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JOINT SPECTRUM FOR QUASI-SOLVABLE LIE ALGEBRAS OF OPERATORS

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ABSTRACT. Given a complex Banach space X and a joint spectrum for complex solvable finite dimensional Lie algebras of operators defined on X, we extend this joint spectrum to quasi-solvable Lie algebras of operators, and we prove the main spectral properties of the extended joint spectrum. We also show that this construction is uniquely determined by the original joint spectrum.

1. Introduction.

Given a Banach space X, Z. Słodkowski and W. Zelazko studied in [9] the main spectral properties of joint spectra of commuting, either finite or infinite, families of operators defined on X. In addition, in order to prove the projection property for these joint spectra, they showed that if a joint spectrum is defined only for finite families of mutually commuting operators, and if it has the projection property, then, by means of the notion of inverse limit, this joint spectrum can be uniquely extended to a joint spectrum defined on the family of all subsets of $\mathcal{L}(X)$ consisting of pairwise commuting operators. Moreover, this generalized joint spectrum also has the main spectral properties, i.e., it is a compact nonempty set and the projection property still holds.

On the other hand, in the last years some joint spectra for operators generating nilpotent and solvable Lie algebras were introduced. For example, in [5] was considered the first non-commutative version of the Taylor joint spectrum, [10], for nilpotent Lie algebras of operators. Working independently, in [2] and in [3] we extended the Taylor, and the Słodkowki joint spectra, [10] and [8], of finite commuting tuples of operators to complex solvable finite dimensional Lie algebras of

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operators. Moreover, in [1] it was introduced a new concept of spectrum for complex solvable finite dimensional Lie algebras of operators, which agrees with one of [5] and [2] in the case of a nilpotent Lie algebra, but in the solvable case differs, in general, from the one of [2]. In addition, the joint spectrum introduced in [1] may be extended to some infinite dimensional Lie algebras of operators, in a similar way as Z. Słodkowski and W. Zelazko did in [9] for the commutative case.

In this article we consider another non commutative variant of the construction developed by Z. Słodkowski and W. Zelazko in [9]. In fact, we consider a joint spectrum defined for complex solvable finite dimensional Lie algebras of operators, such as the Taylor and the Słodkowski joint spectra, and we extend it to quasi-solvable Lie algebra of operators, see section 2. In addition, we prove the main spectral properties for our joint spectrum, and we show that this construction is uniquely determined by the original joint spectrum. On the other hand, this joint spectrum will be, in general, different from the one considered in [1].

The paper is organized as follows. In Section 2 we recall several definitions and results which we need for our work, and in Section 3 we prove our main result.

2. Preliminaries.

Let us begin with the definition of joint spectrum which we shall consider.

Definition 1. *Let X be a Banach space. A joint spectrum is a function, σ, which assigns, to each complex solvable finite dimensional Lie algebra L of operators defined on X, a compact nonempty subset $\sigma(L)$ of characters, such that if H is a Lie ideal of L, and π is the restriction map $\pi: L^* \to H^*$, then the projection property for ideals holds, i.e., $\pi(\sigma(L)) = \sigma(H)$.*

The joint spectra that we are considering are the Taylor joint spectrum and the Słodkowski joint spectra for complex solvable finite dimensional Lie algebras of operators, see [8], [10], [2], [3] and [6].

We now recall the definition of a quasi-solvable Lie algebra.

Definiton 2. *A quasi-solvable Lie algebra $\mathcal{L}$ is a complex Lie algebra such that $\mathcal{L} = \sum_{\alpha\in I} I_\alpha$, where I is an index set, and for each $\alpha \in I$; I_α is a complex solvable finite dimensional ideal of $\mathcal{L}$.*

In order to see the main poperties of the quasi-solvable Lie algebras and their behaviour under representations in Banach spaces, we refer to [7] and [11].

Our main result concers with the notion of inverse limit. We recall the most important facts related to this notion; for a complete exposition see [4].

Definition 3. *An inverse system of sets and maps* $\{\mathcal{X}, \pi\}$, *over a directed set* $(M, <)$, *is a function which attaches to each* $\alpha \in M$ *a set* $\mathcal{X}_\alpha$, *and to each pair* α *and* β *such that* $\alpha < \beta$ *in* M, *a map* $\pi_\alpha^\beta\colon \mathcal{X}_\beta \to \mathcal{X}_\alpha$ *such that*

$$\pi_\alpha^\alpha = Id_M, \qquad \pi_\alpha^\beta \circ \pi_\beta^\gamma = \pi_\alpha^\gamma,$$

where α, β, γ *belong to* M *and* $\alpha < \beta < \gamma$.

Definition 4. *Let* $\{\mathcal{X}, \pi\}$ *be an inverse system of sets and maps over a directed set* $(M, <)$. *Then, the inverse limit* $\mathcal{X}_\infty$ *is the subset of the product* $\Pi_{\alpha \in M} X_\alpha$ *consisting of those elements* $x = (x_\alpha)_{\alpha \in M}$, *such that for each relation* $\alpha < \beta$ *in* M, $\pi_\alpha^\beta(x_\beta) = x_\alpha$.

If all the sets $\mathcal{X}_\alpha$ are topological space, then to $\mathcal{X}_\infty$ is assigned the topology as subspace of $\Pi_{\alpha \in M}\mathcal{X}_\alpha$. Naturally, the projections

$$\pi_\alpha\colon \mathcal{X}_\infty \to \mathcal{X}_\alpha, \qquad \pi_\alpha(x) = x_\alpha,$$

are continuous maps. Moreover, if for each $\alpha \in I$, $\mathcal{X}_\alpha$ is a nonempty compact space, then $\mathcal{X}_\infty$ is also nonempty and compact, see [4;VIII,3.3,3.6].

3. The Main Result.

In this section we extend the joint spectrum which we have defined from complex solvable finite dimensional Lie algebras of operators to quasi-solvable Lie algebras of operators. As we have said, in order to define this joint spectrum and to prove its main spectral properties, we work with the notion of inverse limit as in [9] and in [1]. On the other hand, we first give a definition of the joint spectrum which depends on a particular presentation of the algebra $\mathcal{L}$, and then we show that this definition is the correct one. We proceed as follows.

Let us consider X a Banach space, $\mathcal{L}$ a complex quasi-solvable Lie subalgebra of $\mathcal{L}(X)$ and an index set I such that $\mathcal{L} = \sum_{\alpha \in I} I_\alpha$, where, for each $\alpha \in I$, I_α is a complex solvable finite dimensional ideal of $\mathcal{L}$, and such that I, with the inclusion, is a directed set, i.e., if I_α and I_β are solvable ideals of $\mathcal{L}$ such that α and β belong to I, then there is a solvable ideal I_γ, $\gamma \in I$, such that $I_\alpha \cup I_\beta \subseteq I_\gamma$. For example, if $\mathbb{I}$ is the set of all complex solvable finite dimensional ideals of $\mathcal{L}$, then $\mathbb{I}$ is a directed set.

In addition, if I is as above, let us consider the family of maps $\pi = \{\pi^{\beta}_{\alpha}: I^*_{\beta} \to I^*_{\alpha}\}$, where α and β belong to I, $\alpha < \beta$, and π is the usual pojection, i.e., the restriction map. Let us also consider the family of sets $\mathcal{X} = \{\sigma(I_\alpha)\}_{\alpha\in I}$. Then, by the projection property of the joint spectrum, we have that $\{\mathcal{X}, \pi\}$ is an inverse system of topological spaces. Now, we may state our definition of the joint spectrum for quasi-solvable Lie algebras.

Definition 5. *Let X, $\mathcal{L}$, I, $(I_\alpha)_{\alpha\in I}$ and $\{\mathcal{X}, \pi\}$ be as above. The joint spectrum of the quasi-solvable algebra $\mathcal{L}$, relative to the presentation of $\mathcal{L}$ defined by I and by $(I_\alpha)_{\alpha\in I}$, is the inverse limit of the inverse system $\{\mathcal{X}, \pi\}$, and it is denoted by*

$$\sigma(\mathcal{L}, (I_\alpha)_{\alpha\in I}) = \mathcal{X}_\infty.$$

We observe that this definition depends on the set I and on a particular presentation of $\mathcal{L}$, however, Proposition 3 shows that the extended joint spectrum is independent of the presentation of $\mathcal{L}$, and Theorem 6 that it is uniquely determined by its properties.

On the other hand, by [4;VIII,3.6] we have that $\sigma(\mathcal{L}, (I_\alpha)_{\alpha\in I})$ is a compact nonempty subset of $\Pi_{\alpha\in I}\sigma(I_\alpha)$. Let us now study in more detail the properties of the introduced joint spectrum.

Proposition 1. *Let X, $\mathcal{L}$, I, $(I_\alpha)_{\alpha\in I}$ and $\{\mathcal{X}, \pi\}$ be as above. Then, the joint spectrum $\sigma(\mathcal{L}, (I_\alpha)_{\alpha\in I})$ may be identified with a subset of the characters of $\mathcal{L}$.*

Proof.

Let $(f_\alpha)_{\alpha\in I}$ belongs to $\sigma(L, (I_\alpha)_{\alpha\in I})$, and let us associate to this element the function f defined by $f \mid I_\alpha = f_\alpha$. Let us see that f is a well defined character of $\mathcal{L}$.

First of all let us consider $x \in I_\alpha \cap I_\beta$, where α and $\beta \in I$. Thus, since I is a directed set, there is an ideal I_γ, $\gamma \in I$, such that $I_\alpha \cup I_\beta \subseteq I_\gamma$. Then, since $\{\mathcal{X}, \pi\}$ is an inverse system we have that $f_\alpha(x) = f_\gamma(x) = f_\beta(x)$.

Now if $x \in \mathcal{L}$, let us present it as $x = \sum_{j=1}^{j=n} x_{\alpha_j} = \sum_{k=1}^{k=m} x_{\alpha'_k}$, where $x_{\alpha_j} \in I_{\alpha_j}$, $x_{\alpha'_k} \in I_{\alpha'_k}$, and α_j and α'_k belong to I, $1 \le j \le n$, $1 \le k \le m$. Since I is a directed set, there is a finite dimensional solvable ideal of $\mathcal{L}$, I_β, $\beta \in I$, such that $\cup_{j=1}^{j=n} I_{\alpha_j} \cup \cup_{k=1}^{k=m} I_{\alpha'_k} \subseteq I_\beta$. Then

$$\sum_{j=1}^{j=n} f_{\alpha_j}(x_{\alpha_j}) = f_\beta(x) = \sum_{k=1}^{k=m} f_{\alpha_k'}(x_{\alpha'_k}).$$

Thus, f is a well defined map.

By a similar argument it is easy to see that f is a character of $\mathcal{L}$. Moreover, by construction, the above assigment is an injective identification.

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Proposition 2. *Let X, $\mathcal{L}$, I, $(I_\alpha)_{\alpha\in I}$ and $\{\mathcal{X},\pi\}$ be as above. Then*

$$\sigma(\mathcal{L},(I_\alpha)_{\alpha\in I}) = \{f/f \text{ is a character of } \mathcal{L}, \text{ and for each } \alpha\in I,\ f\mid I_\alpha\in\sigma(I_\alpha)\}.$$

Proof.

By the definition of $\sigma(\mathcal{L},(I_\alpha)_{\alpha\in I})$ and by Proposition 1, we have that the joint spectrum is contained in the right hand set of the identity.

On the other hand, if f is a character of $\mathcal{L}$ such that $f\mid I_\alpha\in\sigma(I_\alpha)$, then by the projection property of the joint spectrum, $(f_\alpha)_{\alpha\in I}$ belongs to $\sigma(\mathcal{L},(I_\alpha)_{\alpha\in I})$. However, by Proposition 1, $(f_\alpha)_{\alpha\in I}$ is identified with f.

∎

As a consequence of the Proposition 2 we have that the joint spectrum is independent of a particular presentation of the quasi-solvable Lie algebra $\mathcal{L}$.

Proposition 3. *Let X and $\mathcal{L}$ be as above, and I_j, $j=1,\,2$, two directed index sets such that $\mathcal{L}=\sum_{\alpha_1\in I_1} I_{\alpha_1}=\sum_{\alpha_2\in I_2} I_{\alpha_2}$, where I_{α_j}, $j=1,\,2$, are complex solvable finite dimensional ideals of $\mathcal{L}$. Then*

$$\sigma(\mathcal{L},(I_{\alpha_1})_{\alpha_1\in I_1})=\sigma(\mathcal{L},(I_{\alpha_2})_{\alpha_2\in I_2}).$$

Proof.

It is enough to see that if $\mathbb{I}$ the set of all complex solvable finite dimensional ideals of $\mathcal{L}$, then $\sigma(\mathcal{L},(I_\alpha)_{\alpha\in I})=\sigma(\mathcal{L},(I_\alpha)_{\alpha\in\mathbb{I}})$, where I is a directed index set and $(I_\alpha)_{\alpha\in I}$ is a particular presentation of $\mathcal{L}$.

First of all, by Proposition 2 it is clear that $\sigma(\mathcal{L},(I_\alpha)_{\alpha\in\mathbb{I}})\subseteq\sigma(\mathcal{L},(I_\alpha)_{\alpha\in I})$.

On the other hand, if $f\in\sigma(\mathcal{L},(I_\alpha)_{\alpha\in I})$, let us consider I_{α_0} an arbitrary complex solvable finite dimensional ideal of $\mathcal{L}$. Since $\mathcal{L}=\sum_{\alpha\in I} I_\alpha$, and I, with the inclusion, is a directed set, there is $\beta\in I$ such that $I_{\alpha_0}\subseteq I_\beta$. Now, since $f\in\sigma(\mathcal{L},(I_\alpha)_{\alpha\in I})$, by Proposition 2 we have that $f\mid I_\beta\in\sigma(I_\beta)$. However, by the projection property of the joint spectrum, $\sigma(I_\beta)\mid I_{\alpha_0}=\sigma(I_{\alpha_0})$. Then, $f\mid I_{\alpha_0}\in\sigma(I_{\alpha_0})$ and $f\in\sigma(\mathcal{L},(I_\alpha)_{\alpha\in\mathbb{I}})$.

∎

In the following propositions we prove two of the most important properties of the joint spectrum for quasi-solvable Lie algebras of operators.

Proposition 4. *Let X, $\mathcal{L}$, I, $(I_\alpha)_{\alpha\in I}$ and $\{\mathcal{X},\pi\}$ be as above. Then, if L is a complex solvable finite dimensional ideal of $\mathcal{L}$,*

$$\sigma(L,(L\cap I_\alpha)_{\alpha\in I}) = \sigma(L).$$

Proof.

First of all we observe that $L=\sum_{\alpha\in I} L\cap I_\alpha$.

Indeed, it is clear that $\sum_{\alpha\in I} L\cap I_\alpha \subseteq L$. On the other hand, if $x\in L\subseteq\mathcal{L}$, there are $\alpha_i\in I$ and $x_i\in I_{\alpha_i}$, $i=1,\ldots,n$, such that $x=\sum_{i=1}^n x_i$. However, since I is a directed set with the inclusion, there is $\alpha\in I$ such that $\cup_{i=1}^n I_{\alpha_i}\subseteq I_\alpha$. Thus, $x\in L\cap I_\alpha\subseteq\sum_{\alpha\in I} L\cap I_\alpha$.

Now, since $L=\sum_{\alpha\in I} L\cap I_\alpha$, we may construct the inverse limit set.

In addition, since L is a finite dimensional ideal of $\mathcal{L}$, and I is a directed set, there is a solvable finite dimensional ideal of $\mathcal{L}$, I_β, $\beta\in I$, such that $L=L\cap I_\beta$. Now, if f is a character of L, by construction of $\sigma(L,(L\cap I_\alpha)_{\alpha\in I})$ and of Proposition 2 we have that $f\mid L\cap I_\beta\in\sigma(L\cap I_\beta)$. Since $L=L\cap I_\beta$, $f\mid L\cap L_\beta=f$, then $\sigma(L,(L\cap I_\alpha)_{\alpha\in I})\subseteq\sigma(L)$.

On the other hand, if $f\in\sigma(L)$, by the projection property for ideals of the joint spectrum, $f\mid L\cap I_\alpha\in\sigma(L\cap I_\alpha)$, thus, as f is a character of L, by Proposition 2 we have the reverse contention.

∎

Proposition 5. *Let X, $\mathcal{L}$, I, $(I_\alpha)_{\alpha\in I}$ and $\{\mathcal{X},\pi\}$ be as above. If $\mathcal{H}$ is an ideal of $\mathcal{L}$. Then, the joint spectrum has the projection property for ideals, i.e.,*

$$\sigma(\mathcal{L},(I_\alpha)_{\alpha\in I})\mid\mathcal{H} = \sigma(\mathcal{H},(\mathcal{H}\cap I_\alpha)_{\alpha\in I}).$$

Proof.

First of all, let us consider the directed set I. Then, an easy calculation shows that $\mathcal{H}=\sum_{\alpha\in I} H\cap I_\alpha$. Thus, we may consider the set $\sigma(\mathcal{H},(\mathcal{H}\cap I_\alpha)_{\alpha\in I})$.

Now we consider the inverse systems $\{\mathcal{X},\pi\}$ and $\{\mathcal{X}',\pi'\}$ defined by $\mathcal{X}_\alpha=\sigma(I_\alpha)$, $\mathcal{X}'_\alpha=\sigma(\mathcal{H}\cap I_\alpha)$, and π and π' are the families of projection maps defined between the corresponding spectral sets. Then, if we consider the map $Id: I \to I$, and if for each $\alpha \in I$ we also consider $P_\alpha: \sigma(I_\alpha) \to \sigma(\mathcal{H} \cap I_\alpha)$, the canonical restriction map, then, it is easy to verify that the identity map of I and the family $(P_\alpha)_{\alpha \in I}$ is a map of inverse systems, see [4;VIII,2.3]. Thus, we have a well defined and continuos map $P_\infty: \sigma(\mathcal{L}, (I_\alpha)_{\alpha \in I}) \to \sigma(\mathcal{H}, (\mathcal{H} \cap I_\alpha)_{\alpha \in I})$, see [4;VIII,3.10,3.11,3.13]. However, by the identification of Proposition2 we have that

$$\sigma(\mathcal{L}, (I_\alpha)_{\alpha \in I}) \mid \mathcal{H} = P_\infty(\sigma(\mathcal{L}, (I_\alpha)_{\alpha \in I})) \subseteq \sigma(\mathcal{H}, (\mathcal{H} \cap I_\alpha)_{\alpha \in I}).$$

On the other hand, let us suppose that there is $f \in \sigma(\mathcal{H}, (\mathcal{H} \cap I_\alpha)_{\alpha \in I}) \setminus P_\infty(\sigma(\mathcal{L}, (I_\alpha)_{\alpha \in I})$. Since $\sigma(\mathcal{L}, (I_\alpha)_{\alpha \in I})$ is a compact set and P_∞ is a continous map, there is an open set U, which may be chosen in the base of the topology of $\sigma(\mathcal{H}, (\mathcal{H} \cap I_\alpha)_{\alpha \in I})$, such that $f \in U$ and that $U \cap P_\infty(\sigma(\mathcal{L}, (I_\alpha)_{\alpha \in I}))$ is the empty set.

In addition, by [4;VIII,3.1,3.3,3.9] we have two well defined families of surjective and continuous maps, $(\pi_\alpha)_{\alpha \in I_\alpha}$ and $(\pi'_\alpha)_{\alpha \in I_\alpha}$, which satisfies

$$\pi_\alpha: \sigma(\mathcal{L}, (I_\alpha)_{\alpha \in I}) \to \sigma(I_\alpha),$$

$$\pi'_\alpha: \sigma(\mathcal{H}, (\mathcal{H} \cap I_\alpha)_{\alpha \in I}) \to \sigma(\mathcal{H} \cap I_\alpha).$$

Moreover, it is easy to see that $\pi'_\alpha \circ P_\infty = P_\alpha \circ \pi_\alpha$, [4;VIII,3.11].

Now, by [4;VIII,3.12] we know that there is an $\alpha \in I$ and V in the topology of $\sigma(\mathcal{H} \cap I_\alpha)$ such that $U = \pi'^{-1}_\alpha(V)$. Since P_α and π_α are surjective and continous maps, $(P_\alpha \circ \pi_\alpha)^{-1}(V)$ is an open nonempty set. Then, $(\pi'_\alpha \circ P_\infty)^{-1}(V) = (P_\infty)^{-1}(U)$ is an open nonempty set, which is impossible for $U \cap P_\infty(\sigma(\mathcal{L}, (I_\alpha)_{\alpha \in I}))$ is the empty set.

∎

We now state our main result.

Theorem 6. *Let X be a complex Banach space and $\sigma(.)$ a joint spectrum for complex solvable finite dimensional Lie algebras of operators defined on X. Then, for each complex quasi-solvable Lie subalgebra $\mathcal{L}$ of $\mathcal{L}(X)$, there is an uniquely well defined map, also denoted by $\sigma(.)$, such that the following conditions are fullfilled.*

i) $\sigma(\mathcal{L})$ is a subset of characters of $\mathcal{L}$ and a compact nonempty subset of $\Pi_{\alpha \in \mathbb{I}} \sigma(I_\alpha)$, where $\mathbb{I}$ denotes the set of all complex solvable finite dimensional ideals of $\mathcal{L}$,

ii) if $\mathcal{H}$ is a complex solvable finite dimensional ideal of $\mathcal{L}$, then $\sigma(\mathcal{H})$ coincides with the joint spectrum of $\mathcal{H}$ defined in the finite dimensional case, *iii) if $\mathcal{M}$ is a subalgebra of $\mathcal{L}$, and $\mathcal{H}$ is a Lie ideal of $\mathcal{M}$, then for the joint spectrum the projection property for ideals holds., i.e.,*

$$\sigma(\mathcal{M}) \mid \mathcal{H} = \sigma(\mathcal{H}).$$

Proof.

By the propositions we have proved, in order to see the existence of such joint spectrum, it is enough to consider the set $\mathbb{I}$ of all complex solvable finite dimensional ideals of $\mathcal{L}$, and the presentation of $\mathcal{L} = \sum_{\alpha\in\mathbb{I}} I_\alpha$. In fact, if $\mathcal{M}$ is a Lie subalgebra, or ideal, of $\mathcal{L}$, an easy calculation shows that $\mathcal{M} = \sum_{\alpha\in\mathbb{I}} \mathcal{M} \cap I_\alpha$, thus, we may define $\sigma(\mathcal{M})$ as

$$\sigma(\mathcal{M}) = \sigma(\mathcal{M}, (\mathcal{M} \cap I_\alpha)_{\alpha\in\mathbb{I}}).$$

With this definition, the joint spectrum satisfies properties i)-iii).

In order to prove that this map is uniquely determined, we proceed as follows.

Let us suppose that $\tilde{\sigma}$ is an assigment which satisfies the previous conditions. Then, by Proposition 2, and by conditions i)-iii), $\tilde{\sigma}(\mathcal{L}) \subseteq \sigma(\mathcal{L})$.

On the other hand, if $f \in \sigma(\mathcal{L}) \setminus \tilde{\sigma}(\mathcal{L})$, since both joint spectra are compact subsets of $\Pi_{\alpha\in\mathbb{I}}\sigma(I_\alpha)$, there is an open set U, which may be chosen in the base of the topology of $\sigma(\mathcal{L})$, such that $f \in U$ and $U \cap \tilde{\sigma}(\mathcal{L})$ is the empty set. Moreover, by [4;VIII,3.12] there is $\alpha \in \mathbb{I}$ and V in the topology of $\sigma(I_\alpha)$ such that $U = \pi_\alpha^{-1}(V)$.

Now, by condition ii) we have that $f \mid I_\alpha \in \sigma(I_\alpha) = \tilde{\sigma}(\mathcal{L}) \mid I_\alpha$. Thus, by condition iii), there is $g \in \tilde{\sigma}(\mathcal{L})$ such that $g \mid I_\alpha = f \mid I_\alpha$. However, since $g \mid I_\alpha = f \mid I_\alpha$, g belongs to U, which is impossible for $g \in \tilde{\sigma}(\mathcal{L})$.

∎

As we have pointed out, this construction may be applied to the Taylor and the Słodkowski joint spectra for complex solvable finite dimensional Lie algebras of operators, see [2], [3] and [6], and then we extend these joint spectra to quasi solvable Lie algebras of operators. Moreover, this construction gives a non commutative version of the one developed by Z. Słodkowski and W. Zelazko in [9], which, in general, differs from the one considered by D. Beltita in [1]. In fact, in the solvable finite dimensional case the joint spectrum of [1] does not, in general, coincide with the one of [2], [3] and [6].

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