left(\sum_{k\in Z}\left(M\left(\frac{|\hat{D}_k f|}{\psi((2A)^{-k})}\right)^{r'}(x)\right)^{\frac{q}{r'}}\right)^{1/q}, \tag{4.71}$$

since, by the same argument that in the previous case $k \geq j$, we can assert that $\sum_{j>0}(2A)^{j(s_2-i_1)}\omega((2A)^{-j}) < \infty$ if either $s_2 = i_1 = 0$, when $\psi(t) = 1$, or $s_2 - i_1 \leq \alpha$ in the other case. Then the proof follows in exactly the same way than before to get that

$$\|S_2\|_{L^p(w)} \leq C\|f\|_{\dot{F}_p^{\psi,q}(w)}. \tag{4.72}$$

Nevertheless, if condition $T^*1 = 0$ is not required then, from inequality (4.52), we still have that $|T_{k,j}(\hat{D}_j f)(x)| \leq C(M|\hat{D}_j f|^{r'}(x))^{\frac{1}{r'}}$. Then to estimate $S_2(x)$, the constant appearing in (4.70) must be replaced by $\sum_{j>0}(2A)^{j(s_2-i_1)} < \infty$ whenever $i_1 - s_2 > 0$. From there on, the proof is the same as before. ◇

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Silvia Hartzstein, silviah@math.unl.edu.ar
Beaatriz E. Viviani, viviani@ceride.gov.ar

Recibido : 21 de Diciembre de 2000
Aceptado : 26 de Diciembre de 2000